QUANTIFICATION IN NONCLASSICAL LOGIC
Preface

If some 30 years ago we had been told that we would write a large book on quantification in nonclassical logic, none of us would have taken it seriously: first — because at that time there was no hope of our effective collaboration; second — because in nonclassical logic too much had to be done in the propositional area, and few people could find the energy for active research in predicate logic.

In the new century the situation is completely different. Connections between Moscow and London became easy. The title of the book is not surprising, and we are now late with the first big monograph in this field. Indeed, at first we did not expect we had enough material for two (or more) volumes. But we hope readers will be able to learn the subject from our book and find it quite fascinating.

Let us now give a very brief overview of the existing systematic expositions of nonclassical first-order logic. None of them aims at covering the whole of this large field. The first book on the subject was [Rasiowa and Sikorski, 1963], where the approach used by the authors was purely algebraic. Many important aspects of superintuitionistic first-order logics can be found in the books written in the 1970–80s: [Dragalin, 1988] (proof theory; algebraic, topological, and relational models; realisability semantics); [Gabbay, 1981] (model theory; decision problem); [van Dalen, 1973], [Troelstra and van Dalen, 1988] (realisability and model theory). The book of Novikov [Novikov, 1977] (the major part of which is a lecture course from the 1950s) addresses semantics of superintuitionistic logics and also includes some material on modal logic.

Still, predicate modal logic was partly neglected until the late 1980s. The book [Harel, 1979] and its later extended version [Harel and Tiuryn, 2000] study particular dynamic modal logics. [Goldblatt, 1984] is devoted to topos semantics; its main emphasis is on intuitionistic logic, although modal logic is also considered. The book [Hughes and Cresswell, 1996] makes a thorough study of Kripke semantics for first-order modal logics, but it does not consider other semantics or intermediate logics. Finally, there is a monograph [Gabbay et al., 2002], which, among other topics, investigates first-order modal and intermediate logics from the ‘many-dimensional’ viewpoint. It contains recent profound results on decidable fragments of predicate logics.

The lack of unifying monographs became crucial in the 1990s, to the extent that in the recent book [Fitting and Mendelsohn, 1998] the area of first-order modal logic was unfairly called ‘under-developed’. That original book contains
interesting material on the history and philosophy of modal logic, but due to its obvious philosophical flavour, it leaves many fundamental mathematical problems and results unaddressed. Still there the reader can find various approaches to quantification, tableau systems and corresponding completeness theorems. So there remains the need for a foundational monograph not only addressing areas untouched by all current publications, but also presenting a unifying point of view.

A detailed description of this Volume can be found in the Introduction below. It is worth mentioning that the major part of the material has never been presented in monographs. One of its sources is the paper on completeness and incompleteness, a brief version of which is [Shehtman and Skvortsov, 1990]; the full version (written in 1983) has not been published for technical reasons. The second basic paper incorporated in our book is [Skvortsov and Shehtman, 1993], where so-called metaframe (or simplicial) semantics was introduced and studied. We also include some of the results obtained after 1980 by G. Corsi, S. Ghilardi, H. Ono, T. Shimura, D. Skvortsov, N.-Y. Suzuki, and others.

However, because of lack of space, we had to exclude some interesting material, such as a big chapter on simplicial semantics, completeness theorems for topological semantics, hyperdoctrines, and many other important matters. Other important omissions are the historical and the bibliographical overviews and the discussion of application fields and many open problems. Moreover, the cooperation between the authors was not easy, because of the different viewpoints on the presentation. There may be also other shortcomings, like gaps in proofs, wrong notation, wrong or missing references, misprints etc., that remain uncorrected — but this is all our responsibility.

We would be glad to receive comments and remarks on all the defects from the readers. As we are planning to continue our work in Volume 2, we still hope to make all necessary corrections and additions in the real future.

At present the reader can find the list of corrections on our webpages
http://www.dcs.kcl.ac.uk/staff/dg/
http://lpcs.math.msu.su/~shehtman

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One of the authors points out that he disagrees with some of the notation and the style of some proofs in the final version.
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Introduction

Quantification and modalities have always been topics of great interest for logicians. These two themes emerged from philosophy and language in ancient times; they were studied by traditional informal methods until the 20th century. Then the tools became highly mathematical, as in the other areas of logic, and modal logic as well as quantification (mainly on the basis of classical first-order logic) found numerous applications in Computer Science.

At the same time many other kinds of nonclassical logics were investigated. In particular, intuitionistic logic was created by L. Brouwer at the beginning of the century as a new basis for mathematical reasoning. This logic, as well as its extensions (superintuitionistic logics), is also very useful for Computer Science and turns out to be closely related to modal logics.

(A) The introduction of quantifier axioms to classical logic is fairly straightforward. We simply add the following obvious postulates to the propositional logic:

1. $\forall xA(x) \supset A(t)$,
   where $t$ is 'properly' substituted for $x$

2. \[
\frac{A \supset B(x)}{A \supset \forall xB(x)},
\]
   where $x$ is not free in $A$

3. $A(t) \supset \exists xA(x)$,

4. \[
\frac{B(t) \supset A}{\exists xB(x) \supset A},
\]
   where $x$ is not free in $A$.

The passage from the propositional case of a logic $L$ to its quantifier case works for many logics by adding the above axioms to the respective propositional axioms — for example, the intuitionistic logic, standard modal logics $\mathbf{S4, S5, K}$ etc. We may need in some cases to make some adjustment to account for constant domains, $\forall x(A \lor B(x)) \supset (A \lor \forall xB(x))$ in case of intuitionistic logic and the Barcan formula, $\forall x\Box A(x) \supset \Box \forall xA(x)$ in the case of modal logics. On the whole the correspondence seems to be working.
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The recipe goes on as follows:
take the propositional semantics and put a domain $D_u$ in each world $u$ or take
the axiomatic formulation and add the above axioms and you maintain corre-
spondence and completeness.

There were some surprises however. Unexpectedly, this method fails for very
simple and well-known modal and intermediate logics: the 'Euclidean logic'
$K5 = K + \Box \Diamond p \supset \Box p$ (see Chapter 6 of this volume), the 'confluence logic'
$S4.2 = S4 + \Diamond \Box p \supset \Box \Diamond p$ and for the intermediate logic $KC = H + \neg p \lor \neg \neg p$
with constant domains, nonclassical intermediate logics of finite depth [Ono,
1983], etc. All these logics are incomplete in the standard Kripke semantics.

In some other cases, completeness theorems hold, but their proofs require
nontrivial extra work — for example, this happens for the logic of linear Kripke
frames $S4.3$ [Corsi, 1989].

This situation puts at least two difficult questions to us: (1) how should
we change semantics in order to restore completeness of 'popular' logics? (2)
how should we extend these logics by new axioms to make them complete in
the standard Kripke semantics? These questions will be studied in our book,
especially in Volume 2, but we are still very far from final answers.

Apparently when we systematically introduce natural axioms and ask for
the corresponding semantics, we may not be able to see what are the natural
semantical conditions (which may not be expressible in first-order logic) and
conversely some natural conditions on the semantics require complex and some-
times non-axiomatisable logics.

The community did not realize all these difficulties. A serious surprise was
the case of relevance logic, where the additional axioms were complex and
seemed purely technical. See [Mares and Goldblatt, 2006], [Fine, 1988], [Fine,
1989]. For some well-known logics there were no attempts of going first-order,
especially for resource logics such as Lambek Calculus.

(B) There are other reasons why we may have difficulties with quantifiers,
for example, in the case of superintuitionistic logics. Conditions on the possible
worlds such as discrete ordering or finiteness may give the connectives them-
selves quantificational power of their own (note that the truth condition for
$A \supset B$ has a hidden world quantifier), which combined with the power of the
explicit quantifiers may yield some pretty complex systems [Skvortsov, 2006].

(C) In fact, a new approach is required to deal with quantifiers in possible
world systems. The standard approach associates domains with each possible
world and what is in the domain depends only on the nature of the world, i.e.
if $u$ is a world, $P$ a predicate, $\theta$ a valuation, then $\theta_u(P)$ is not dependent on
other $\theta_{u'}(P)$, except for some very simple conditions as in intuitionistic logic.

There are no interactive conditions between existence of elements in the
domain and satisfaction in other domains. If we look at some axioms like the
Markov principle
$$\neg \exists x A(x) \supset \exists x \neg A(x),$$
we see that we need to pay attention on how the domain is constructed. This
is reminiscent of the Herbrand universe in classical logic.
(D) There are other questions which we can ask. Given a classical theory \( \Gamma \) (e.g. a theory of rings or Peano arithmetic), we can investigate what happens if we change the underlying logic to intuitionistic or modal or relevant. Then what kind of theory do we get and what kind of semantics? Note we are not dealing now with a variety of logics (modal or superintuitionistic), but with a fixed nonclassical logic (say intuitionistic logic itself) and a variety of theories.

If intuitionistic predicate semantics is built up from classical models, would the intuitionistic predicate theory of rings have semantics built up from classical rings? How does it depend on the formulation (\( \Gamma \) may be classically equivalent to \( \Gamma' \), but not intuitionistically) and what can happen to different formulations? See [Gabbay, 1981].

(E) One can have questions with quantifiers arising from a completely different angle. E.g. in resource logics we pay attention to which assumptions are used to proving a formula \( \phi \).

For example in linear or Lambek logic we have that

\[
\begin{align*}
(1) & \quad \phi \\
(2) & \quad \phi \\
(3) & \quad \phi \to (\phi \to \psi) \nonumber
\end{align*}
\]

can prove \( \psi \) but (2) and (3) alone cannot prove \( \psi \); because of resource considerations, we need two copies of \( \phi \). Such logics are very applicable to the analysis and modelling of natural language [van Benthem, 1991]. So what shall we do with \( \forall x \phi(x) \)? Do we divide our resource between all instances \( \phi(t_1), \phi(t_2), \ldots \) of \( \phi \)? These are design questions which translate into technical axiomatic and semantical questions.

How do we treat systems which contain more than one type of nonclassical connective? Any special problems with regard to adding quantifiers? See, for example, the theory of bunched implications [O’Heann and Pyms, 1999].

(F) The most complex systems with regards to quantifiers are LDS, Labelled Deductive Systems (this is a methodology for logic, cf. [Gabbay, 1996; Gabbay, 1998]). In LDS formulas have labels, so we write \( t : \phi \), where \( t \) is a label and \( \phi \) is a formula. Think of \( t \) as a world or a context. (This label can be integrated and in itself be a formula, etc.) Elements now have visa rules for migrating between labels and need to be annotated, for example as \( a^t_s \), the element \( a \) exists at world \( s \), but was first created (or instantiated) in world \( t \). Surprisingly, this actually helps with the proof theory and semantics for quantifiers, since part of the semantics is brought into the syntax. See [Viganò, 2000]. So it is easier to develop, say, theories of Hilbert \( \varepsilon \)-symbol using labels. \( \varepsilon \)-symbols axioms cannot be added simple mindedly to intuitionistic logic, it will collapse [Bell, 2001].

(G) Similarly, we must be careful with modal logic. We have not even begun thinking about \( \varepsilon \)-symbols in resource logics (consider \( \varepsilon x \phi(x) \), if there is sensitivity for the number of copies of \( \phi \), then are we to be sensitive also to copies of elements?).

(H) In classical logic there is another direction to go with quantifiers, namely the so-called generalised quantifiers, for example \( (\text{many } x) \phi(x) \) ('there are many
x such that $A(x')$), or (uncountably many $x)A(x)$ or many others. Some of these can be translated as modalities as van Lambalgen has shown [Alechina, van Lambalgen, 1994], [van Lambalgen, 1996]. Such quantifiers (at least for the finite case) exist in natural language. They are very important and they have not been exported yet to nonclassical logics (only through the modalities e.g. $\Diamond_n A$ ('$A$ is true in $n$ possible worlds'), see [Gabbay, Reynolds and Finger, 2000], [Peters and Westerstahl, 2006]).

Volume 1 of these books concentrates on the landscape described in (A) above, i.e., correspondence between axioms for modal or intuitionistic logic and semantical conditions and vice versa.

Even for such seemingly simple questions we have our hands full. The table of contents for future volumes shows what to be addressed in connection with (B)-(H). It is time for nonclassical logic to pay full attention to quantification. Up to now the focus was mainly propositional. Now the era of the quantifier has begun!

This Volume includes results in nonclassical first-order logic obtained during the past 40 years. The main emphasis is model-theoretic, and we confine ourselves with only two kinds of logics: modal and superintuitionistic. Thus many interesting and important topics are not included, and there remains enough material for future volumes and future authors.

Figure 1. Chapters dependency structure

Let us now briefly describe the contents of Volume 1. It consists of three parts. Part I includes basic material on propositional logic and first-order syntax.

Chapter 1 contains definitions and results on syntax and semantics of nonclassical propositional logics. All the material can be found elsewhere, so the proofs are either sketched or skipped.

Chapter 2 contains the necessary syntactic background for the remaining parts of the book. Our main concern is the precise notion of substitution based
on re-naming of variables. This classical topic is well known to all students in logic. However none of the existing definitions fits well for our further purposes, because in some semantics soundness proofs may be quite intricate. Our approach is based on the idea that re-naming of bound variables creates different synonymous (or 'congruent') versions of the same predicate formula. These versions are generated by a 'scheme' showing the reference structure of quantifiers. (Schemes are quite similar to formulas in the sense of [Bourbaki, 1968].) Now variable substitutions (acting on schemes or congruence classes) can be easily arranged in an appropriate congruent version. After this preparation we introduce two main types of first-order logics to be studied in the book — modal and superintuitionistic, and prove syntactic results that do not require involved proof theory, such as deduction theorems, Glivenko theorem etc.

In Part II (Chapters 3 - 5) we describe different semantics for our logics and prove soundness results.

Chapter 3 considers the simplest kinds of relational semantics. We begin with the standard Kripke semantics and then introduce two its generalisations, which are equivalent: Kripke frames with equality and Kripke sheaves. The first one (for the intuitionistic case) is due to [Dragalin, 1973], and the second version was first introduced in [Shehtman and Skvortsov, 1990]. Soundness proofs in that chapter are not obvious, but rather easy. We mention simple incompleteness results showing that Kripke semantics is weaker than these generalisations. Further incompleteness theorems are postponed until Volume 2. We also prove results on Löwenheim – Skolem property and recursive axiomatisability using translations to classical logic from [Ono, 1972/73] and [van Benthem, 1983].

Chapter 4 studies algebraic semantics. Here the main objects are Heyting-valued (or modal-valued) sets. In the intuitionistic case this semantics was studied by many authors, see [Dragalin, 1988], [Fourman and Scott, 1979], [Goldblatt, 1984]. Nevertheless, our soundness proof seems to be new. Then we show that algebraic semantics can be also obtained from presheaves over Heyting (or modal) algebras. We also show that for the case of topological spaces the same semantics is given by sheaves and can be defined via so-called 'fibrewise models'. These results were first stated in [Shehtman and Skvortsov, 1990], but the proofs have never been published so far. They resemble the well-known results in topos theory, but do not directly follow from them.

In Chapter 5 we study Kripke metaframes, which are a many-dimensional generalisation of Kripke frames from [Skvortsov and Shehtman, 1993] (where they were called 'Cartesian metaframes'). The crucial difference between frames and metaframes is in treatment of individuals. We begin with two particular cases of Kripke metaframes: Kripke bundles [Shehtman and Skvortsov, 1990] and C-sets (sheaves of sets over (pre)categories) [Ghilardi, 1989]. Their predecessor in philosophical logic is 'counterpart theory' [Lewis, 1968]. In a Kripke bundle individuals may have several 'inheritors' in the same possible world, while in a C-set instead of an inheritance relation there is a family of maps. In

2The first author is happy to fulfill his promise given in the preface of [Gabbay, 1981]: "It would require further research to be able to present a general theory [of topological models, second order Beth and Kripke models] possibly using sheaves".

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Kripke metaframes there are additional inheritance relations between tuples of individuals.

The proof of soundness for metaframes is rather laborious (especially for the intuitionistic case) and is essentially based on the approach to substitutions from Chapter 2. This proof has never been published in full detail. Then we apply soundness theorem to Kripke bundle and functor semantics. The last section of Chapter 5 gives a brief introduction to an important generalisation of metaframe semantics – so called 'simplicial semantics'. The detailed exposition of this semantics is postponed until Volume 2.

Part III (Chapters 6-7) is devoted to completeness results in Kripke semantics. In Kripke semantics many logics are incomplete, and there is no general powerful method for completeness proofs, but still we describe some approaches.

In Chapter 6 we study Kripke frames with varying domains. First, we introduce different types of canonical models. The simplest kind is rather well-known, cf. [Hughes and Cresswell, 1996], but the others are original (due to D. Skvorstov). We prove completeness for intermediate logics of finite depth [Yokota, 1989], directed frames [Corsi and Ghilardi, 1989], linear frames [Corsi, 1992]. Then we elucidate the methods from [Skvortsov, 1995] for axiomatising some 'tabular' logics (i.e., those with a fixed frame of possible worlds).

Chapter 7 considers logics with constant domains. We again present different canonical models constructions and prove completeness theorems from [Hughes and Cresswell, 1996]. Then we prove general completeness results for subframe and cofinal subframe logics from [Tanaka and Ono, 1999], [Shimura, 1993], [Shimura, 2001], Takano's theorem on logics of linearly ordered frames [Takano, 1987] and other related results.

Here are chapter headings in preparation for later volumes:

Chapter 8. Simplicial semantics
Chapter 9. Hyperdoctrines
Chapter 10. Completeness in algebraic and topological semantics
Chapter 11. Translations
Chapter 12. Definability
Chapter 13. Incompleteness
Chapter 14. Simulation of classical models
Chapter 15. Applications of semantical methods
Chapter 16. Axiomatisable logics
Chapter 17. Further results on Kripke-completeness
Chapter 18. Fragments of first-order logics
Chapter 19. Propositional quantification
Chapter 20. Free logics
Chapter 21. Skolemisation
Chapter 22. Conceptual quantification
Chapter 23. Categorical logic and toposes
Chapter 24. Quantification in resource logic
Chapter 25. Quantification in labelled logics.
Chapter 26. e-symbols and variable dependency
Chapter 27. Proof theory

Some guidelines for the readers. Reading of this book may be not so easy. Parts II, III are the most important, but they cannot be understood without Part I.

For the readers who only start learning the field, we recommend to begin with sections 1.1–1.5, then move to sections 2.1, 2.2, the beginning parts of sections 2.3, 2.6, and next to 2.16. After that they can read Part II and sometimes go back to Chapters 1, 2 if necessary. We do not recommend them to go to Chapter 5 before they learn about Kripke sheaves. Those who are only interested in Kripke semantics can move directly from Chapter 3 to Part III.

An experienced reader can look through Chapter 1 and go to sections 2.1–2.5 and the basic definitions in 2.6, 2.7. Then he will be able to read later Chapters starting from Chapter 3.
Notation convention

We use logical symbols both in our formal languages and in the meta-language. The notation slightly differs, so the formal symbols $\land$, $\rightarrow$, $\equiv$ correspond to the metasymbols $\&$, $\Rightarrow$, $\Leftrightarrow$; and the formal symbols $\lor$, $\exists$, $\forall$ are also used as metasymbols.

In our terminology we distinguish functions and maps. A function from $A$ to $B$ is a binary relation $F \subseteq A \times B$ with domain $A$ satisfying the functionality condition ($xFy \& xFz \Rightarrow x = z$), and the triple $f = (F, A, B)$ is then called a map from $A$ to $B$. In this case we use the notation $f : A \rightarrow B$.

Here is some other set-theoretic notation and terminology.

- $2^X$ denotes the power set of a set $X$;
- we use $\subseteq$ for inclusion, $\subset$ for proper inclusion;
- $R \circ S$ denotes the composition of binary relations $R$ and $S$:
\[
R \circ S := \{(x, y) \mid \exists z (xRz \& zSy)\};
\]
- $R^{-1}$ is the converse of a relation $R$;
- $Id_W$ is the equality relation in a set $W$;
- $id_W$ is the identity map on a set $W$ (i.e. the triple $(Id_W, W, W)$);
- for a relation $R \subseteq W \times W$, $R(V)$, or just $RV$, denotes the image of a set $V \subseteq W$ under $R$, i.e. $\{y \mid \exists x \in V \ xRy\}$; $R(x)$ or $Rx$ abbreviates $R(\{x\})$;
- $\text{dom}(R)$, or $\text{pr}_1(R)$, denotes the domain of a relation $R$, i.e., $\{x \mid \exists y \ xRy\}$;
- $\text{rng}(R)$, or $\text{pr}_2(R)$, denotes the range of a relation $R$, i.e., $\{y \mid \exists x \ xRy\}$;
- for a subset $X \subseteq Y$ there is the inclusion map $j_{XY} : X \rightarrow Y$ (which is usually denoted just by $j$) sending every $x \in X$ to itself;
- $R \upharpoonright V$ denotes the restriction of a relation $R$ to a subset $V$, i.e. $R \upharpoonright V = R \cap (V \times V)$, and $f \upharpoonright V$ denotes the restriction of a map $f$ to $V$;
- for a relation $R$ on a set $X$ $R^{-} := R - Id_X$ is the 'irreflexivisation' of $R$;
- $|X|$ denotes the cardinality of a set $X$;
- $I_n$ denotes the set $\{1, \ldots, n\}$; $I_0 := \emptyset$;
- $X^\infty$ denotes the set of all finite sequences with elements in $X$;
- $(X_i \mid i \in I)$ (or $(X_i)_{i \in I}$) denotes the family of sets $X_i$ with indices in the set $I$;
- $\bigsqcup_{i \in I} X_i$ denotes the disjoint union of the family $(X_i)_{i \in I}$, i.e. $\bigcup_{i \in I} X_i \times \{i\}$;
• $\omega$ is the set of natural numbers, and $T_\omega$ denotes $\omega^\infty$;

• $\Sigma_{mn} = (I_n)^{I_m}$ denotes the set of all maps $\sigma : I_m \rightarrow I_n$ (for $m, n \in \omega$);

• $\Upsilon_{mn}$ denotes the set of all injective maps in $\Sigma_{mn}$;

• $\Upsilon_n$ is the abbreviation for $\Upsilon_{nn}$, the set of all permutations of $I_n$.

Note that we use two different notations for composition of maps: the composition of $f : A \rightarrow B$ and $g : B \rightarrow C$ is denoted by either $g \cdot f$ or $f \circ g$. So $(f \circ g)(x) = (g \cdot f)(x) = g(f(x))$.

Obviously, $\Sigma_{mn} \neq \emptyset \iff n > 0$ or $m = 0$,
$\Upsilon_{mn} \neq \emptyset \iff n \geq m$.

A map $f : I_m \rightarrow I_n$ (for fixed $n$) is presented by the table

$$
\begin{pmatrix}
1 & \ldots & m \\
\vdots & \ddots & \vdots \\
1 & \ldots & f(m)
\end{pmatrix}
$$

We use a special notation for some particular maps.

• Transpositions $\sigma_{ij}^n \in \Upsilon_n$ for $n \geq 2$, $1 \leq i < j \leq n$.

$$
\sigma_{ij}^n := 
\begin{pmatrix}
1 & \ldots & i & \ldots & j & \ldots & n \\
1 & \ldots & j & \ldots & i & \ldots & n
\end{pmatrix}
$$

In particular, simple transpositions are $\sigma_i^n := \sigma_{1i}^n$ for $1 < i \leq n$;

• Standard embeddings (inclusion maps).

$\sigma_{+mn} \in \Upsilon_{mn}$, for $0 \leq m \leq n$ is defined by the table

$$
\begin{pmatrix}
1 & \ldots & m \\
\vdots & \ddots & \vdots \\
1 & \ldots & m
\end{pmatrix}
$$

In particular, there are simple embeddings $\sigma_{+m}^m := \sigma_{+m,m+1}^m$ for $m \geq 0$;
$\emptyset_n := \sigma_{+0n}^n$ is the empty map $I_0 \rightarrow I_n$ (and obviously, $\Sigma_{0n} = \{\emptyset_n\}$).

• Facet embeddings $\delta_{i}^n \in \Upsilon_{n-1,n}$ for $n > 0$.

$$
\delta_{i}^n := 
\begin{pmatrix}
1 & \ldots & i - 1 & i & \ldots & n - 1 \\
1 & \ldots & i - 1 & i + 1 & \ldots & n
\end{pmatrix}
$$

In particular, $\delta_{n}^n = \sigma_{+}^{n-1}$.

• Standard projections $\sigma_{-mn}^m \in \Sigma_{mn}$ for $m \geq n > 0$.

$$
\sigma_{-mn}^m := 
\begin{pmatrix}
1 & \ldots & n & \ldots & m \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
1 & \ldots & n & \ldots & n
\end{pmatrix}
$$

In particular, simple projections are $\sigma_{-n}^n := \sigma_{-n+1,n}^n$ for $n > 0$. 


It is well-known that (for \( n > 1 \)) every permutation \( \sigma \in \Sigma_n \) is a composition of (simple) transpositions. One also can easily show that every map from \( \Sigma_{mn} \) is a composition of simple transpositions, simple embeddings, and simple projections. In particular, every injection (from \( \Sigma_{mn} \)) is a composition of simple transpositions and simple embeddings, and every surjection is a composition of simple transpositions and simple projections, cf. [Gabriel and Zisman, 1967].

The identity map in \( \Sigma_n \) is \( \text{id}_n := \text{id}_{n} = \sigma^n_{n} = \sigma_{-n} \), and it is obvious that \( \text{id}_n = \sigma^n_{j} \circ \sigma^n_{i} \) whenever \( n \geq 2, j < i \).

Let also \( \lambda^i_j \in \Sigma_{1n} \) be the map sending 1 to \( i \); let \( \lambda^n_{ij} \in \Sigma_{2n} \) be the map with the table

\[
\begin{pmatrix}
1 & 2 \\
\ i & j \\
\end{pmatrix}
\]

For every \( \sigma \in \Sigma_{mn} \) we define its simple extension \( \sigma^+ \in \Sigma_{m+1,n+1} \) such that

\[
(\sigma^+)^+(i) := \begin{cases} 
\sigma(i) & \text{for } i \in I_m, \\
1 & \text{if } i = m + 1. 
\end{cases}
\]

In particular, for any \( n \) we have \( (\sigma^n_+)^+ = \delta^n_{n+2} \in \Sigma_{n+1,n+2} \):

\[
(\sigma^n_+)^+(i) := \begin{cases} 
i & \text{for } i \in I_n, \\
1 & \text{if } i = n + 1. 
\end{cases}
\]

We do not make any difference between words of length \( n \) in an alphabet \( D \) and \( n \)-tuples from \( D^n \). So we write down a tuple \( (a_1, \ldots, a_n) \) also as \( a_1 \ldots a_n \).

- \( \lambda \) denotes the void sequence;
- \( \ell(\alpha) \) (or \( |\alpha| \)) denotes the length of a sequence \( \alpha \);
- \( \alpha \beta \) denotes the join (the concatenation) of sequences \( \alpha, \beta \); we often write \( x_1 \ldots x_n \) rather than \( (x_1, \ldots, x_n) \) (especially if \( n = 1 \)), and also \( \alpha x \) or \( (\alpha, x) \) rather than the dubious notation \( \alpha(x) \);
- For a letter \( c \) put

\[
c^k := c \ldots c. 
\]

For an arbitrary set \( S \), every tuple \( a = (a_1, \ldots, a_n) \in S^n \) can be regarded as a function \( I_n \rightarrow S \). We usually denote the range of this function, i.e. the set \( \{a_1, \ldots, a_n\} \) as \( r(a) \). Sometimes we write \( b \in a \) instead of \( b \in r(a) \). Every map \( \sigma : I_m \rightarrow I_n \) acts on \( S^n \) via composition:

\[
a \cdot \sigma = \sigma \circ a = (a_{\sigma(1)}, \ldots, a_{\sigma(m)}). 
\]

Thus every map \( \sigma \in \Sigma_{mn} \) gives rise to the map \( \pi_\sigma : S^n \rightarrow S^m \) sending \( a \) to \( a \cdot \sigma \). In the particular case, when \( \sigma = \delta^n_i \) is a facet embedding and \( a \in S^n \), we also use the notation \( \pi^n_i := \pi_{\delta^n_i} \) and

\[
\pi^n_i a := a - a_i := \hat{a}_i := a \cdot \delta^n_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n). 
\]

Hence we obtain
Lemma 0.0.1  

(1) \[ \pi_{\tau} \cdot \pi_{\sigma} = \pi_{\sigma \cdot \tau}, \]
whenever \( a \in S^n, \sigma \in \Sigma_{mn}, \tau \in \Sigma_{km}. \)

(2) If \( \sigma \) is a permutation (\( \sigma \in \Theta_n \)), then \( \pi_{\sigma} \) is a permutation of \( S^n \) and \( \pi_{\sigma^{-1}} = (\pi_{\sigma})^{-1}. \)

Proof  (1) Since composition of maps is associative, we have
\[ a \cdot (\sigma \cdot \tau) = (a \cdot \sigma) \cdot \tau. \]

(2) \[ \pi_{\sigma} \cdot \pi_{\sigma^{-1}} = \pi_{\sigma^{-1}} \cdot \pi_{\sigma} = \pi_{\text{id}_n} = \text{id}_{S^n}. \]

We use the following relations on \( n \)-tuples:

(\( \sigma \))

a sub b iff \( \forall i, j \ (a_i = a_j \Rightarrow b_i = b_j). \)

Lemma 0.0.2 Let \( S \neq \emptyset, \sigma \in \Sigma_{mn}. \) Then
\[ \pi_{\sigma}[S^n] = \{ a \in S^m \mid \sigma \text{ sub } a \}, \]
where \( \sigma \text{ sub } a \) denotes the property \( \forall i, j \ (\sigma(i) = \sigma(j) \Rightarrow a_i = a_j), \) cf. (\( \sigma \)).

Proof  In fact, if \( a = b \cdot \sigma \), then obviously \( \sigma(j) = \sigma(k) \) implies \( a_j = a_k \). On the other hand, if \( \sigma \text{ sub } a \), then \( a = b \cdot \sigma \) for some \( b \); just put \( b_{\sigma(i)} := a_i \) and add arbitrary \( b_k \) for \( k \notin r(\sigma). \)

Lemma 0.0.3 For \( |S| > 1, \sigma \in \Sigma_{mn}, \sigma \) is injective iff \( \pi_{\sigma} : S^n \rightarrow S^m \) is surjective.\(^3\)

Proof  If \( \sigma \) is injective, then for any \( a \in S^n, \sigma(i) = \sigma(j) \Rightarrow i = j \Rightarrow a_i = a_j \), i.e. \( \sigma \text{ sub } a \). Hence by Lemma 0.0.2, \( \pi_{\sigma} \) is surjective.

The other way round, if \( \sigma(i) = \sigma(j) \) for some \( i \neq j \), take \( a \in S^m \) such that \( a_i \neq a_j \). Then \( \sigma \text{ sub } a \) is not true, i.e. \( a \notin \pi_{\sigma}[S^n] \) by 0.0.2.

Lemma 0.0.4 For \( |S| > 1, \sigma \in \Sigma_{mn}, \sigma \) is surjective iff \( \pi_{\sigma} \) is injective.

Proof  Suppose \( \sigma : I_m \rightarrow I_n \) is surjective and \( a, b \in S^n, \pi_{\sigma}a \neq \pi_{\sigma}b. \) If \( \pi_{\sigma}a \) and \( \pi_{\sigma}b \) differ at the \( j \)th component, then \( a_i \neq b_i \) for \( i \in I_n \) such that \( \sigma(i) = j \). On the other hand, let \( \sigma \in \Sigma_{mn} \) be non-surjective, \( j \in I_m - \text{rng}(\sigma) \). Let \( c, d \in S, \ c \neq d \). Take \( a = c^n; \ b = c^{j-1}dc^{n-j}. \) Then \( a \neq b \) and \( \pi_{\sigma}a = \pi_{\sigma}b = c^m. \)

Hence we obtain

Lemma 0.0.5 For \( |S| > 1, \sigma \in \Sigma_{mn}, \sigma \) is bijective iff \( \pi_{\sigma} : S^n \rightarrow S^m \) is bijective.

\(^3\)Clearly, if \( |S| = 1, \) then \( \pi_{\sigma} \) is bijective for every \( \sigma \in \Sigma_{mn}. \)
We further simplify notation in some particular cases. Let $\pi_i^n := \pi_{i^n}$, so facet embedding $\delta_i^n$ eliminates the $i$th component from an $n$-tuple $a \in S^n$. Let also

$$
\pi^n_- := \pi_{\sigma_-^n}, \quad \pi^n_+ := \pi_{\sigma_+^n},
$$

where $\sigma^n_- \in \Sigma_{n,1,n}$ is a simple projection, $\sigma^n_+ \in \Sigma_{n,n+1}$ is a simple embedding. Thus

$$
\pi^n_-(a_1, \ldots, a_n) = (a_1, \ldots, a_n, a_n) \text{ for } n > 0,
$$

$$
\pi^n_+(a) = a - a_{n+1} = (a_1, \ldots, a_n) \text{ for } a = (a_1, \ldots, a_n, a_{n+1}) \in D^{n+1}, \ n \geq 0.
$$

We say that a sequence $a \in D^n$ is distinct, if all its components $a_i$ are different.

Lemma 0.0.6 If $\sigma, \tau : I_m \rightarrow I_n$, $\sigma \neq \tau$ and $|S| \geq n$, then $a \cdot \sigma \neq a \cdot \tau$ for any distinct $a \in S^n$.

Proof If for some $i$, $\tau(i) \neq \sigma(i)$, then $a_{\sigma(i)} \neq a_{\tau(i)}$. \hfill ■

Lemma 0.0.7  
(1) For $\tau \in \Sigma_{mn}$, $\sigma \in \Sigma_{km}$,

$$
(\tau \cdot \sigma)^+ = \tau^+ \cdot \sigma^+.
$$

(2) For $\sigma \in \Sigma_{mn}$,

$$
\sigma^+ \cdot \sigma^m = \sigma^m \cdot \sigma.
$$

Proof Straightforward. \hfill ■

Lemma 0.0.8  
(1) Let $a \in S^n$, $b \in S^m$, $r(b) \subseteq r(a)$. Then $b = a \cdot \sigma$ for some $\sigma \in \Sigma_{mn}$.

(2) Moreover, if $b$ is distinct, then $\sigma$ is an injection.

Proof Put $\sigma(i) = j$ for some $j$ such that $b_j = a_j$. \hfill ■

---

In other words, $b$ is obtained by renumbering a subsequence of $a$. 

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Part I

Preliminaries
Chapter 1

Basic propositional logic

This chapter contains necessary information about propositional logics. We give all the definitions and formulate results, but many proofs are sketched or skipped. For more details we address the reader to textbooks and monographs in propositional logic: [Goldblatt, 1987], [Chagrov and Zakharyaschev, 1997], [Blackburn, de Rijke and Venema, 2001], also see [Gabbay, 1981], [Dragalin, 1988], [van Benthem, 1983].

1.1 Propositional syntax

1.1.1 Formulas

We consider $N$-modal (propositional) formulas\(^1\) built from the denumerable set $PL = \{p_1, p_2, \ldots \}$ of proposition letters, the classical propositional connectives $\land, \lor, \rightarrow, \leftrightarrow$ and the unary modal connectives $\Box_1, \ldots, \Box_N$; the derived connectives are introduced in a standard way as abbreviations:

\[
\neg A := (A \supset \bot), \quad \top := (\bot \lor \bot),
\]

\[
(A \equiv B) := ((A \supset B) \land (B \supset A)),
\]

\[
\Diamond_i A := \neg \Box_i \neg A \quad \text{for } i = 1, \ldots, N.
\]

To simplify notation, we write $p, q, r$ instead of $p_1, p_2, p_3$. We also use standard agreements about bracketing: the principal brackets are omitted; $\land$ is stronger than $\lor$, which is stronger than $\rightarrow$ and $\leftrightarrow$. Sometimes we use dots instead of brackets; so, e.g. $A \supset B \supset C$ stands for $(A \supset (B \supset C))$.

For a sequence of natural numbers $\alpha = k_1 \ldots k_r$ from $I_N^\infty$, $\Box_\alpha$ abbreviates $\Box_{k_1} \ldots \Box_{k_r}$. $\Box_\bot$ denotes the identity operator, i.e. $\Box_\bot A = A$ for every formula $A$. If $\alpha = k_1 \ldots k_r$, $\Box_\alpha$ is also denoted by $\Box^r_k$ (for $r \geq 0$).

Similarly, we use the notations $\Diamond_\alpha$, $\Diamond^r_k$.

\(^1\)1-modal formulas are also called monomodal, 2-modal formulas are called bimodal. Some authors prefer the term 'unimodal' to 'monomodal'.
As usual, for a finite set of formulas $\Gamma$, $\bigwedge \Gamma$ denotes its conjunction and $\bigvee \Gamma$ its disjunction; the empty conjunction is $\top$ and the empty disjunction is $\bot$. We also use the notation (for arbitrary $\Gamma$)

$$\square_a \Gamma := \{ \square_a A \mid A \in \Gamma \}.$$ 

If $n = 1$, we write $\Box$ instead of $\square_1$ and $\Diamond$ instead of $\Diamond_1$.

The degree (or the depth) of a modal formula $A$ (denoted by $d(A)$) is defined by induction:

$$d(p_k) = d(\bot) = 0,$$

$$d(A \land B) = d(A \lor B) = d(A \supset B) = \max (d(A), d(B)),$$

$$d(\Box_i A) = d(A) + 1.$$ 

$\mathcal{LP}_N$ denotes the set of all $N$-modal formulas; $\mathcal{LP}_0$ denotes the set of all formulas without modal connectives; they are called classical (or intuitionistic$^2$).

An $N$-modal (propositional) substitution is a map $S : \mathcal{LP}_N \rightarrow \mathcal{LP}_N$ preserving $\bot$ and all but finitely many proposition letters and commuting with all connectives, i.e. such that

- $S(\bot) = \bot$;
- $\{ k \mid S(p_k) \neq p_k \}$ is finite;
- $S(A \land B) = S(A) \land S(B)$;
- $S(A \lor B) = S(A) \lor S(B)$;
- $S(A \supset B) = S(A) \supset S(B)$;
- $S(\Box_i A) = \Box_i S(A)$.

Let $q_1, \ldots, q_k$ be different proposition letters. A substitution $S$ such that $S(q_i) = A_i$ for $i \leq k$ and $S(q) = q$ for any other $q \in PL$, is denoted by $[A_1, \ldots, A_k/q_1, \ldots, q_k]$. A substitution of the form $[A/q]$ is called simple. It is rather clear that every substitution can be presented as a composition of simple substitutions.

Later on we often write $SA$ instead of $S(A)$; this formula is called the substitution instance of $A$ under $S$, or the $S$-instance of $A$. For a set of formulas $\Gamma$, $Sub(\Gamma)$ (or $Sub_N(\Gamma)$, if we want to specify the language) denotes the set of all substitution instances of formulas from $\Gamma$.

An intuitionistic substitution is nothing but a 0-modal substitution.

$^2$In this book intuitionistic and classical formulas are syntactically the same; the only difference between them is in semantics.
1.1.2 Logics

In this book a logic (in a formal sense) is a set of formulas. We say that a logic $L$ is closed under the rule

$$
\frac{A_1, \ldots, A_n}{B}
$$

(or that this rule is admissible in $L$) if $B \in L$, whenever $A_1, \ldots, A_n \in L$. A (normal propositional) $N$-modal logic is a subset of $\mathcal{LP}_N$ closed under arbitrary $N$-modal substitutions, modus ponens $\left( \frac{A, A \supset B}{B} \right)$, necessitation $\left( \frac{A}{\Box i A} \right)$ and containing all classical tautologies and all the formulas

$$
AK_i := \Box_i (p \supset q) \supset (\Box_i p \supset \Box_i q),
$$

where $1 \leq i \leq N$.

$K_N$ denotes the minimal $N$-modal logic, and $K$ denotes $K_1$. Sometimes we call $N$-modal logics (or formulas) just 'modal', if $N$ is clear from the context.

The smallest $N$-modal logic containing a given $N$-modal logic $\Lambda$ and a set of $N$-modal formulas $\Gamma$ is denoted by $\Lambda + \Gamma$; for a formula $A$, $\Lambda + A$ is an abbreviation for $\Lambda + \{A\}$. We say that the logic $K_N + \Gamma$ is axiomatised by the set $\Gamma$. A logic is called finitely axiomatisable (respectively, recursively axiomatisable) if it can be axiomatised by a finite (respectively, recursive) set of formulas. It is well-known that a logic is recursively axiomatisable iff it is recursively enumerable. A logic $\Lambda$ is consistent if $\bot \notin \Lambda$.

Here is a list of some frequently used modal formulas and modal logics:

$$
\begin{align*}
AT &:= \Box p \supset p, \\
A4 &:= \Box p \supset \Box \Box p, \\
AD &:= \Box \Box p \supset \Box p, \\
AM &:= \Box \Box p \supset \Box \Box p \quad \text{(McKinsey formula),} \\
A2 &:= \Diamond \Diamond p \supset \Box \Diamond p, \\
A3 &:= \Box (p \land \Box p \supset q) \lor \Box (q \land \Box q \supset p), \\
AGrz &:= \Box (\Box (p \supset \Box p) \supset p) \supset p \quad \text{(Grzegorczyk formula),} \\
AL &:= \Box (\Box p \supset p) \supset \Box p \quad \text{(Löb formula),} \\
A5 &:= \Diamond \Box p \supset \Box p, \\
AB &:= \Diamond \Box p \supset p, \\
At_1 &:= \Diamond_1 \Box_2 p \supset p, \\
At_2 &:= \Diamond_2 \Box_1 p \supset p, \\
AW_n &:= \bigvee_{i=0}^{n} \Box \left( p_i \land \Box p_i \supset \bigvee_{j \neq i} p_j \right), \\
Alt_n &:= \bigvee_{i=0}^{n} \Box \left( p_i \supset \bigvee_{j \neq i} p_j \right).
\end{align*}
$$
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\[
\begin{align*}
D &= K + \Box T, & T &= K + AT, \\
K_4 &= K + A_4, & S_4 &= K_4 + AT, & D_4 &= D + A_4, \\
S_{4.1} &= S_4 + AM, & D_{4.1} &= D_4 + AM, \\
S_{4.2} &= S_4 + A_2, & S_{4.3} &= S_4 + (\Box p \supset q) \lor (\Box q \supset p), \\
K_{4.3} &= K_4 + A_3, & Grz &= S_4 + AGrz \quad \text{(Grzegorczyk logic)}, \\
S_5 &= S_4 + AB, & GL &= K + AL, \quad \text{(Gödel–Löb logic)}, \\
K_5 &= K + A_5, & K.t &= K_2 + At_1 + At_2.
\end{align*}
\]

The corresponding \( N \)-modal versions are denoted by \( D_N, T_N \) etc.; so for example,

\[
D_N := K_N + \{ \Box_i T \mid 1 \leq i \leq N \},
\]

and so on.

A superintuitionistic logic is a set of intuitionistic formulas closed under intuitionistic substitutions and modus ponens, and containing the following well-known axioms:

\[
\begin{align*}
(Ax 1) & \quad p \supset (q \supset p), \\
(Ax 2) & \quad (p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r)), \\
(Ax 3) & \quad p \wedge q \supset p, \\
(Ax 4) & \quad p \wedge q \supset q, \\
(Ax 5) & \quad p \supset (q \supset p \wedge q), \\
(Ax 6) & \quad p \supset p \lor q, \\
(Ax 7) & \quad q \supset p \lor q, \\
(Ax 8) & \quad (p \supset r) \supset ((q \supset r) \supset (p \lor q \supset r)), \\
(Ax 9) & \quad \bot \supset p.
\end{align*}
\]

The smallest superintuitionistic logic is exactly the intuitionistic (or Heyting) propositional logic; it is denoted by \( H \).

The notation \( \Lambda + \Gamma \) and the notions of finite axiomatisability, etc. are used for superintuitionistic logics as well.

An \( m \)-formula is a formula without occurrences of letters \( p_i \) for \( i > m \). \( \mathcal{L} \mathcal{P}_N \mid m \) denotes the set of all \( N \)-modal \( m \)-formulas. If \( \Lambda \) is a modal or a superintuitionistic logic, \( \Lambda \mid m \) denotes its restriction to \( m \)-formulas. The sets \( \Lambda \mid m \) are called bounded logics.

An extension of an \( N \)-modal logic \( \Lambda \) is an arbitrary \( N \)-modal logic containing \( \Lambda \); extensions of a logic \( \Lambda \) are also called \( \Lambda \)-logics.

Members of a logic are also called its theorems; moreover, we use the notation \( \Lambda \vdash A \) as synonymous for \( A \in \Lambda \). A formula \( A \) in the language of a logic \( \Lambda \) is \( \Lambda \)-consistent if \( \neg A \not\in \Lambda \). An \( N \)-modal propositional theory is a set of \( N \)-modal propositional formulas. Such a theory is \( \Lambda \)-consistent if all conjunctions
over its finite subsets are \( \Lambda \)-consistent and \( \Lambda \)-complete if it is maximal among \( \Lambda \)-consistent theories (in the same language).

In the intuitionistic case we also consider double theories that are pairs of sets of intuitionistic formulas. For a superintuitionistic logic \( \Lambda \), a double theory \((\Gamma, \Delta)\) is called \( \Lambda \)-consistent if for any finite sets \( \Gamma_0 \subseteq \Gamma, \Delta_0 \subseteq \Delta, \Delta \cap \Gamma_0 \supseteq \bigvee \Delta_0 \). A \( \Lambda \)-consistent double theory \((\Gamma, \Delta)\) is called \( \Lambda \)-complete if \( \Gamma \cup \Delta = \mathcal{L}\mathcal{P}_0 \).

Let us fix names for some particular intuitionistic formulas and superintuitionistic logics:

\[
\begin{align*}
EM & := p \lor \neg p \text{ (the law of the excluded middle);} \\
AJ & := \neg p \lor \neg \neg p \text{ (the weak law of the excluded middle);} \\
AJ^- & := \neg \neg p \lor (\neg \neg p \supset p); \\
AP_{n+1} & := EM; \\
AP_n & := p_n \lor (p_n \supset AP_{n-1}) \text{ (for } n > 1); \\
KP & := (\neg p \supset q \lor r) \supset (\neg p \supset q) \lor (\neg p \supset r) \text{ (Kreisel-Putnam formula);} \\
Br_n & := \left( \bigwedge_{i=0}^{n} (p_i \lor \bigvee_{j \neq i} p_j) \lor \bigvee_{j \neq i} p_j \right) \lor \bigvee_{i=0}^{n} p_i \text{ (Gabbay-De Jongh formulas);} \\
AZ & := (p \lor q) \lor (q \lor p); \\
AIW_n & := \bigvee_{i=0}^{n} (p_i \lor \bigvee_{j \neq i} p_j); \\
IG_n & := \bigvee_{0 \leq i < j \leq n} (p_i \equiv p_j); \\
HJ & := KC := H + AJ (Jankov's logic); \\
LC & := H + AZ (Dummett's logic); \\
CL & := H + EM (classical, or Boolean logic).
\end{align*}
\]

The following inclusions are well-known:

\[ H \subset HJ \subset LC \subset CL, \]

\[ H + AP_{n+1} \subset H + AP_n. \]

A superintuitionistic logic \( \Sigma \) is called consistent if \( \bot \notin \Sigma \); \( \Sigma \) is said to be intermediate if \( H \subseteq \Sigma \subseteq CL \). It is well-known that every consistent superintuitionistic propositional logic is intermediate.

**Lemma 1.1.1** (1) Some theorems of \( K_N \):

\[
\begin{align*}
\Box_i (p \land q) & \equiv \Box_i p \land \Box_i q; \\
\Diamond_i (p \lor q) & \equiv \Diamond_i p \lor \Diamond_i q; \\
\Box_i p \land \Diamond_i q & \supset \Diamond_i (p \land q); \\
\Box_i (p \supset q) & \supset (\Diamond_i p \supset \Diamond_i q); \\
\Diamond_i (p \supset q) & \supset (\Box_i p \supset \Box_i q); \\
\Box_i (p \equiv q) & \supset (\Box_i p \equiv \Box_i q).
\end{align*}
\]
(2) The following rules are admissible in every modal logic:

Monotonicity rules

\[
\begin{align*}
A & \supset B \\
\Box_i A & \supset \Box_i B
\end{align*}
\]

Replacement rules

\[
\begin{align*}
A & \equiv B \\
\Box_i A & \equiv \Box_i B
\end{align*}
\]

(3) Some theorems of S₄:

\[
\begin{align*}
\Diamond \Diamond p & \equiv \Diamond p; \\
\Box (\Box p \lor \Box q) & \equiv \Box p \lor \Box q.
\end{align*}
\]

(4) A theorem of S₄₂:

\[
\Diamond \Box \left( \bigwedge_i p_i \right) \equiv \bigwedge_i \Diamond \Box p_i.
\]

Lemma 1.1.2 Some theorems of H:

(1) \( p \supset \neg \neg p \),

(2) \( \neg \neg (p \land q) \equiv \neg \neg p \land \neg \neg q \),

(3) \( \neg \neg (p \equiv q) \supset (\neg \neg p \equiv \neg \neg q) \),

(4) \( (p \supset q) \supset (\neg q \supset \neg p) \),

(5) \( \bigwedge_{i=1}^r ((p_i \supset q) \supset q) \equiv \left( \bigwedge_{i=1}^r p_i \supset q \right) \supset q \).

Lemma 1.1.3 (Propositional replacement rule) The following rule is admissible in every modal or superintuitionistic logic:

\[
A \equiv A' \\
[A/p]B \equiv [A'/p]B.
\]

We can write this rule more loosely as

\[
A \equiv A' \\
C(\ldots A \ldots) \equiv C(\ldots A' \ldots),
\]

i.e. in any formula \( C \) we can replace some occurrences of a subformula \( A \) with its equivalent \( A' \).

To formulate the next theorem, we introduce some notation. For an \( N \)-modal formula \( B, r \geq 0 \), let

\[
\Box \leq^r B := \bigwedge \{ \Box_\alpha B \mid \alpha \in I_N^\infty, l(\alpha) \leq r \};
\]

for a finite set of \( N \)-modal formulas \( \Delta \), let

\[
\Box \leq^r \Delta := \bigwedge \{ \Box \leq^r B \mid B \in \Delta \}.
\]
Theorem 1.1.4 (Deduction theorem)

(1) Let $\Sigma$ be a superintuitionistic logic, $\Gamma \cup \{A\}$ a set of intuitionistic formulas. Then:

$$A \in (\Sigma + \Gamma) \iff (A \Rightarrow A') \in \Sigma$$

for some finite $\Delta \subseteq \text{Sub}(\Sigma)$.

(II) Let $\Lambda$ be an $N$-modal logic, $\Gamma \cup \{A\}$ a set of $N$-modal formulas. Then $A \in (\Lambda + \Gamma)$ iff

$$\left( \bigwedge_{r \leq \alpha} A \Rightarrow A' \right) \in \Lambda$$

for some $r \geq 0$ and some finite $\Delta \subseteq \text{Sub}(\Gamma)$.

(III) Let $\Lambda$ be a 1-modal logic, $\Gamma \cup \{A\} \subseteq \mathcal{L}_{P_1}$. Then $A \in (\Lambda + \Gamma)$ iff

1. $(A \Rightarrow A') \in \Lambda$ for some $r \geq 0$ and some finite $\Delta \subseteq \text{Sub}(\Gamma)$
   — in the general case;
2. $(A \Rightarrow A') \in \Lambda$ for some $r \geq 0$ and some finite $\Delta \subseteq \text{Sub}(\Gamma)$
   — provided $T \subseteq \Lambda$;
3. $(\Lambda \Rightarrow \square \alpha A') \in \Lambda$ for some finite $\Delta \subseteq \text{Sub}(\Gamma)$
   — provided $K4 \subseteq \Lambda$;
4. $(\bigwedge_{r \leq \alpha} A) \in \Lambda$ for some finite $\Delta \subseteq \text{Sub}(\Gamma)$
   — provided $S4 \subseteq \Lambda$.

Similarly one can simplify the claim (2) for the case when $\Lambda$ is an $N$-modal logic containing $T_N$, $K4_N$, or $S4_N$; we leave this as an exercise for the reader. But let us point out that for the case when $S4_N \subseteq \Lambda$, $n > 1$, $\square \alpha$ is not necessarily an $S4$-modality, and it may happen that for any $\Delta$, $A \in (\Lambda + \Gamma)$ is not equivalent to $\left( \bigwedge_{i=1}^{n} \square_i \Delta \Rightarrow A' \right) \in \Lambda$.

Corollary 1.1.5

1. For superintuitionistic logics:

$$ (\Sigma + \Gamma) \cap (\Sigma + \Gamma') = \Sigma + \{A \vee A' \mid A \in \Gamma, A' \in \Gamma' \} $$

if formulas from $\Gamma$ and $\Gamma'$ do not have common proposition letters.

2. For $N$-modal logics:

$$ (\Lambda + \Gamma) \cap (\Lambda + \Gamma') = \Lambda + \{\Box \alpha A \vee \Box \alpha A' \mid A \in \Gamma, A' \in \Gamma' ; \alpha, \alpha' \in \mathcal{I}_{\infty} \} $$

if formulas from $\Gamma$ and $\Gamma'$ do not have common proposition letters.

3. For 1-modal logics:

$$ (\Lambda + \Gamma) \cap (\Lambda + \Gamma') = \Lambda + \{\Box^r A \vee \Box^s A' \mid A \in \Gamma, A' \in \Gamma' ; r, s \geq 0 \} $$

if formulas from $\Gamma$ and $\Gamma'$ do not have common proposition letters.

In some particular cases this presentation can be further simplified:
(a) for logics above $T$:
\[(\Lambda + \Gamma) \cap (\Lambda + \Gamma') = \Lambda + \{\square^r A \vee \square^s A' \mid A \in \Gamma, A' \in \Gamma'; \ r \geq 0\};\]

(b) for logics above $K4$:
\[(\Lambda + \Gamma) \cap (\Lambda + \Gamma') = \Lambda + \{\square^r A \vee \square^s A' \mid A \in \Gamma, A' \in \Gamma'; \ r, s \in \{0, 1\}\};\]

(c) for logics above $S4$:
\[(\Lambda + \Gamma) \cap (\Lambda + \Gamma') = \Lambda + \{\square A \vee \square A' \mid A \in \Gamma, A' \in \Gamma'\}.

Therefore we have:

**Proposition 1.1.6**

(1) The set of superintuitionistic logics $S$ is a complete well-distributive lattice:

\[\Lambda \cap \sum_{i \in I} \Lambda_i = \sum_{i \in I} (\Lambda \cap \Lambda_i).\]

Here the sum of logics $\sum_{i \in I} \Lambda_i$ is the smallest logic containing their union.

The set of finitely axiomatisable and the set of recursively axiomatisable superintuitionistic logics are sublattices of $S$.

(2) The set of $N$-modal logics $\mathcal{M}_N$ is a complete well-distributive lattice; the set of recursively axiomatisable $N$-modal logics is a sublattice of $\mathcal{M}_N$.

**Proof** In fact, for example, in the intuitionistic case, both parts of the equality are axiomatised by the same set of formulas

\[\left\{A \vee B \mid A \in \Lambda, B \in \bigcup_{i \in I} \Lambda_i\right\}.\]

**Remark 1.1.7** Although the set of all finitely axiomatisable 1-modal logics is not closed under finite intersections [van Benthem, 1983], this is still the case for finitely axiomatisable extensions of $K4$, cf. [Chagrov and Zakharyaschev, 1997].

**Theorem 1.1.8 (Glivenko theorem)** For any intermediate logic $\Sigma$

\[\neg A \in H \iff \neg A \in \Sigma \iff \neg A \in CL.\]

For a syntactic proof see [Kleene, 1952]. For another proof using Kripke models see [Chagrov and Zakharyaschev, 1997], Theorem 2.47.

**Corollary 1.1.9** If $A \in CL$, then $\neg \neg A \in H$.

**Proof** In fact, $A \in CL$ implies $\neg \neg A \in CL$, so we can apply the Glivenko theorem.
1.2 Algebraic semantics

For modal and intermediate propositional logics several kinds of semantics are known. Algebraic semantics is the most general and straightforward; it interprets formulas as operations in an abstract algebra of truth-values. Actually this semantics fits for every propositional logic with the replacement property; completeness follows by the well-known Lindenbaum theorem.

Relational (Kripke) semantics is nowadays widely known; here formulas are interpreted in relational systems, or Kripke frames. Kripke frames correspond to a special type of algebras, so Kripke semantics is reducible to algebraic. Neighbourhood semantics (see Section 1.17) is in between relational and algebraic.

Let us begin with algebraic semantics.

**Definition 1.2.1** A Heyting algebra is an implicative lattice with the least element:

\[ \Omega = (\Omega, \land, \lor, \rightarrow, 0). \]

More precisely, \((\Omega, \land, \lor)\) is a lattice with the least element 0, and \(\rightarrow\) is the implication in this lattice, i.e. for any \(a, b, c\)

\[
    c \land a \leq b \text{ iff } c \leq (a \rightarrow b).
\]

(*)

(Here \(\leq\) is the standard ordering in the lattice, i.e. \(a \leq b\) iff \(a \land b = a\).)

Recall that negation in Heyting algebras is \(\neg a := a \rightarrow 0\) and \(1 = a \rightarrow a\) is the greatest element.

Note that (*) can be written as

\[
    a \rightarrow b = \max\{c \mid c \land a \leq b\}.
\]

In particular,

\[
    a \rightarrow b = 1 \text{ iff } a \leq b.
\]

Also recall that an implicative lattice is always distributive:

\[
    (a \lor b) \land c = (a \land c) \lor (b \land c),
\]

\[
    (a \land b) \lor c = (a \lor c) \land (b \lor c).
\]

A lattice is called complete if joins and meets exist for every family of its elements:

\[
    \bigvee_{j \in J} a_j := \min\{b \mid \forall j \in J a_j \leq b\}, \quad \bigwedge_{j \in J} a_j := \max\{b \mid \forall j \in J b \leq a_j\}.
\]

A complete lattice is implicative iff it is well-distributive, i.e., the following holds:

\[
    a \land \left( \bigvee_{j \in J} a_j \right) = \bigvee_{j \in J} (a \land a_j).
\]

\(^3\)Cf. [Rasiowa and Sikorski, 1963; Borceaux, 1994].
So every complete well-distributive lattice can be turned into a Heyting algebra. Let us prove two useful properties of Heyting algebras.

**Lemma 1.2.2**

\[ c \land \bigwedge_{j \in J} (a_j \to b_j) \leq \bigwedge_{j \in J} (a_j \to b_j \land c). \]

**Proof** We have to prove

\[ c \land \bigwedge_{j \in J} (a_j \to b_j) \leq a_k \to b_k \land c, \]

which is equivalent (by 1.2.1(*)) to

\[ c \land a_k \land \bigwedge_{j \in J} (a_j \to b_j) \leq b_k \land c. \]

But this follows from

\[ a_k \land \bigwedge_{j \in J} (a_j \to b_j) \leq b_k. \]

The latter holds, since by 1.2.1(*), it is equivalent to

\[ \bigwedge_{j \in J} (a_j \to b_j) \leq a_k \to b_k. \]

\[ \square \]

**Lemma 1.2.3**

\[ u \to \bigwedge_{i \in I} v_i = \bigwedge_{i \in I} (u \to v_i). \]

**Proof** \((\leq)\)

\[ (u \to \bigwedge_{i \in I} v_i) \land u \leq \bigwedge_{i \in I} v_i \leq v_i, \]

hence

\[ u \to \bigwedge_{i \in I} v_i \leq u \to v_i, \]

and thus

\[ u \to \bigwedge_{i \in I} v_i \leq \bigwedge_{i \in I} (u \to v_i). \]

\((\geq)\)

\[ u \land \bigwedge_{i \in I} (u \to v_i) \leq u \land (u \to v_i) \leq v_i, \]

hence

\[ u \land \bigwedge_{i \in I} (u \to v_i) \leq \bigwedge_{i \in I} v_i, \]
and thus
\[
\bigwedge_{i \in I} (u \rightarrow v_i) \leq u \rightarrow \bigwedge_{i \in I} v_i.
\]

**Lemma 1.2.4**

\[
\bigwedge_{i \in I} (v_i \rightarrow u) = \bigvee_{i \in I} v_i \rightarrow u.
\]

**Proof** (\(\geq\))

\(v_i \leq \bigvee_{i \in I} v_i\) implies

\[
\bigvee_{i \in I} v_i \rightarrow u \leq v_i \rightarrow u;
\]

hence

\[
\bigvee_{i \in I} v_i \rightarrow u \leq \bigwedge_{i \in I} (v_i \rightarrow u).
\]

(\(\leq\)) Since

\[
\bigwedge_{i \in I} (v_i \rightarrow u) \leq v_i \rightarrow u,
\]

it follows that for any \(i \in I\)

\[
v_i \land \bigwedge_{i \in I} (v_i \rightarrow u) \leq u.
\]

Hence

\[
\left(\bigvee_{i \in I} v_i\right) \land \bigwedge_{i \in I} (v_i \rightarrow u) = \bigvee_{i \in I} (v_i \land \bigwedge_{i \in I} (v_i \rightarrow u)) \leq u.
\]

Eventually

\[
\bigwedge_{i \in I} (v_i \rightarrow u) \leq \bigvee_{i \in I} v_i \rightarrow u.
\]

A Boolean algebra is a particular case of a Heyting algebra, where \(a \lor \neg a = 1\). In this case \(\lor, \land, \rightarrow, \neg\) are usually denoted by \(\cup, \cap, \exists, \neg\). Then we can consider \(\cup, \cap, \neg, 0, 1\) (and even \(\cup, \neg, 0\)) as basic and define \(a \equiv b := \neg a \lor b\).

We also use the derived operation (equivalence)

\[
a \leftrightarrow b := (a \rightarrow b) \land (b \rightarrow a)
\]

in Heyting algebras and its analogue

\[
a \equiv b := (a \equiv b) \lor (b \equiv a)
\]

in Boolean algebras.
Definition 1.2.5 An $N$-modal algebra is a structure
\[ \Omega = (\Omega, \cap, \cup, -, 0, 1, \square_1, \ldots, \square_N), \]
such that its nonmodal part
\[ \Omega^b := (\Omega, \cap, \cup, -, 0, 1) \]
is a Boolean algebra, and $\square_i$ are unary operations in $\Omega$ satisfying the identities:
\[ \square_i(a \cap b) = \square_i a \cap \square_i b, \]
\[ \square_i 1 = 1. \]
$\Omega$ is called complete if the Boolean algebra $\Omega^b$ is complete.

We also use the dual operations
\[ \Diamond_i a = \neg \square_i a. \]

For 1-modal algebras we write $\square, \Diamond$ rather than $\square_1, \Diamond_1$ (cf. Section 1.1.1).

Definition 1.2.6 A topo-Boolean (or interior, or S4-) algebra is a 1-modal algebra satisfying the inequalities
\[ \square a \leq a, \square a \leq \square \square a. \]
In this case $\square$ is called the interior operation and its dual $\Diamond$ the closure op-eration. An element $a$ is said to be open if $\square a = a$ and closed if $\Diamond a = a$.

Proposition 1.2.7 The open elements of a topo-Boolean algebra $\Omega$ constitute a Heyting algebra:
\[ \Omega^0 = (\Omega^0, \cap, \cup, \rightarrow, 0), \]
in which $a \rightarrow b = \square(a \lor b)$. Moreover, if $\Omega$ is complete then $\Omega^0$ also is, and
\[ \bigvee_{j \in J} a_j = \bigcup_{j \in J} a_j, \quad \bigwedge_{j \in J} a_j = \square \left( \bigcap_{j \in J} a_j \right) \]

Proof Cf. [McKinsey and Tarski, 1944]; [Rasiowa and Sikorski, 1963].

Following [Esakia, 1979], we call $\Omega^0$ the pattern of $\Omega$. It is known that every Heyting algebra is isomorphic to some algebra $\Omega^0$ [Rasiowa and Sikorski, 1963]

Definition 1.2.8 A valuation in an $N$-modal algebra $\Omega$ is a map $\varphi : PL \rightarrow \Omega$. The valuation $\varphi$ has a unique extension to all $N$-modal formulas such that
\begin{enumerate}
  \item $\varphi(\bot) = 0$;
  \item $\varphi(A \land B) = \varphi(A) \cap \varphi(B)$;
\end{enumerate}
1.2. ALGEBRAIC SEMANTICS

(3) \( \varphi(A \lor B) = \varphi(A) \cup \varphi(B) \);

(4) \( \varphi(A \supset B) = \varphi(A) \supset \varphi(B) \);

(5) \( \varphi(\Box_i A) = \Box_i \varphi(A) \).

The pair \((\Omega, \varphi)\) is then called an (algebraic) model over \(\Omega\). An \(N\)-modal formula \(A\) is said to be true in the model \((\Omega, \varphi)\) if \(\varphi(A) = 1\) (notation: \((\Omega, \varphi) \models A\)); \(A\) is called valid in the algebra \(\Omega\) (notation: \(\Omega \models A\)) if it is true in every model over \(\Omega\).

**Lemma 1.2.9** Let \(\Omega\) be an \(N\)-modal algebra, \(S\) a propositional substitution. Let \(\varphi, \eta\) be valuations in \(\Omega\) such that for any \(B \in PL_k\)

\((\bigodot)\ \eta(B) = \varphi(SB)\).

Then \((\bigodot)\) holds for any \(N\)-modal \(k\)-formula \(B\).

**Proof** Easy, by induction on the length of \(B\).

**Lemma 1.2.10 (Soundness lemma)** The set

\[ ML(\Omega) := \{ A \in \mathcal{P}_N \mid \Omega \models A \} \]

is a modal logic.

**Proof** First note that \(ML(\Omega)\) is substitution closed. In fact, assume \(\Omega \models A\), and let \(S\) be a propositional substitution. To show that \(\Omega \models SA\), take an arbitrary valuation \(\varphi\) in \(\Omega\), and consider a new valuation \(\eta\) according to \((\bigodot)\) from Lemma 1.2.9. So we obtain

\[ \varphi(SA) = \eta(A) = 1, \]

i.e. \(\Omega \models SA\).

The classical tautologies are valid in \(\Omega\), because they hold in any Boolean algebra. The validity of \(AK_i\) follows by a standard argument. In fact, note that in a modal algebra \(\Box_i\) is monotonic:

\((*)\quad x \leq y \Rightarrow \Box_i x \leq \Box_i y,\)

because \(x \leq y\) implies

\[ \Box_i x = \Box_i(x \land y) = \Box_i x \land \Box_i y. \]

Now since

\[ (a \lor b) \land a \leq b, \]

by monotonicity \((*)\), we have

\[ \Box(a \lor b) \land \Box a \leq \Box b, \]

which implies

\[ \Box(a \lor b) \leq (\Box a \lor \Box b), \]

This yields the validity of \(AK_i\).

Finally, modus ponens and necessitation preserve validity, since in a modal algebra \(1 \leq a\) implies \(a = 1\), and \(\Box_i 1 = 1\).
Definition 1.2.11 \( \text{ML}(\Omega) \) is called the modal logic of the algebra \( \Omega \).

We also define the modal logic of a class \( C \) of \( N \)-modal algebras

\[
\text{ML}(C) := \bigcap \{ \text{ML}(\Omega) \mid \Omega \in C \}.
\]

Note that \( \text{ML}(\Omega) \) is consistent iff the algebra \( \Omega \) is nondegenerate, i.e. iff \( 0 \neq 1 \) in \( \Omega \).

Definition 1.2.12 A valuation in a Heyting algebra \( \Omega \) is a map \( \varphi : PL \rightarrow \Omega \).

It has a unique extension \( \varphi^I : \mathcal{L} \rightarrow \Omega \) such that

1. \( \varphi^I(\bot) = 0 \);
2. \( \varphi^I(A \land B) = \varphi^I(A) \land \varphi^I(B) \);
3. \( \varphi^I(A \lor B) = \varphi^I(A) \lor \varphi^I(B) \);
4. \( \varphi^I(A \supset B) = \varphi^I(A) \rightarrow \varphi^I(B) \).

As in the modal case, the pair \((\Omega, \varphi)\) is called an (algebraic) model over \( \Omega \). An intuitionistic formula \( A \) is said to be true in \((\Omega, \varphi)\) if \( \varphi^I(A) = 1 \) (notation: \((\Omega, \varphi) \models A\)); \( A \) is called valid in the algebra \( \Omega \) (notation: \( \Omega \models A \)) if it is true in every model over \( \Omega \).

We easily obtain an intuitionistic analogue of Lemma 1.2.9:

Lemma 1.2.13 Let \( \Omega \) be a Heyting algebra, \( S \) a propositional substitution. Let \( \varphi, \eta \) be valuations in \( \Omega \) such that for any \( B \in PL \)

\[ (\clubsuit) \quad \eta^I(B) = \varphi^I(SB). \]

Then \((\clubsuit)\) holds for any intuitionistic formula \( B \).

Similarly we have

Lemma 1.2.14 (Soundness lemma) For a Heyting algebra \( \Omega \), the set

\[
\text{IL}(\Omega) := \{ A \in \mathcal{L} \mid \Omega \models A \}
\]

is a superintuitionistic logic.

Definition 1.2.15 \( \text{IL}(\Omega) \) is called the superintuitionistic logic of the algebra \( \Omega \). Similarly to the modal case, we define the superintuitionistic logic of a class \( C \) of Heyting algebras

\[
\text{IL}(C) := \bigcap \{ \text{IL}(\Omega) \mid \Omega \in C \}.
\]

Definition 1.2.16 A valuation \( \varphi \) in an \( S4 \)-algebra \( \Omega \) is called intuitionistic if it is a valuation in \( \Omega^0 \) i.e. if its values are open.
Definition 1.2.17 G"odel–Tarski translation is the map \((-)^T\) from intuitionistic to 1-modal formulas defined by the following clauses:
\[
\begin{align*}
\bot^T &= \bot; \\
q^T &= \square q \text{ for every proposition letter } q; \\
(A \land B)^T &= A^T \land B^T; \\
(A \lor B)^T &= A^T \lor B^T; \\
(A \supset B)^T &= \square(A^T \supset B^T).
\end{align*}
\]

Lemma 1.2.18 \((\square A^T \equiv A^T) \in S4\) for any intuitionistic formula \(A\).

Proof Easy by induction; for the cases \(A = B \lor C\), \(A = B \land C\) use Lemma 1.1.1.

Lemma 1.2.19 Let \(\Omega\) be an S4-algebra.

(1) Let \(\varphi, \psi\) be valuations in \(\Omega\) such that for any \(q \in PL\)
\[
\varphi(q) = \square \psi(q).
\]
Then for any intuitionistic formula \(A\),
\[
\varphi^I(A) = \psi(A^T).
\]
In particular,
\[
\varphi^I(A) = \varphi(A^T),
\]
if \(\varphi\) is intuitionistic.

(2) For any intuitionistic formula \(A\),
\[
\Omega^0 \models A \text{ iff } \Omega \models A^T.
\]

Proof
(1) By induction. Consider only the case \(A = B \supset C\). Suppose
\[
\varphi^I(B) = \psi(B^T), \quad \varphi^I(C) = \psi(C^T).
\]
Then we have
\[
\begin{align*}
\varphi^I(B \supset C) &= \varphi^I(B) \supset \varphi^I(C) = \psi(B^T) \supset \psi(C^T) = \square(\psi(B^T) \supset \psi(C^T)) \\
&= \psi(\square(B^T \supset C^T)) = \psi((B \supset C)^T).
\end{align*}
\]
(2) (Only if.) Assume \(\Omega^0 \models A\). Let \(\psi\) be an arbitrary valuation in \(\Omega\), and let \(\varphi\) be the valuation in \(\Omega^0\) such that
\[
\varphi(q) = \square \psi(q)
\]
for every \(q \in PL\). By (1) and our assumption, we have:
\[
\psi(A^T) = \varphi^I(A) = 1.
\]
Hence \(\Omega \models A^T\).

(If.) Assume \(\Omega \models A^T\). By (1), for any valuation \(\varphi\) in \(\Omega^0\) we have \(\varphi^I(A) = \varphi(A^T) = 1\). Hence \(\Omega^0 \models A\).
Let us now recall the Lindenbaum algebra construction. For an $N$-modal or superintuitionistic logic $\Lambda$, the relation $\sim_\Lambda$ between $N$-modal (respectively, intuitionistic) formulas such that

$$A \sim_\Lambda B \iff (A \equiv B) \in \Lambda$$

is an equivalence.

Let $[A]$ be the equivalence class of a formula $A$ modulo $\sim_\Lambda$.

**Definition 1.2.20** The Lindenbaum algebra $Lind(\Lambda)$ of a modal logic $\Lambda$ is the set $\mathcal{LP}_N/\sim_\Lambda$ with the operations

- $[A] \cap [B] := [A \land B],$
- $[A] \cup [B] := [A \lor B],$
- $-[A] := [\neg A],$
- $0 := [\bot],$
- $1 := [\top],$
- $\Box_i[A] := [\Box_i A].$

**Theorem 1.2.21** For an $N$-modal logic $\Lambda$

1. $Lind(\Lambda)$ is an $N$-modal algebra;
2. $ML(Lind(\Lambda)) = \Lambda.$

**Definition 1.2.22** The Lindenbaum algebra $Lind(\Sigma)$ of a superintuitionistic logic $\Sigma$, is the set $\mathcal{LP}_0/\sim_\Sigma$ with the operations

- $[A] \land [B] := [A \land B],$
- $[A] \lor [B] := [A \lor B],$
- $[A] \rightarrow [B] := [A \supset B],$
- $0 := [\bot].$

**Theorem 1.2.23** For a superintuitionistic logic $\Sigma$,

1. $Lind(\Sigma)$ is a Heyting algebra;
2. $IL(Lind(\Sigma)) = \Sigma.$

**Definition 1.2.24** A set of modal formulas $\Gamma$ is valid in a modal algebra $\Omega$ (notation: $\Omega \models \Gamma$) if all these formulas are valid; similarly for intuitionistic formulas and Heyting algebras. In this case $\Omega$ is called a $\Gamma$-algebra. The set of all $\Gamma$-algebras is called an algebraic variety defined by $\Gamma$. 
1.3. RELATIONAL SEMANTICS (THE MODAL CASE)

Algebraic varieties can be characterised in algebraic terms, due to the well-known Birkhoff theorem [Birkhoff, 1979] (which holds also in a more general context):

**Theorem 1.2.25** A class of modal or Heyting algebras is an algebraic variety iff it is closed under subalgebras, homomorphic images and direct products.

Since every logic is complete in algebraic semantics, there is the following duality theorem.

**Theorem 1.2.26** The poset $M_N$ of $N$-modal propositional logics (ordered by inclusion) is dually isomorphic to the set of all algebraic varieties of $N$-modal algebras; similarly for superintuitionistic logics and Heyting algebras.

1.3 Relational semantics (the modal case)

1.3.1 Introduction

First let us briefly recall the underlying philosophical motivation. For more details, we address the reader to [Fitting and Mendelsohn, 2000]. In relational (or Kripke) semantics formulas are evaluated in 'possible worlds' representing different situations. Depending on the application area of the logic, worlds can also be called 'states', 'moments of time', 'pieces of information', etc. Every world $w$ is related to some other worlds called 'accessible from $w$', and a formula $\Box A$ is true at $w$ iff $A$ is true at all worlds accessible from $w$; dually, $\Diamond A$ is true at $w$ iff $A$ is true at some world accessible from $w$.

This corresponds to the ancient principle of Diodorus Cronus saying that

*The possible is that which either is or will be true*

So from the Diodorean viewpoint, possible worlds are moments of time, with the accessibility relation $\leq$ 'before' (nonstrict).

For polymodal formulas we need several accessibility relations corresponding to different necessity operators.

For the intuitionistic case, Kripke semantics formalises the 'historical approach' to intuitionistic truth by Brouwer. Here worlds represent stages of our knowledge in time. According to Brouwer's truth-preservation principle, the truth of every formula is inherited in all later stages. $\neg A$ is true at $w$ iff the truth of $A$ can never be established afterwards, i.e. iff $A$ is not true at $w$ and always later. Similarly, $A \supset B$ is true at $w$ iff the truth of $A$ implies the truth of $B$ at $w$ and always later. See [Dragalin, 1988], [van Dalen, 1973] for further discussion.

1.3.2 Kripke frames and models

Now let us recall the main definitions in detail.
Definition 1.3.1 An $N$-modal (propositional) Kripke frame is an $(N+1)$-tuple $F = (W, R_1, \ldots, R_N)$, such that $W \neq \emptyset$, $R_i \subseteq W \times W$. The elements of $W$ are called possible worlds (or points), $R_i$ are the accessibility relations.

Quite often we write $u \in F$ rather than $u \in W$.

For a Kripke frame $F = (W, R_1, \ldots, R_N)$ and a sequence $\alpha \in I_N^\infty$ we define the relation $R_\alpha$ on $W$:

$$R_{i_1 \ldots i_k} := R_{i_1} \circ \ldots \circ R_{i_k}, \quad R_\lambda := Id_W.$$  

(Recall that $\lambda$ is a void sequence, $Id_W$ is the equality relation, see the Introduction.)

Every $N$-modal Kripke frame $F = (W, R_1, \ldots, R_N)$ corresponds to an $N$-modal algebra

$$MA(F) := (2^W, \cap, \cup, -, \emptyset, W, \Box_1, \ldots, \Box_N),$$

where $\cap, \cup, -$ are the standard set-theoretic operations on subsets of $W$, and

$$\Box_i V := \{ x | R_i(x) \subseteq V \}.$$  

$MA(F)$ is called the modal algebra of the frame $F$.

Definition 1.3.2 A valuation in a set $W$ (or in a frame with the set of worlds $W$) is a valuation in $MA(F)$, i.e. a map $\theta : PL \rightarrow 2^W$. A Kripke model over a frame $F$ is a pair $M = (F, \theta)$, where $\theta$ is a valuation in $F$. $\theta$ is extended to all formulas in the standard way, according to Definition 1.2.8:

1. $\theta(\bot) = \emptyset$;
2. $\theta(A \land B) = \theta(A) \cap \theta(B)$;
3. $\theta(A \lor B) = \theta(A) \cup \theta(B)$;
4. $\theta(A \supset B) = \theta(A) \supset \theta(B)$;
5. $\theta(\Box_i A) = \Box_i \theta(A) = \{ u | R_i(u) \subseteq \theta(A) \}$.

For a formula $A$, we also write: $M, w \models A$ (or just $w \models A$) instead of $w \in \theta(A)$, and say that $A$ is true at the world $w$ of the model $M$ (or that $w$ forces $A$).

The above definition corresponds to the well-known inductive definition of forcing in a Kripke model given by (1)–(6) in the following lemma.

Lemma 1.3.3  
1. $M, u \models q$ iff $u \in \theta(q)$ (for $q \in PL$);
2. $M, u \not\models \bot$;
3. $M, u \models B \land C$ iff $(M, u \models B$ and $M, u \models C)$;
4. $M, u \models B \lor C$ iff $(M, u \models B$ or $M, u \models C)$;
(5) \( M, u \models B \supset C \iff (M, u \models B \implies M, u \models C) \);

(6) \( M, u \models \Box_i B \iff \forall v \in R_i(u) \ M, v \models B \);

(7) \( M, u \models \neg B \iff M, u \not\models B \);

(8) \( M, u \models \Diamond_i B \iff \exists v \in R_i(u) \ M, v \models B \).

(9) \( M, u \models \Box_\alpha B \iff \forall v \in R_\alpha(u) \ M, v \models B \);

(10) \( M, u \models \Diamond_\alpha B \iff \exists v \in R_\alpha(u) \ M, v \models B \).

**Definition 1.3.4** An \( m \)-bounded Kripke model over a Kripke frame \( F = (W, R_1, \ldots, R_N) \) is a pair \((F, \theta)\), in which \( \theta : \{p_1, \ldots, p_m\} \to 2^W \); \( \theta \) is called an \( m \)-valuation. In this case \( \theta \) is extended only to \( m \)-formulas, according to Definition 1.3.2.

**Definition 1.3.5** A modal formula \( A \) is true in a model \( M \) (notation: \( M \models A \)) if it is true at every world of \( M \); \( A \) is satisfied in \( M \) if it is true at some world of \( M \). A formula is called refutable in a model if it is not true.

**Definition 1.3.6** A modal formula \( A \) is valid in a frame \( F \) (notation: \( F \models A \)) if it is true in every model over \( F \). A set of formulas \( \Gamma \) is valid in \( F \) (notation: \( F \models \Gamma \)) if every \( A \in \Gamma \) is valid. In the latter case we also say that \( F \) is a \( \Gamma \)-frame. The (Kripke frame) variety of \( \Gamma \) (notation: \( V(\Gamma) \)) is the class of all \( \Gamma \)-frames.

A formula \( A \) is valid at a world \( x \) in a frame \( F \) (notation: \( F, x \models A \)) if it is true at \( x \) in every model over \( F \); similarly for a set of formulas. A nonvalid formula is called refutable (in a frame or at a world).

A formula \( A \) is satisfiable at a world \( w \) of a frame \( F \) (or briefly, at \( F, w \)) if there exists a model \( M \) over \( F \) such that \( M, w \models A \).

Since by Definitions 1.2.8 and 1.3.2, \( \theta(A) \) is the same in \( F \) and in \( MA(F) \), we have

**Lemma 1.3.7** For any modal formula \( A \) and a Kripke frame \( F \)

\[ F \models A \iff MA(F) \models A. \]

Thus 1.2.10 implies:

**Lemma 1.3.8** (Soundness lemma)

(1) For a Kripke frame \( F \), the set

\[ ML(F) := \{ A \mid F \models A \} \]

is a modal logic.
(2) For a class $C$ of $N$-modal frames, the set
\[ \text{ML}(C) := \bigcap \{ \text{ML}(F) \mid F \in C \} \]
is an $N$-modal logic.

**Definition 1.3.9** $\text{ML}(F)$ (respectively, $\text{ML}(C)$) is called the modal logic of $F$ (respectively, of $C$), or the modal logic determined by $F$ (by $C$), or complete w.r.t. $F$ ($C$).

For a Kripke model $M$, the set
\[ \text{MT}(M) := \{ A \mid M \models A \} \]
is called the modal theory of $M$.

$\text{MT}(M)$ is not always a modal logic; it is closed under $MP$ and $\Box$-introduction but not necessarily under substitution.

The following is a trivial consequence of definitions and the soundness lemma.

**Lemma 1.3.10** For an $N$-modal logic $\Lambda$ and a set of $N$-modal formulas $\Gamma$, $V(\Lambda + \Gamma) = V(\Lambda) \cap V(\Gamma)$. In particular, $V(K_N + \Gamma) = V(\Gamma)$.

Let us describe varieties of some particular modal logics:

**Proposition 1.3.11**

- $V(D)$ consists of all serial frames, i.e. of the frames $(W, R)$ such that $\forall x \exists y xRy$;
- $V(T)$ consists of all reflexive frames;
- $V(K4)$ consists of all transitive frames;
- $V(S4)$ consists of all quasi-ordered (or pre-ordered) sets, i.e. reflexive transitive frames;
- $V(S4.1)$ consists of all S4-frames with McKinsey property:
  \[ \forall x \exists y (xRy \land R(y) = \{ y \}) ; \]
- $V(S4.2)$ consists of all S4-frames with Church–Rosser property (or confluent, or piecewise directed):
  \[ \forall x, y, z (xRy \land xRz \Rightarrow \exists t (yRt \land zRt)) , \]
  or equivalently,
  \[ R^{-1} \circ R \subseteq R \circ R^{-1} ; \]
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- \( \mathbf{V}(\mathbf{K4.3}) \) consists of all piecewise linear (or nonbranching) \( \mathbf{K4} \)-frames, i.e. such that
  \[ \forall x, y, z \ (xRy \& xRz \Rightarrow (y = z \lor yRx \lor zRy)), \]
  or equivalently
  \[ R^{-1} \circ R \subseteq I_W \cup R \cup R^{-1}; \]
- \( \mathbf{V}(\mathbf{K4 + AW}_n) \) consists of all transitive frames of width \( \leq n \),
- \( \mathbf{V}(\mathbf{Grz}) \) consists of all Nötherian posets, i.e. of those without infinite ascending chains \( x_1R^-x_2R^-x_3 \ldots \),
- \( \mathbf{V}(\mathbf{S5}) \) consists of all frames, where accessibility is an equivalence relation.

Due to these characterisations, an \( N \)-modal logic is called reflexive (respectively, serial, transitive) if it contains \( \mathbf{T}_N \) (respectively, \( \mathbf{D}_N \), \( \mathbf{K4}_N \)).

Definition 1.3.12 Let \( \Lambda \) be a modal logic.
- \( \Lambda \) is called Kripke-complete if it is determined by some class of frames;
- \( \Lambda \) has the finite model property (f.m.p.) if it is determined by some class of finite frames;
- \( \Lambda \) has the countable frame property (c.f.p.) if it is determined by some class of countable frames.\(^6\)

The following simple observation readily follows from the definitions.

Lemma 1.3.13

(1) A logic \( \Lambda \) is Kripke-complete (respectively, has the c.f.p./f.m.p.) iff each of its nontheorems is refutable in some \( \Lambda \)-frame (respectively, in a countable/finite \( \Lambda \)-frame).

(2) \( \mathbf{ML}(\mathbf{V}(\Lambda)) \) is the smallest Kripke-complete extension of \( \Lambda \); so \( \Lambda \) is Kripke-complete iff \( \Lambda = \mathbf{ML}(\mathbf{V}(\Lambda)) \).

All particular propositional logics mentioned above (and many others) are known to be Kripke-complete. Kripke-completeness was proved for large families of propositional logics; Section 1.9 gives a brief outline of these results. However not all modal or intermediate propositional logics are complete in Kripke semantics; counterexamples were found by S. Thomason, K. Fine, V. Shehtman, J. Van Benthem, cf. [Chagrov and Zakharyaschev, 1997]. But incomplete propositional logics look rather artificial; in general one can expect that a ‘randomly chosen’ logic is compete.

Nevertheless every logic is ‘complete w.r.t. Kripke models’ in the following sense.

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\(^4\)See Section 1.9.
\(^5\)Recall that \( xR^-y \) iff \( xRy \& x \neq y \), see Introduction.
\(^6\)‘countable’ means ‘of cardinality \( \leq \aleph_0 \)’.
**Definition 1.3.14** An $N$-modal Kripke model $M$ is exact for an $N$-modal logic $\Lambda$ if $\Lambda = MT(M)$.

**Proposition 1.3.15** Every propositional modal logic has a countable exact model.

This follows from the canonical model theorem by applying the standard translation, see below.

### 1.3.3 Main constructions

**Definition 1.3.16** If $F = (W, R_1, \ldots, R_N)$ is a frame, $V \subseteq W$, then the frame

$$F \upharpoonright V := (V, R_1 \upharpoonright V, \ldots, R_N \upharpoonright V)$$

is called a subframe of $F$ (the restriction of $F$ to $V$).

If $M = (F, \theta)$ is a Kripke model, then

$$M \upharpoonright V := (F \upharpoonright V, \theta \upharpoonright V),$$

where $(\theta \upharpoonright V)(q) := \theta(q) \cap V$ for every $q \in PL$, is called its submodel (the restriction to $V$).

A set $V \subseteq W$ is called stable (in $F$) if for every $i, R_i(V) \subseteq V$. In this case the subframe $F \upharpoonright V$ and the submodel $M \upharpoonright V$ are called generated.

**Definition 1.3.17** $F' = (V, R'_1, \ldots, R'_N)$ is called a weak subframe of $F = (W, R_1, \ldots, R_N)$ if $R'_i \subseteq R_i$ for every $i$ and $V \subseteq W$. Then for a Kripke model $M = (F, \theta)$, $M' = (F', \theta \upharpoonright V)$ is called a weak submodel. If also $W = V$, $F'$ is called a full weak subframe of $F$.

We use the signs $\subseteq$, $\prec$, $\prec\prec$, $\subseteq$ to denote subframes, generated subframes, weak subframes, and full weak subframes, respectively; the same for submodels.

**Definition 1.3.18** Let $F, M$ be the same as in the previous definition. The smallest stable subset $W \upharpoonright u$ containing a given point $u \in W$ is called the cone generated by $u$; the corresponding subframe $F \upharpoonright u := F \upharpoonright (W \upharpoonright u)$ is also called the cone (in $F$) generated by $u$, or the subframe generated by $u$; similarly for the submodel $M \upharpoonright u := M \upharpoonright (W \upharpoonright u)$. A frame $F$ (respectively, a Kripke model $M$) is called rooted (with the root $u$) if $F = F \upharpoonright u$ (respectively, $M = M \upharpoonright u$).

We skip the simple proof of the following

**Lemma 1.3.19** $W \upharpoonright u = R^*(u)$, where $R^*$ is the reflexive transitive closure of $(R_1 \cup \ldots \cup R_N)$, i.e. $R^* = \bigcup_{\alpha \in I_N^\infty} R_\alpha$.

**Definition 1.3.20** A path of length $m$ from $u$ to $v$ in a frame $F = (W, R_1, \ldots, R_N)$ is a sequence $(u_0, j_0, u_1, \ldots, j_m-1, u_m)$ such that $u_0 = u$, $u_m = v$, and $u_i R_{j_i} u_{i+1}$ for $i = 0, \ldots, m - 1$. 
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For the particular case $N = 1$ we have $j_i = 1$ for any $i$, so we can denote a path just by $(u_0, u_1, \ldots, u_m)$.

Now Lemma 1.3.19 can be reformulated as follows:

**Lemma 1.3.21** $x \in F \uparrow u$ iff there exists a path from $u$ to $x$ in $F$.

**Definition 1.3.22** The temporalisation of a propositional Kripke frame $F = (W, R_1, \ldots, R_N)$ is the frame $F^\pm := (W, R_1, \ldots, R_N, R_1^{-1}, \ldots, R_N^{-1})$. A non-oriented path in $F$ is a path in $F^\pm$.

**Definition 1.3.23** Let $F = (W, R_1, \ldots, R_N)$ be a propositional Kripke frame. A subset $V \subseteq W$ is called connected (in $F$) if it is stable in $F^\pm$, i.e. both $R_i$- and $R_i^{-1}$-stable for every $i = 1, \ldots, N$. $F$ itself is called connected if $W$ is connected in $F$. A cone in $F^\pm$ (as a subset) is called a (connected) component of $F$.

**Lemma 1.3.24**

1. The component containing $x \in F$ (i.e. the cone $F^\pm \uparrow x$) consists of all $y \in F$ such that there exists a non-oriented path from $x$ to $y$.

2. The components of $F$ make a partition.

**Proof**

1. Readily follows from Lemma 1.3.21.

2. Follows from (1) and the observation that

$$\{(x, y) \mid \text{there exists a nonoriented path from } x \text{ to } y\}$$

is an equivalence relation on $W$.

The following is well-known:

**Lemma 1.3.25** (Generation lemma) Let $V$ be a stable subset in $F$, $M = (F, \theta)$ a Kripke model. Then

1. For any $u \in V$, for any modal formula $A$,

$$M \uparrow V, u \models A \iff M, u \models A;$$

2. $\mathit{ML}(F) \subseteq \mathit{ML}(F \uparrow V)$.

The same holds for bounded models, with obvious changes.

We also have:

**Lemma 1.3.26**

1. $\mathit{ML}(F) = \bigcap_{u \in F} \mathit{ML}(F \uparrow u)$.

2. $\mathit{MT}(M) = \bigcap_{u \in M} \mathit{MT}(M \uparrow u)$.

**Proof**
(1) By Lemma 1.3.25(2), $\text{ML}(F) \subseteq \bigcap_{u \in F} \text{ML}(F \uparrow u)$ for any $u \in F$, and thus

$$\text{ML}(F) \subseteq \bigcap_{u \in F} \text{ML}(F \uparrow u).$$

Now let $A \notin \text{ML}(F)$. Then, by definition, there is a Kripke model $M$ over $F$ and a world $u$ such that $M, u \not\models A$. By Lemma 1.3.25(1) $M \uparrow u, u \not\models A$, and so $F \uparrow u \not\models A$.

(2) By 1.3.25(1), $\text{MT}(M) \subseteq \bigcap_{u \in M} \text{MT}(M \uparrow u)$, so

$$\text{MT}(M) \subseteq \bigcap_{u \in M} \text{MT}(M \uparrow u).$$

On the other hand, if $A \notin \text{MT}(M)$, then $M, u \not\models A$ for some $u \in M$; hence $M \uparrow u, u \not\models A$ by 1.3.25 (1), thus $A \notin \text{MT}(M \uparrow u).

\[\Box\]

**Definition 1.3.27** The disjoint sum (or disjoint union) of a family of frames $F_j = (W_j, R_{1j}, \ldots, R_{nj})$, for $j \in J$, is the frame $\bigcup_{j \in J} F_j := (W, R_1, \ldots, R_N)$, where

$$W = \bigcup_{j \in J} W_j := \bigcup_{j \in J}(W_j \times \{j\}),

(x, j)R_i(y, j') \iff j = j' \& xR_{ij}y.$$  

Obviously, in this case $F'_i := \left(\bigcup_{j \in J} F_j\right) \uparrow (W_i \times \{i\})$ is a generated subframe of $\bigcup_{j \in J} F_j$ isomorphic to $F_i$.

Hence we obtain

**Lemma 1.3.28** $\text{ML}\left(\bigcup_{j \in J} F_j\right) = \bigcap_{j \in J} \text{ML}(F_j)$.

**Proof** Let $F := \bigcup_{j \in J} F_j$. By Lemma 1.3.26,

$$\text{ML}(F) = \bigcap_{u \in F} \text{ML}(F \uparrow u) = \bigcap_{j \in J} \bigcap_{u \in F_j} \text{ML}(F \uparrow (u, j)).$$

But the embedding of $F_j$ in $F$ yields an isomorphism $F_j \uparrow u \cong F \uparrow (u, j)$, hence

$$\bigcap_{u \in F_j} \text{ML}(F \uparrow (u, j)) = \bigcap_{u \in F_j} \text{ML}(F_j \uparrow u) = \bigcap_{j \in J} \text{ML}(F_j)$$

by Lemma 1.3.26, which implies the main statement. \[\Box\]
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Remark 1.3.29 One can show that $MA \left( \bigcup_{j \in J} F_j \right) \cong \prod_{j \in J} MA(F_j)$. One can also show that $ML \left( \prod_{j \in J} \Omega_j \right) = \bigcap_{j \in J} ML(\Omega_j)$ for a family of modal algebras $(\Omega_j)_{j \in J}$. Together with Lemma 1.3.7, this yields an alternative proof of 1.3.28.

Definition 1.3.30 Let $F = (W, R_1, \ldots, R_N), F' = (W, R'_1, \ldots, R'_N)$ be two frames. A map $f : W \to W'$ is called a morphism from $F$ to $F'$ (notation: $f : F \to F'$) if it satisfies the following conditions:

1. $\forall u, v \in W \forall i (uRiv \Rightarrow f(u)R'_iv(f(v)))$ (monotonicity);
2. $\forall u \in W \forall v' \in W' \forall i (f(u)R'_iv' \Rightarrow \exists v (uRiv & f(v) = v'))$ (lift property).

A surjective morphism is called a $p$-morphism (notation: $f : F \to F'$).

$f$ is called an isomorphism from $F$ onto $F'$ (notation: $f : F \cong F'$) if it is a monotonic bijection and $f^{-1}$ is also monotonic.

As usual, $F$ and $F'$ are called isomorphic (notation: $F \cong F'$) if there exists an isomorphism from $F$ onto $F'$.

We write $F \rightarrow F'$ if there exists a $p$-morphism from $F$ onto $F'$.

Note that the conjunction $(1) \& (2)$ is equivalent to ‘cone preservation’:

$$\forall u \forall i f(R_i(u)) = R'_i(f(u)).$$

Definition 1.3.31 Let $M = (F, \theta), M' = (F', \theta')$ be two Kripke models. A $p$-morphism (respectively, an isomorphism) of frames $f : F \to F'$ is said to be a $p$-morphism (respectively, an isomorphism) from $M$ (on)to $M'$ if for every $q \in PL, u \in W$

$$M, u \models q \text{ iff } M', f(u) \models q.$$  

As in the case of frames, morphisms of models are denoted by $\to$, $p$-morphisms by $\rightarrow$, isomorphisms by $\cong$.

Lemma 1.3.32

1. The composition of frame morphisms $F \to F'$ and $F' \to F''$ is a frame morphism $F \to F''$; similarly for $p$-morphisms of frames and for $p$-morphisms of Kripke models.

2. A subframe $F' \subseteq F$ is generated iff the inclusion map is a morphism $F' \to F$; similarly for Kripke models.

3. A restriction of a frame morphism to a generated subframe is a frame morphism and moreover, a restriction to a cone is a $p$-morphism onto a cone; similarly for Kripke models.

4. $\cong$ is an equivalence relation between Kripke frames; the same for Kripke models.
Proof

(1) Easily follows from cone preservation.

(2) In fact, the cone preservation for \( j \) is equivalent to \( R'(x) = R(x) \) for \( x \in F' \).

(3) Follows from (1) and (2); note that the restriction of \( f : F \to G \) to \( F' \subseteq F \) is the composition \( f \cdot j \), where \( j : F' \to F \) is the inclusion map. The image of a cone is a cone by cone preservation.

(4) Trivial, since a composition of isomorphisms is an isomorphism.

\( \blacksquare \)
Lemma 1.3.33 (Morphism lemma)

(1) Every morphism of Kripke models is reliable, i.e. if \( f : M \rightarrow M' \), then for any \( u \in M \), for any modal formula \( A \),
\[
M, u \vDash A \text{ iff } M', f(u) \vDash A.
\]

(2) If \( f : M \rightarrow M' \), then for any modal formula \( A \),
\[
M \vDash A \text{ iff } M' \vDash A.
\]

(3) If \( F \rightarrow F' \) then \( ML(F) \subseteq ML(F') \).

Proposition 1.3.34 Every variety \( V(\Gamma) \) is closed under generated subframes, \( p \)-morphic images, and disjoint sums.

Proof Follows from 1.3.25, 1.3.28 and 1.3.33.

Remark 1.3.35 This proposition resembles the 'only if' part of the Birkhoff theorem 1.2.25. The analogy becomes clear if we note the duality between Kripke frames and their modal algebras — generated subframes correspond to homomorphic images of algebras, \( p \)-morphic images to subalgebras, and disjoint sums to direct products. However the converse to 1.3.34 is not true. A precise model-theoretic characterisation of Kripke frame varieties (using ultrafilter extensions) was given by van Benthem, cf. [van Benthem, 1983], [Blackburn, de Rijke and Venema, 2001]. We shall return to this topic in Volume 2.

Exercise 1.3.36 Show the following analogue of 1.2.26: the poset of Kripke-complete \( N \)-modal propositional logics is dually isomorphic to the poset of all \( N \)-modal Kripke frame varieties.

Proposition 1.3.37 Let \( (F_i \mid i \in I) \) be a family consisting of all different components of a propositional Kripke frame \( F \). Then \( F \cong \bigsqcup_{i \in I} F_i \).

Proof An isomorphism is given by the map \( f \) sending \( x \) to \( (x, i) \) whenever \( x \in F_i \). This map is well-defined, since different components are disjoint by 1.3.24.

If \( xR_{ji}y \) and \( x \in F_i \), then \( y \in F_i \) as well, so \( f(x) = (x, i)R_{ji}(y, i) = f(y) \).

The implication \( f(x)R_{ji}f(y) \Rightarrow xR_{ji}y \) is trivial.

Proposition 1.3.38 Let \( (F_i \mid i \in I) \) be a family of all different components of a propositional Kripke frame \( F \).

Then

(1) for any morphism \( f : F \rightarrow G \), every \( f_i := f \upharpoonright F_i \) is a morphism;

(2) for any family of morphisms \( f_i : F_i \rightarrow G \), the joined map \( f := \bigcup_{i \in I} f_i \) is a morphism \( F \rightarrow G \).
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Proof

(1) Since every component is a generated subframe, this follows from Lemma 1.3.32(3).

(2) Let $R_j$ be a relation in $F$, $S_j$ the corresponding relation in $G$. For $x \in F_i$ we have $R_j(x) \subseteq F_i$, so

$$f[R_j(x)] = f_i[R_j(x)] = S_j(f_i(x)) = S_j(f(x)),$$

since $f \upharpoonright F_i = f_i$ and $f_i$ is a morphism.

In this book we are especially interested in $\mathbf{K4}$- and $\mathbf{S4}$-frames; by Lemma 1.3.11, $\mathbf{K4}$-frames are transitive, and $\mathbf{S4}$-frames are quasi-ordered sets.

For a $\mathbf{K4}$-frame $(W, R)$ there is the equivalence relation $\approx_R := (R \cap R^{-1}) \cup \text{Id}_W$ in $W$; its equivalence classes are called $(R\text{-})$clusters, and we can consider the quotient set as a frame.

Definition 1.3.39 Let $F = (W, R)$ be a $\mathbf{K4}$-frame. Let $W^\sim := W/\approx_R$ be the set of all $R$-clusters, and let $u^\sim$ be the cluster of $u$. Then the frame $F^\sim := (W^\sim, R^\sim)$, where $R^\sim := \{(u^\sim, v^\sim) \mid uRv\}$, is called the skeleton of $F$.

A singleton cluster $\{u\}$ is called trivial (respectively, degenerate) if $u$ is reflexive (respectively, irreflexive).

Lemma 1.3.40

(1) $R^\sim$ is transitive and antisymmetric; if $R$ is reflexive, then $R^\sim$ is a partial order.

(2) The map $u \mapsto u^\sim$ is a $p$-morphism from $F$ onto $F^\sim$.

Proof Straightforward.

Lemma 1.3.41 If $f : F \to G$ and $F$ is connected, then $G$ is connected.

Proof If $(u_0, j_0, \ldots, j_{m-1}, u_m)$ is a path from $u$ to $v$ in $F$, then $(f(u_0), j_0, \ldots, f(u_m))$ is a path from $f(u)$ to $f(v)$ in $G$.

1.3.4 Conical expressiveness

Definition 1.3.42 An $N$-modal propositional logic $L$ is called conically expressive if there exists a propositional $N$-modal formula $C(p)$ with a single proposition letter $p$ such that for any propositional Kripke model $M$ with $M \models L$, for any $u \in M$,

$$M, u \models C(p) \iff M \upharpoonright u \models p.$$ 

Obviously, an extension of a conically expressive logic (in the same language) is conically expressive. A simple example of a conically expressive logic is $\mathbf{K4}$; the corresponding $C(p)$ is $p \land \Box p$. 

Exercise 1.3.43

(a) Show that \( K + \Box^2p \supset \Box^4p \) is conically expressive.

(b) Recall that \( K_4.t = K_2 + \Diamond_1 \Box_2 \supset p + \Diamond_1 \Box_1p \supset p + \Box_1p \supset \Box_2^2p \). Show that \( K_4.t + \Box_1 \Box_2p \supset \Box_2 \Box_1p \) is conically expressive.

Now given \( C \) from Definition 1.3.42, for a formula \( A \in \mathcal{L}_{PN} \) put

\[ \Box^* A := [A/p]C(p). \]

Let us show that \( \Box^* \) behaves like an \( S_4 \)-modality in logics containing \( L \).

**Lemma 1.3.44** For any \( N \)-modal Kripke model \( M \) such that \( M \vDash \Lambda \) for any \( u \in M \), for any \( N \)-modal formula \( A \)

\[ M, u \vDash \Box^* A \iff \forall v \in R^*(u) \ M, v \vDash A, \]

where \( R^* \) is the same as in Lemma 1.3.19.

**Proof** Similar to soundness of the substitution rule. Let \( M_0 \) be a Kripke model over the same frame as \( M \), such that for any \( v \in M \)

\[ M_0, v \vDash p \iff M, v \vDash A. \]

Then by induction we obtain for any propositional formula \( X(p) \) for any \( v \in M \).

\[ M_0, v \vDash X(p) \iff M, v \vDash X(A). \]

Hence for any \( u \in M \) and for \( C \) from 1.3.42

\[ M_0, u \vDash C(p) \iff M, u \vDash C(A) (= \Box^* A), \]

and so by 1.3.42,

\[ M, u \vDash \Box^* A \iff M_0 \uparrow u \vDash p. \]

But by Lemma 1.3.19 and the choice of \( M_0 \)

\[ M_0 \uparrow u \vDash p \iff \forall v \in R^*(u) \ M_0, v \vDash p \iff \forall v \in R^*(u) \ M, v \vDash A. \]

Hence the claim follows.

**Lemma 1.3.45** If a modal logic \( \Lambda \) is conically expressive and \( \Box^* \) is defined as above, then the rule \( \begin{array}{c} A \\ \hline \Box^* A \end{array} \) is admissible in \( \Lambda \).

**Proof** Since by 1.13.3, \( \Lambda \) has an exact model, it suffices to show that for any Kripke model \( M \), for any \( A, M \vDash A \) implies \( M \vDash \Box^* A \). But this readily follows from 1.3.44.

**Lemma 1.3.46** If an \( N \)-modal propositional logic \( \Lambda \) is conically expressive, then the following formulas are in \( \Lambda \) (for \( k \leq N \)):
(1) $\Box^*(p \supset q) \supset \Box^*p \supset \Box^*q$;

(2) $\Box^*p \supset p$;

(3) $\Box^*p \supset \Box^*\Box^*p$;

(4) $\Box^*p \supset \Box_1\Box^*p$;

(5) $\Box^*p \supset \Box_k p$.

Proof Again we can consider an exact model and show that these formulas are true at every world. This easily follows from 1.3.44, the reflexivity and the transitivity of $R^*$, and the inclusions $R^* \supset R \circ R^*$, $R_k \subseteq R^*$.

\section*{1.4 Relational semantics (the intuitionistic case)}

Definition 1.4.1 Assume that $F = (W, R)$ is an S4-frame, i.e. $MA(F)$ is a topo-Boolean algebra. Then the algebra of its open elements (or, equivalently, of stable subsets of $F$)

$$HA(F) := MA(F)^0$$

is called the Heyting algebra of $F$.

S4-frames are also called intuitionistic propositional.

An intuitionistic valuation in $F$ is defined as a valuation in $HA(F)$, i.e. as a valuation in $W$ with the following truth-preservation (or monotonicity) property (for every $q \in PL$):

$$(TP_0) \quad \forall u, v \in W \ (uRv \land u \in \theta(q) \Rightarrow v \in \theta(q)).$$

The corresponding Kripke model $M = (F, \theta)$ is also called intuitionistic. $\theta$ is extended to the map $\theta^I : IF \rightarrow HA(F)$ according to Definition 1.2.12.

Definition 1.4.2 For an intuitionistic Kripke model $M = (W, R, \theta)$ we define the intuitionistic forcing relation between worlds and intuitionistic formulas as follows:

$$M, u \Vdash A := u \in \theta^I(A).$$

Now we readily obtain an alternative inductive definition of intuitionistic forcing, cf. Lemma 1.3.3.

Lemma 1.4.3 Intuitionistic forcing has the following properties:

- $M, u \Vdash q$ iff $u \in \theta(q)$ (for $q \in PL$);
- $M, u \Vdash \bot$;
- $M, u \Vdash B \land C$ iff $M, u \Vdash B \land M, u \Vdash C$;
- $M, u \Vdash B \lor C$ iff $M, u \Vdash B \lor M, u \Vdash C$;
1.4. RELATIONAL SEMANTICS (THE INTUITIONISTIC CASE)

- $M, u \models B \supset C$ iff $\forall v \in R(u) (M, v \models B \Rightarrow M, v \models C)$;
- $M, u \models \neg B$ iff $\forall v \in R(u) M, v \not\models B$;
- $M, u \models B \equiv C$ iff $\forall v \in R(u) (M, v \models B \Leftrightarrow M, v \models C)$.

**Lemma 1.4.4** In intuitionistic Kripke models the truth-preservation holds for any formula $A$:

$$(TP) \quad \forall u, v \in F (uRv \& M, u \models A \Rightarrow M, v \models A).$$

**Proof** Trivial, since $\theta^I(A)$ is stable.

**Definition 1.4.5** An intuitionistic formula $A$ is said to be valid in an S4-frame $F$ (notation: $F \models A$) if $HA(F) \models A$, i.e. if $A$ is true in every intuitionistic model over $F$.

Similarly, one can reformulate the definitions of satisfiability, etc. (1.3.5, 1.3.6) for the intuitionistic case.

We also have a relational analogue of Lemma 1.2.19, for which we need the following

**Definition 1.4.6** Let $F$ be an S4-frame, $M$ a Kripke model over $F$. The pattern of $M$ is the Kripke model $M_0$ over $F$ such that for any $u \in F, q \in PL$

$$M_0, u \models q \iff M, u \models \Box q.$$  

Obviously, $M_0$ is an intuitionistic Kripke model; $M_0 = M$ if $M$ is itself intuitionistic.

**Lemma 1.4.7** Under the conditions of Definition 1.4.6, we have

1. for any $u \in F$ and intuitionistic formula $A$

$$M_0, u \models A \iff M, u \models A^T,$$

2. $F \models A \iff F \models A^T$.

**Proof** Obvious from 1.2.19.

Together with Lemma 1.2.14 and the trivial observation that $F \not\models \bot$, this implies

**Lemma 1.4.8** Let $F$ be an S4-frame. Then the set of all intuitionistic formulas valid in $F$ is an intermediate logic.

This logic is called the intermediate logic of $F$ and denoted by $IL(F)$.

For a set of intuitionistic formulas $\Gamma$, an intuitionistic $\Gamma$-frame is an intuitionistic propositional frame, in which $\Gamma$ is intuitionistically valid. The class of all these frames is called the intuitionistic (Kripke frame) variety of $\Gamma$ and denoted by $V^I(\Gamma)$.

Definition 1.3.9 is obviously transferred to the intuitionistic case. Respectively the notation $ML$, $MT$ changes to $IL$, $IT$. Lemmas 1.3.10 and 1.3.13 and Proposition 1.3.15 also have intuitionistic versions; the reader can easily formulate them.
Lemma 1.4.9 Let $F$ be an $S_4$-frame, $M$ an intuitionistic model over $F$.

(1) If $M_1$ is a generated submodel of $M$, then $M_1$ also is intuitionistic, and for any $A \in \mathcal{LP}_0$, $u \in M_1$

\[ M_1, u \models A \iff M, u \models A. \]

(2) If $F_1$ is a generated subframe of $F$ then

\[ \text{IL}(F) \subseteq \text{IL}(F_1). \]

Proof

(1) From the generation lemma 1.3.25 and the definition of intuitionistic forcing (1.4.1).

(2) From Lemma 1.3.25(2) and Lemma 1.4.7(2).

Lemma 1.4.10 Let $F'$ be a subframe of a Kripke $S_4$-frame $F$. Then every intuitionistic valuation $\theta'$ in $F'$ can be extended to an intuitionistic valuation in $F$.

Proof In fact, take a valuation $\theta$ in $F$ such that $\theta(q) = R(\theta'(q))$ for any $q \in \mathcal{PL}$.

The three subsequent lemmas readily follow from Lemmas 1.4.7, 1.3.26, 1.3.28 and 1.3.33.

Lemma 1.4.11 For an $S_4$-frame $F$

\[ \text{IL}(F) = \bigcap_{u \in F} \text{IL}(F \upharpoonright u). \]

Lemma 1.4.12 For $S_4$-frames $F_i$, $i \in I$

\[ \text{IL} \left( \bigcup_{i \in I} F_i \right) = \bigcap_{i \in I} \text{IL}(F_i). \]

Lemma 1.4.13 (Morphism lemma) Let $M$, $M'$ be intuitionistic Kripke models, and let $F$, $F'$ be $S_4$-frames.

(1) If $f : M \rightarrow M'$, then $f$ is reliable: for any $u \in M$, for any intuitionistic formula $A$,

\[ M, u \models A \iff M', f(u) \models A. \]

(2) If $f : M \rightarrow M'$, then

\[ M \models A \iff M' \models A. \]

(3) If $F \rightarrow F'$, then $\text{IL}(F) \subseteq \text{IL}(F')$. 
Lemma 1.4.14 Let $F$ be an S4-frame. Then

$$IL(F) = IL(F^\sim).$$

Proof If $\varphi$ is an intuitionistic valuation in $F$, consider the valuation $\varphi^\sim$ in $F^\sim$ defined by

$$\varphi^\sim(q) := \{u^\sim \mid u \in \varphi(q)\}.$$ 

It is clear that $\varphi^\sim$ is well-defined and intuitionistic. By Lemma 1.3.40, it follows that the map sending $u$ to $u^\sim$ is a $p$-morphism of Kripke models, and thus

$$(F, \varphi) \models A \iff (F^\sim, \varphi^\sim) \models A$$

by Lemma 1.4.13. To complete the proof, note that an arbitrary intuitionistic valuation $\psi$ in $F^\sim$ can be presented as $\varphi^\sim$, with

$$\varphi(q) = \{u \mid u^\sim \in \psi(q)\}.$$

\[ \blacksquare \]

For a class $C$ of S4-frames let $C^\sim$ be the closure of $\{F^\sim \mid F \in C\}$ under isomorphism, and let Posets be the class of all posets.

Lemma 1.4.15

1. $IL(C) = IL(C^\sim)$.
2. $V^I(\Gamma)^\sim = V^I(\Gamma) \cap \text{Posets}$ for any set of intuitionistic formulas $\Gamma$.
3. Every Kripke-complete intermediate logic is determined by some class of posets: $L = IL(V^I(L) \cap \text{Posets})$.

Proof

1. Follows readily from 1.4.14.

2. In fact, $F \Vdash \Gamma \iff F^\sim \Vdash \Gamma$ by 1.4.14, so $V^I(\Gamma)^\sim \subseteq V^I(\Gamma)$. The other way round, if a poset $G \in V^I(\Gamma)$, then $G \cong G^\sim \in V^I(\Gamma)^\sim$.

3. Note that $L = IL(V^I(L))$ for a complete $L$ and apply (1), (2).

\[ \blacksquare \]

So instead of $V^I(\Gamma)$ we can use the reduced intuitionistic variety

$$V^\sim(\Gamma) := V^I(\Gamma) \cap \text{Posets}.$$ 

Now we obtain an analogue of 1.3.34:

Proposition 1.4.16 Intuitionistic Kripke frame varieties and reduced intuitionistic Kripke frame varieties are closed under generated subframes, $p$-morphic images, and disjoint sums.
Proof  For intuitionistic varieties this follows from 1.4.9, 1.4.12 and 1.4.13. For reduced intuitionistic varieties we can also apply 1.4.15(2) and note that the class of posets is closed under the same three operations.

Let us describe reduced intuitionistic varieties for some intuitionistic formulas.

**Proposition 1.4.17**

- \( V^\sim(EM) = V^\sim(CL) \) consists of all trivial frames, i.e. frames of the form \((W, Id_W)\);
- \( V^\sim(AJ) = V^\sim(HJ) = V(S4.2)^\sim \) consists of all confluent posets;
- \( V^\sim(AZ) = V^\sim(LC) = V(S4.3)^\sim \) consists of all nonbranching posets;
- \( V^\sim(AP_n) \) consists of all posets of depth \( \leq n \);
- \( V^\sim(AIW_n) \) consists of all posets of width \( \leq n \);
- \( V^\sim(AG_n) \) consists of all posets of \( n \) depth; \( \forall x \in F, |HA(F \uparrow x)| \leq n. \)

**Definition 1.4.18** A quasi-morphism between \( S4 \)-frames \( F = (W, R) \) and \( F' = (W', R') \) is a monotonic map \( h : W \rightarrow W' \) with the quasi-lift property:

\[
\forall x \in W \forall y' \in W' (h(x)R'y' \Rightarrow \exists y (xRy \& h(y) \approx R' y')).
\]

For intuitionistic Kripke models \( M = (F, \theta), M' = (F', \theta') \) a quasi-morphism from \( M \) to \( M' \) is a quasi-morphism of their frames such that for any \( q \in PL \)

\[
M, x \models q \iff M', h(x) \models q.
\]

A quasi-p-morphism is a surjective quasi-morphism.

The following is clear:

**Lemma 1.4.19** A quasi-(p)-morphism of frames \( h : F \rightarrow F' \) gives rise to a quasi-(p)-morphism of their skeletons \( h^\sim : F^\sim \rightarrow F'^\sim \) such that \( h^\sim(u^\sim) = h(u)^\sim. \)

Quasi p-morphisms are reliable for intuitionistic formulas:

**Lemma 1.4.20**

1. If \( h \) is a quasi-morphism from \( M \) to \( M' \), then for any \( x \in M \), for any intuitionistic formula \( A \),

\[
M, x \models A \iff M', h(x) \models A.
\]

2. If there exists a quasi-p-morphism from \( F \) onto \( F' \) then \( IL(F) \subseteq IL(F') \).

**Proof** By Lemmas 1.4.20 and 1.4.14.

---

\(^7\text{See Section 1.15.}\)

\(^8\text{See Section 1.9.}\)
1.5 Modal counterparts

The following lemma can be easily proved by induction.

**Lemma 1.5.1** Let \( S = [C/p] \) be an intuitionistic substitution, \( ST := [CT/p] \). Then \( S4 \vdash (SA)^T \equiv ST A^T \) for any intuitionistic formula \( A \).

This lemma implies

**Proposition 1.5.2** For any 1-modal logic \( \Lambda \supseteq S4 \) the set
\[ T\Lambda := \{ A \in \mathcal{L} \mathcal{P}_0 \mid A^T \in \Lambda \} \]
is a superintuitionistic logic.

**Lemma 1.5.3** For any S4-algebra \( \Omega \), \( TML(\Omega) = IL(\Omega^0) \).

**Proof** This is a reformulation of 1.2.19(2).

**Definition 1.5.4** The above defined logic \( T\Lambda \) is called the superintuitionistic fragment of \( \Lambda \); the logic \( \Lambda \) is called a modal counterpart of \( T\Lambda \).

For a set \( \Gamma \) of intuitionistic formulas let \( \Gamma^T := \{ A^T \mid A \in \Gamma \} \).

**Theorem 1.5.5**\(^9\) Every propositional superintuitionistic logic \( L = H + \Gamma \) has the smallest modal counterpart: \( \tau(L) := S4 + \Gamma^T \).

**Theorem 1.5.6**\(^10\) Every propositional superintuitionistic logic has the greatest modal counterpart. In particular, the greatest modal counterpart of \( H \) is \( Grz \).

The greatest modal counterpart of \( L \) is denoted by \( \sigma(L) \).

**Theorem 1.5.7** (Blok–Esakia)

1. \( \sigma(H + \Gamma) = Grz + \Gamma^T \).

2. The correspondence between superintuitionistic logics and their greatest modal counterparts is an order isomorphism between superintuitionistic logics and modal logics above \( Grz \).

See [Chagrov and Zakharyaschev, 1997] for the proof of 1.5.7 (as well as 1.5.5 and 1.5.6).

**Proposition 1.5.8** If a modal logic \( \Lambda \supseteq S4 \) is Kripke-complete, then \( T\Lambda \) is also Kripke-complete. More precisely, \( TML(C) = IL(C) \).

---

\(^9\)[Dummett and Lemmon, 1959].

\(^10\)[Maksimova and Rybakov, 1974], [Esakia, 1979].
Proof In fact, $^T \text{ML}(C) = \bigcap_{F \in C} ^T \text{ML}(F) = \bigcap_{F \in C} \text{IL}(F) = \text{IL}(C)$. The second equality follows from 1.4.7.

Lemma 1.5.9 For any intermediate logic $L$,

$$V(\tau(L)) = V^I(L).$$

Proof An exercise.

Theorem 1.5.10 (Zakharyaschev) The map $\tau$ preserves Kripke-completeness, so if an intermediate logic $L$ is Kripke-complete, then

$$\tau(L) = \text{ML}(V^I(L)).$$

For the proof see [Chagrov and Zakharyaschev, 1997]

Remark 1.5.11 Unlike $\tau$, the map $\sigma$ does not preserve Kripke-completeness; a counterexample can be found in [Shehtman, 1980].

1.6 General Kripke frames

'General Kripke frame semantics' from [Thomason, 1972]\textsuperscript{11} (cf. also [Chagrov and Zakharyaschev, 1997]) is an extended version of Kripke semantics, which is equivalent to algebraic semantics.

Definition 1.6.1 A general modal Kripke frame is a modal Kripke frame together with a subalgebra of its modal algebra, i.e. $\Phi = (F, W)$, where $W \subseteq MA(F)$ is a modal subalgebra. $W$ is called the modal algebra of $\Phi$ and also denoted by $MA(\Phi)$; its elements are called interior sets of $\Phi$.

So for $F = (W, R_1, \ldots, R_N)$, $W$ should be a non-empty set of subsets closed under Boolean operations and $\square_i : V \rightarrow \square_i V$ (Section 1.3). Instead of $\square_i$ one can use its dual $\diamond_i V := R_i^{-1} V$.

Definition 1.6.2 A valuation in a general Kripke frame $\Phi = (F, W)$ is a valuation in $MA(\Phi)$. A Kripke model over $\Phi$ is just a Kripke model $M = (F, \theta)$ over $F$, in which $\theta$ is a valuation in $\Phi$. A modal formula $A$ is valid in $\Phi$ (notation: $\Phi \models A$) if it is true in every Kripke model over $\Phi$; similarly for a set of formulas.

Analogous definitions are given in the intuitionistic case.

Definition 1.6.3 A general intuitionistic Kripke frame is $\Phi = (F, W)$, where $F$ is an intuitionistic Kripke frame, $W \subseteq HA(F)$ is a Heyting subalgebra. $W$ is called the Heyting algebra of $\Phi$ and denoted by $HA(\Phi)$; its elements are called interior sets.

\textsuperscript{11}In that paper it was called 'first-order semantics'.
Definition 1.6.4 An intuitionistic valuation in a general intuitionistic Kripke frame \( \Phi = (F, \mathcal{W}) \) is a valuation in \( HA(\Phi) \). An intuitionistic Kripke model over \( \Phi \) is of the form \( M = (F, \theta) \), where \( \theta \) is an intuitionistic valuation in \( \Phi \). An intuitionistic formula \( A \) is valid in \( \Phi \) (notation: \( \Phi \models A \)) if it is true in every intuitionistic Kripke model over \( \Phi \); similarly for a set of formulas.

Obviously we have analogues of 1.3.7, 1.3.8, 1.4.7 and 1.4.8:

Lemma 1.6.5 For any modal formula \( A \) and a general Kripke frame \( \Phi \)
\[
\Phi \models A \iff MA(\Phi) \models A;
\]
analogously, for intuitionistic \( A \) and \( \Phi \)
\[
\Phi \vdash A \iff HA(\Phi) \models A;
\]

Lemma 1.6.6

1. For a general modal Kripke frame \( \Phi \) the set
\[
ML(\Phi) := \{ A \mid \Phi \models A \}
\]
is a modal logic; if \( \Phi \) is intuitionistic, then
\[
IL(\Phi) := \{ A \mid \Phi \vdash A \}
\]
is an intermediate logic, and moreover, \( IL(\Phi) = TML(\Phi) \).

2. For a class \( C \) of \( N \)-modal general frames the set
\[
ML(C) := \bigcap \{ ML(\Phi) \mid \Phi \in C \}
\]
is an \( N \)-modal logic; if the frames are intuitionistic, then
\[
IL(C) := \bigcap \{ IL(\Phi) \mid \Phi \in C \}
\]
is an intermediate logic, and \( IL(C) = TML(C) \).

Lemma 1.6.7

1. For an \( N \)-modal Kripke model \( M = (F, \theta) \) the set of all definable sets
\( \mathcal{W}_M := \{ \theta(A) \mid A \in \mathcal{L}P_n \} \) is a subalgebra of \( MA(F) \).

2. Similarly, for an intuitionistic Kripke model \( M = (F, \theta) \), \( \mathcal{W}_M^I = \{ \theta^I(A) \mid A \in \mathcal{L}P_0 \} \) is a subalgebra of \( HA(F) \).

Proof (Modal case) In fact, by definition
\[
\theta(\neg A) = -\theta(A), \ \theta(A \land B) = \theta(L)(A) \cap \theta(L)(B), \ \theta(\Box_i A) = \Box_i \theta(A).
\]
So we define the corresponding general frames

**Definition 1.6.8** For a Kripke model \( M = (F,\theta) \), the general frame \( GF(M) := (F, W_M) \) (or \( GF'(M) := (F, W'_M) \) in the intuitionistic case) is called associated.

**Lemma 1.6.9** \( GF(M) \models A \) iff all modal substitution instances of \( A \) are true in \( M \); similarly for the intuitionistic case.

**Proof** (If.) For \( M = (F,\theta) \), let \( \eta \) be a valuation in \( F \) such that \( \eta(p_i) = \theta(B_i) \) for every \( i \). An easy inductive argument shows that

\[
\eta(A) = \theta([B_1, \ldots, B_n/p_1, \ldots, p_n]A)
\]

for any \( n \)-formula \( A \), cf. Lemma 1.2.9.

(Only if.) The same equality shows that for any modal (or intuitionistic) substitution \( S \), \( \theta(SA) = \eta(A) \) for an appropriate valuation \( \eta \) in \( GF(M) \).

**Definition 1.6.10** If \( \Phi = (F, W) \) is a general Kripke frame, \( F = (W, R_1, \ldots, R_N), V \subseteq W \), then we define the corresponding general subframe:

\[
\Phi \upharpoonright V := (F \upharpoonright V, W \upharpoonright V),
\]

where

\[
W \upharpoonright V := \{ X \cap V \mid X \in W \}.
\]

If \( V \) is stable, \( \Phi \upharpoonright V \) is called a generated (general) subframe. The cone generated by \( u \) in \( \Phi \) is the subframe \( \Phi \upharpoonright u := \Phi \upharpoonright (W \upharpoonright u) \).

The definition of a subframe is obviously sound, because \( W \upharpoonright V \) is a modal algebra of subsets of \( V \) with modal operations \( \Box_t V X := \Box_t X \cap V \).

The following is a trivial consequence of 1.3.25 and 1.3.26.

**Lemma 1.6.11** For a general frame \( \Phi \),

1. if \( V \) is a stable subset, then \( \text{ML}(\Phi) \subseteq \text{ML}(\Phi \upharpoonright V) \);
2. \( \text{ML}(\Phi) = \bigcap_{u \in \Phi} \text{ML}(\Phi \upharpoonright u) \).

**Exercise 1.6.12** Define morphisms of general frames and prove their properties.

### 1.7 Canonical Kripke models

**Definition 1.7.1** The canonical Kripke frame for an \( N \)-modal propositional logic \( \Lambda \) is \( F_\Lambda := (W_\Lambda, R_{1,\Lambda}, \ldots, R_{N,\Lambda}) \), where \( W_\Lambda \) is the set of all \( \Lambda \)-complete theories, \( xR_{i,\Lambda}y \) iff for any formula \( A \), and \( \Box_i A \in x \) implies \( A \in y \).

The canonical model for \( \Lambda \) is \( M_\Lambda = (F_\Lambda, \theta_\Lambda) \), where

\[
\theta_\Lambda(p_i) := \{ x \mid p_i \in x \}.
\]
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Analogously, the canonical frame for a bounded modal logic $\Lambda[m]$ is $F_{\Lambda[m]} := (W_{\Lambda[m]}, R_1, \Lambda[m], \ldots, R_N, \Lambda[m])$, where $W_{\Lambda[m]}$ is the set of all maximal $\Lambda$-consistent sets of $N$-modal $m$-formulas,

$$xR_i,\Lambda[m]y \text{ iff for any } m\text{-formula } A, \square_i A \in x \text{ implies } A \in x.$$ 

The canonical model for $\Lambda[m]$ is $M_{\Lambda[m]} := (F_{\Lambda[m]}, \theta_{\Lambda[m]})$, where $\theta_{\Lambda[m]}(p_i) = \theta_\Lambda(p_i)$ for $i \leq m$.

**Definition 1.7.2** The canonical frame for an intermediate propositional logic $\Sigma$ is $F_{\Sigma} := (W_{\Sigma}, R_{\Sigma})$, where $W_{\Sigma}$ is the set of all $\Sigma$-complete intuitionistic (double) theories, $xR_{\Sigma}y \text{ iff } x \subseteq y$.

The canonical model for $\Sigma$ is $M_{\Sigma} := (F_{\Sigma}, \theta_{\Sigma})$, where

$$\theta_{\Sigma}(p_i) := \{x \mid p_i \in x\}.$$ 

The canonical model theorem for bounded intermediate logics must be now clear, so we skip them.

The following is well-known, cf. [Chagrov and Zakharyaschev, 1997], [Blackburn, de Rijke and Venema, 2001].

**Theorem 1.7.3** (Canonical model theorem) For any $N$-modal or intermediate logic $\Lambda$ and $m$-formula $A$ (of the corresponding kind):

1. $M_{\Lambda}, x \models A$ iff $A \in x$;
2. $M_{\Lambda[m]}, y \models A$ iff $A \in y$;
3. $M_{\Lambda[m]} \models A$ iff $M_{\Lambda} \models A$ iff $A \in \Lambda$.

**Definition 1.7.4** The general canonical frame of an $N$-modal (respectively, intuitionistic) logic $\Lambda$ is $\Phi_\Lambda := GF(M_\Lambda)$ (respectively, $\Phi_\Lambda^i := GF^i(M_\Lambda)$).

**Theorem 1.7.5** (General canonical model theorem) For a modal (respectively, intermediate) propositional logic $\Lambda$ 

$$ML(\Phi_\Lambda) = \Lambda \quad (\text{respectively, } I\text{L}(\Phi_\Lambda) = \Lambda).$$ 

**Proof** By Lemma 1.6.9, $ML(\Phi_\Lambda)$ consists of all formulas $A$ such that $M_\Lambda \models SA$ of any modal substitution $S$. By 1.7.3, the latter is equivalent to $SA \in \Lambda$. Since $\Lambda$ is substitution closed, it follows that $ML(\Phi_\Lambda) = \Lambda$. The same argument works for the intuitionistic case. 

An alternative proof of 1.7.5 can be obtained from the algebraic completeness theorem 1.2.21 and the following observation:

**Proposition 1.7.6** $MA(\Phi_\Lambda) \cong Lind(\Lambda)$ for a modal logic $\Lambda$; 
$HA(\Phi_\Sigma) \cong Lind(\Sigma)$ for an intermediate logic $\Sigma$. 


Proof (Modal case.) The map \( \gamma : \text{Lind}(\Lambda) \rightarrow MA(\Phi_\Lambda) \) sending \([A]\) to \(\theta_\Lambda(A)\) is well-defined, since \(M_\Lambda \models A \equiv B\) whenever \(A \sim_\Lambda B\) by Theorem 1.7.1. \(\gamma\) is obviously surjective and preserves the modal algebra operations. In fact,

\[
\gamma([\square A]) = \gamma([\square A]) = \theta_\Lambda(\square A) = \square \theta_\Lambda(A)
\]

according to Definitions 1.2.20 and 1.3.2, and similarly for the other operations. Finally, note that

\[
\gamma([A]) = \gamma([B]) \iff \theta_\Lambda(A) = \theta_\Lambda(B) \iff M_\Lambda \models A \equiv B \iff (A \equiv B) \in \Lambda \iff [A] = [B]
\]

by 1.7.1 and the definitions. So \(\gamma\) is an isomorphism. ■

Definition 1.7.7 A general frame \(\Phi = ((W, R_1, \ldots, R_N), W)\) (modal or intuitionistic) is called descriptive if it satisfies the following conditions:

- \(\Phi\) is differentiated (distinguishable): for any two different points \(x, y\) there exists an interior set \(U \in W\) separating them, i.e. such that \(x \in U \Leftrightarrow y \in U\);
- tightness:
  \[
  \forall x, y, i \ (\forall U \in W (x \in \square_i U \Rightarrow y \in U) \Rightarrow xR_i y) \quad \text{(in the modal case)},
  \forall x, y (\forall U \in W (x \in U \Rightarrow y \in U) \Rightarrow xR_1 y) \quad \text{(in the intuitionistic case)};
  \]
- compactness:
  every centered subset \(X \subseteq W\) (i.e. such that \(\bigcap X_1 \neq \emptyset\) for any finite \(X_1 \subseteq X\)) has a non-empty intersection (in the modal case);
  if a pair \((X, Y)\) of subsets of \(W\) is centered (i.e. \(\bigcap X_1 \nsubseteq \bigcup Y_1\) for any finite \(X_1 \subseteq X, Y_1 \subseteq Y\)), then \(\bigcap X \nsubseteq \bigcup Y\) (in the intuitionistic case).

A differentiated and tight general frame is called refined.

Lemma 1.7.8 A generated subframe of a refined frame is refined.

Proof Distinguishability is obviously preserved for subframes.

Let us check tightness in the modal case. We use the same notation as in 1.7.7. Suppose \(V \subseteq W\) is stable, \(x, y \in V\) and

\[
(1) \forall U \in W | V (x \in \square_i V \Rightarrow y \in U).
\]

This is equivalent to

\[
\forall U \in W (x \in \square_i (U \cap V) \cap V \Rightarrow y \in U \cap V)
\]

and thus (since \(V\) is stable and \(x, y \in V\)) to

\[
(2) \forall U \in W (x \in \square_i U \Rightarrow y \in U).
\]

If \(\Phi\) is tight, (2) implies \(xR_i y\), therefore (1) also implies \(xR_i y\), which means tightness of \(\Phi | V\). ■
Remark 1.7.9 Compactness is not always preserved for generated subframes; the reader can try to construct a counterexample.

Descriptive frames can also be characterised as canonical frames of modal algebras. These frames resemble canonical frames of modal logics. Recall that a (proper) filter in a Boolean algebra is a \( \leq \)-stable (proper) subset closed under meets. A maximal filter is maximal among proper filters.

Definition 1.7.10 For an \( N \)-modal algebra \( \mathfrak{A} \), consider the set \( \Omega_+ \) of its maximal filters with the accessibility relations:

\[ xR_+y \text{ iff } \forall a (\square_i a \in x \Rightarrow a \in y) \]

and with the interior sets of the form

\[ h(a) := \{ x \in \Omega_+ \mid a \in x \} \]

for all \( a \in \Omega \). The resulting general frame \( \Omega_+ := (\Omega_+, R_1, \ldots, R_N) \) is called the canonical or the dual frame of \( \Omega \).

Recall that a (proper) prime filter in a Heyting algebra is a \( \leq \)-stable (proper) subset \( X \) closed under \( \wedge \) and such that

\[ a \lor b \in X \text{ only if } (a \in X \text{ or } b \in X). \]  

Definition 1.7.11 For a Heyting algebra \( \mathfrak{H} \), consider the set \( \Omega_+ \) of its prime filters with the accessibility relation \( xRy \text{ iff } x \subseteq y \) and with the interior sets

\[ h(a) = \{ x \in \Omega_+ \mid a \in x \} \text{ for } a \in \Omega \]

The general frame \( \Omega_+ := (\Omega_+, R) \) is called the canonical or the dual frame of \( \Omega \).

Theorem 1.7.12 (Tarski–Jonsson)

1. If \( \Omega \) is a modal algebra, then \( MA(\Omega_+) \cong \Omega \).
2. If \( \Omega \) is a Heyting algebra, then \( HA(\Omega_+) \cong \Omega \).

A required isomorphism is the Stone map \( h \) from Definitions 1.7.10 and 1.7.11. In particular, for the Lindenbaum algebra we again obtain the frame isomorphic to \( \Phi_\Lambda \) (with an obvious notion of isomorphism):

Theorem 1.7.13 For a modal or intermediate logic \( \Lambda \)

\[ Lind(\Lambda)_+ \cong \Phi_\Lambda. \]

This is because the maximal (prime) filters of \( Lind(\Lambda) \) exactly correspond to \( \Lambda \)-complete theories.

Theorem 1.7.14 A general Kripke frame is descriptive iff it is isomorphic to some frame \( \Omega_+ \).
This result was proved in [Goldblatt, 1976] and earlier in [Esakia, 1974] for S4-frames. For a more available proof see [Chagrov and Zakharyaschev, 1997].

The subsequent lemma follows from the above, but also has an independent proof based on the properties of \( M_\Lambda \).

**Lemma 1.7.15** Every general canonical frame \( \Phi_\Lambda \) is descriptive.

From the definitions it is evident that for any general Kripke frame \( (F, W) \), \( ML(F) \subseteq ML(F, W) \) (or \( IL(F) \subseteq IL(F, W) \) in the intuitionistic case). Moreover, in many cases these two logics coincide.

**Definition 1.7.16** A set of \( N \)-modal formulas \( \Gamma \) is called d-persistent (respectively, r-persistent) if for any descriptive (respectively, refined) general frame \( (F, W) \), \( (F, W) \models \Gamma \Rightarrow F \models \Gamma \); similarly, for intuitionistic formulas.

The following observations are trivial consequences of soundness:

**Lemma 1.7.17**

1. A modal or intermediate propositional logic is d-persistent (respectively, r-persistent) iff it is axiomatisable by a d-persistent (respectively, r-persistent) set.

2. A sum of d-persistent (respectively, r-persistent) logics is d-persistent (respectively, r-persistent).

**Proposition 1.7.18** Every d-persistent modal or intermediate logic is canonical, and therefore Kripke complete.

**Proof** (Modal case) \( \Phi_\Lambda \models \Lambda \) by 1.7.5, and \( \Phi_\Lambda = (F_\Lambda, W_\Lambda) \) is descriptive by 1.7.15. So for a d-persistent \( \Lambda \) it follows that \( F_\Lambda \models \Lambda \).

Thus for propositional modal logics d-persistence implies canonicity and canonicity implies Kripke completeness. The converse implications are not true — however, all three properties are equivalent for elementary logics, by the Fine–van Benthem theorem, see 1.8.3 below.

### 1.8 First-order translations and first-order definability

Let \( \mathcal{L}_1^N \) be the classical first-order language with equality and binary predicate letters \( R_1, \ldots, R_N \). So classical \( \mathcal{L}_1^N \)-structures are nothing but \( N \)-modal frames. The classical truth of an \( \mathcal{L}_1^N \)-formula \( \varphi \) in a frame \( F \) is denoted by \( F \models \varphi \) as usual.

**Definition 1.8.1** An \( \mathcal{L}_1^N \)-formula \( \varphi \) corresponds to an \( N \)-modal propositional formula \( A \) if for any \( N \)-modal Kripke frame \( F \),
1.8. FIRST-ORDER TRANSLATIONS AND DEFINABILITY

\[ F \models A \text{ (modally)} \iff F \models \varphi \text{ (classically)}. \]

Similarly, an \( \mathcal{L}_1 \)-formula \( \varphi \) corresponds to an intuitionistic propositional formula \( A \) if for any intuitionistic Kripke frame \( F \), \( F \models A \iff F \models \varphi \).

**Definition 1.8.2** The class \( \text{Mod}(\Sigma) \) of all classical models of a first-order theory (i.e. a set of first-order sentences) \( \Sigma \) is called \( \Delta \)-elementary; if \( \Sigma \) is recursive (respectively, finite), this class is called \( R \)-elementary (respectively, elementary). An \( N \)-modal or intermediate logic \( \Lambda \) is called \( \Delta \)-elementary (respectively, \( R \)-elementary, elementary) if the class \( \mathcal{V}(\Lambda) \) (or \( \mathcal{V}^I(\Lambda) \)) is of the corresponding kind in the language \( \mathcal{L}_N \) (or \( \mathcal{L}_1 \)).

A modal or intermediate logic determined by a \( \Delta \)-elementary (respectively, \( R \)-elementary, elementary) class of frames is called quasi-\( \Delta \)-elementary (respectively, quasi-\( R \)-elementary, quasi-elementary).\(^{12}\)

**Theorem 1.8.3** (Fine-van Benthem) Every quasi-\( \Delta \)-elementary modal or intermediate logic (in particular, every Kripke complete \( \Delta \)-elementary logic) is \( d \)-persistent.

For the proof see [van Benthem, 1983], [Chagrov and Zakharyaschev, 1997]. Let us recall the well-known 'standard' translation from modal formulas into first-order formulas, cf. [van Benthem, 1983].\(^{13}\)

Let \( \mathcal{L}_1^* \) be the language obtained by adding countably many monadic predicate letters \( P_1, P_2, \ldots \) to \( \mathcal{L}_1 \). Every \( N \)-modal formula \( A \) is translated into an \( \mathcal{L}_1^* \)-formula \( A^*(t) \) with at most one parameter \( t \), according to the rules:

\[
P_1^* = P_1(t),
\]

\[
\bot^* = \bot,
\]

\[
(A \supset B)^*(t) = A^*(t) \supset B^*(t),
\]

\[
(\Box_k A)^*(t) = \forall x (R_k(t, x) \supset A^*(x)), \text{ where } A^*(x) \text{ is obtained from } A^*(t) \text{ by substituting } x \text{ for } t \text{ (together with renaming of bound variables, if necessary).}^{14}\]

Every Kripke model \( M = (F, \varphi) \) over a frame \( F = (W, \varrho_1, \ldots, \varrho_n) \) clearly corresponds to a classical \( \mathcal{L}_1^* \)-structure \( M^* = (W, \varrho_1, \ldots, \varrho_n, \varphi(p_1), \varphi(p_2), \ldots) \).

**Lemma 1.8.4**

(1) Let \( M = (F, \varphi) \) be a Kripke model over a frame \( F = (W, \varrho_1, \ldots, \varrho_N) \). Then for any \( a \in W \), for any \( N \)-modal formula \( A \),

\[
M, a \models A \iff M^* \models A^*(a).
\]

(2) For any \( N \)-modal frame \( F \), for any \( N \)-modal formula \( A \)

\[
F \models A \iff \forall \varphi \ (F, \varphi)^* \models \forall t A^*(t).
\]

\(^{12}\)According to another terminology, a quasi-\( \Delta \)-elementary logic is called \( \Delta \)-elementarily determined, etc.

\(^{13}\)A similar translation for intuitionistic formulas was first introduced in [Mints, 1967].

\(^{14}\)Variable substitutions for first-order formulas are considered in detail in Chapter 2.
**Proof**

(1) Easy by induction. E.g. in the case \( A = \Box_k B \) we have:

\[
M, a \vDash A \iff \forall b \in g_k(a) \quad M, u_1 \vDash B \\
\iff \forall b \in g_k(a) \quad M^* \vDash B^*(b) \quad \text{(by the induction hypothesis)} \\
\iff M^* \vDash \forall x(R_k(a, x) \supset B^*(x)) \iff M^* \vDash (\Box_k B)^*(a).
\]

(2) \( F \vDash A \iff \forall \varphi \forall a \quad (F, \varphi), a \vDash A \iff \forall \varphi \forall a \quad (F, \varphi)^* \vDash A^*(a) \quad \text{(by (1))} \]

In the next proposition we use the following notation. For an \( \mathcal{L}_{1N} \)-theory \( \Sigma \), \( [\Sigma] \) denotes the set of its \( \mathcal{L}_{1N}^* \)-theorems (in classical first-order logic).

**Proposition 1.8.5**

(1) For any \( \mathcal{L}_{1N} \)-theory \( \Sigma \), \( \text{ML}(\text{Mod}(\Sigma)) \subseteq_1 [\Sigma] \). In particular, every \( \text{quasi-} R \)-elementary modal or intermediate logic is recursively axiomatisable.

(2) Every \( \text{quasi-} \Delta \)-elementary modal or intermediate logic has the c.f.p.

(3) Every finitely axiomatisable \( \text{(quasi-)} \Delta \)-elementary modal or intermediate logic is \( \text{(quasi-)} \)-elementary.

**Proof**

(1) Since all \( \mathcal{L}_{1N}^* \)-models of \( \Sigma \) are exactly the structures of the form \((F, \varphi)^*\), where \( F \vDash \Sigma \), by Lemma 1.8.4 and Gödel's completeness theorem, we obtain

\[\forall \varphi \forall a \quad (F, \varphi), a \vDash A \iff \forall \varphi \forall a \quad (F, \varphi)^* \vDash A^*(a) \quad \text{(by (1))}.\]

Since \( A \) can be uniquely restored from \( A^* \), this proves the 1-reducibility.

Next, if \( \Sigma \) is recursive, then \( \text{ML}(\text{Mod}(\Sigma)) \) is RE as it is 1-reducible to the set of theorems of a recursive classical theory. Now (1) follows by the well-known Craig's lemma, cf. [Boolos, Burgess, and Jeffrey, 2002].

To prove (1) in the intuitionistic case, note that by 1.5.8, \( \text{IL}(C) = T\text{ML}(C) \), so \( \text{IL}(C) \) is 1-reducible to \( \text{ML}(C) \).

(2) Let \( C_0 \) be the class of all countable \( \mathcal{L}_{1N}^* \)-models of \( \Sigma \). Similarly to the above, we obtain that \( A \in \text{ML}(C_0) \) iff for any countable \( \mathcal{L}_{1N}^* \)-structure \( \mathcal{M} \), \( (\mathcal{M} \vDash \Sigma \Rightarrow \mathcal{M} \vDash \forall tA^*(t)) \). By the Löwenheim–Skolem theorem, the latter is equivalent to \( \Sigma \vDash \forall tA^*(t) \), and thus (by (\#)) to \( A \in \text{ML}(\text{Mod}(\Sigma)) \). Therefore \( \text{ML}(\text{Mod}(\Sigma)) = \text{ML}(C_0) \), which proves (2). The intuitionistic version follows readily.

(3) (Modal case) If \( \text{ML}(\text{Mod}(\Sigma)) = K_N + A \) for a formula \( A \), then by (\#), \( \Sigma \vDash \forall tA^*(t) \) in classical logic, which implies \( \Sigma_1 \vDash \forall tA^*(t) \) for some finite \( \Sigma_1 \subseteq \Sigma \). Hence \( \varphi \vDash \forall tA^*(t) \), where \( \varphi = \bigwedge \Sigma_1 \), so \( A \in \text{ML}(\text{Mod}(\varphi)) \) (by (\#)), and thus

\[\text{ML}(\text{Mod}(\Sigma)) = K_N + A \subseteq \text{ML}(\text{Mod}(\varphi)).\]

On the other hand, \( \text{Mod}(\Sigma) \subseteq \text{Mod}(\varphi) \), hence \( \text{ML}(\text{Mod}(\varphi)) \subseteq \text{ML}(\text{Mod}(\Sigma)) \). Therefore \( \text{ML}(\text{Mod}(\Sigma)) = \text{ML}(\text{Mod}(\varphi)) \) is quasi-elementary.
The case when \( K_N + A \) is \( \Delta \)-elementary is left to the reader.

(Intuitionistic case.) Assume that \( L \) is finitely axiomatisable and \( \Delta \)-elementary. Then by definition, \( \tau(L) \) is also finitely axiomatisable. Since by 1.5.9, \( V^I(L) = V(\tau(L)) \), it follows that \( \tau(L) \) is \( \Delta \)-elementary. Then (as we have just proved) \( \tau(L) \) is elementary. Now the equality \( V^I(L) = V(\tau(L)) \) implies the elementarity of \( L \).

It remains to consider the case when an intermediate \( L \) is finitely axiomatisable and quasi-\( \Delta \)-elementary. Then we can argue as in the modal case. In fact, if \( \mathcal{I}L(C) = H + A \) for \( C = \text{Mod}(\Sigma) \), then \( A^T \in \mathcal{M}L(C) \), so by (\$), \( \Sigma \vdash \forall t(A^T)^*(t) \). Then we can replace \( \Sigma \) with a single \( \phi \) and obtain that \( \mathcal{I}L(C) = \mathcal{I}L(\text{Mod}(\phi)) \).

\[ \square \]

1.9 Some general completeness theorems

In this section we recall two general results on Kripke-completeness — the Sahlqvist theorem and the Fine theorem.

**Definition 1.9.1** A Sahlqvist formula is a modal formula of the form \( \square^\alpha(A \supset B) \), where \( \alpha \in I_N^\infty \), \( B \) is a positive modal formula (i.e. built from proposition letters using \( \bot, T, \lor, \land, \Diamond_i, \Box_i \)), and \( A \) is built from proposition letters and their negations using the same connectives, so that subformulas of \( A \) of the form \( C \lor D \) or \( \Diamond_i C \) containing negated proposition letters are not within the scope of any \( \Box_j \).

**Theorem 1.9.2 (Sahlqvist theorem)** Let \( A \) be a modal logic axiomatised by Sahlqvist formulas. Then \( A \) is \( d \)-persistent and \( \Delta \)-elementary.

For the proof see [Chagrov and Zakharyaschev, 1997], [Blackburn, de Rijke and Venema, 2001].

**Definition 1.9.3** A subset \( V \) in a transitive frame \( F = (W, R) \) is called an antichain if its different worlds are incomparable:

\[ \forall x, y \in V(xRy \Rightarrow x = y). \]

The width of \( F \) is the maximal cardinality of finite antichains in cones of \( F \) if it exists and \( \infty \) otherwise.

So for a finite \( n \), \( F \) is of width \( \leq n \) iff

\[ \forall y, x_0, \ldots, x_n \in W (\forall i \ yRx_i \Rightarrow \exists i \ \exists j \neq i \ (x_iR x_j \lor x_jR x_i \lor x_i = x_j)). \]

This property corresponds to the modal formula \( AW_n \) (cf. 1.3.11) and to \( AIW_n \) in the intuitionistic case (cf. 1.4.17).

**Definition 1.9.4** For a finite \( n \), a modal logic of width \( \leq n \) is an extension of \( K4 + AW_n \); an intermediate logic of width \( \leq n \) is an extension of \( H + AIW_n \). All these are called the logics of finite width.
The width of a transitive modal logic $\Lambda$ can be defined explicitly as the least $n$ such that $AW_n \in \Lambda$ (and $\infty$ if $AW_n \notin \Lambda$ for any $n$); similarly for the intuitionistic case.

**Theorem 1.9.5 (Fine theorem)** Every logic of finite width (modal or intermediate) is Kripke-complete and moreover, has the c.f.p.

This result (for the modal case) was first proved in [Fine, 1974]; for a shorter proof see [Chagrov and Zakharyaschev, 1997]. The intuitionistic case follows from the observation that $L \vdash AIW_n$ implies $\tau(L) \vdash AW_n$ together with 1.5.8 and 1.5.5.

### 1.10 Trees and unravelling

Recall that a path in a frame is a sequence of related worlds together with indices of the accessibility relations (Definition 1.3.20).

**Definition 1.10.1** The depth of a world $u$ in $F$ (notation: $d_F(u)$ or $d(u)$ if there is no confusion) is the maximal length of a path from $u$ in $F$ if it exists and $\infty$ otherwise.

Let us introduce two other kinds of ‘paths’ and ‘depths’ in Kripke frames.

**Definition 1.10.2** A distinct path in a frame $F$ is a path, in which all the worlds are different. The distinct depth of $u$ in $F$ (notation: $dd_F(u)$ or $dd(u)$) is the maximal length of a distinct path from $u$ in $F$ if it exists and $\infty$ otherwise.

**Definition 1.10.3** A strict path in a transitive frame $F = (W, R)$ is a path $(u_0, u_1, \ldots, u_m)$, in which $u_{i+1} R u_i$ for any $i < m$ (or equivalently, all the worlds are in different clusters). Respectively, the strict depth of $u$ in $F$ (notation: $sd_F(u)$ or $sd(u)$) is the maximal length of a strict path from $u$ in $F$ if it exists and $\infty$ otherwise.

Obviously, $sd(u) = sd(u) = sd(v)$ if $u \approx_R v$, and $sd_F(u) = sd_F^- (u^-)$.

**Exercise 1.10.4** Show that $d(x) \leq n$ iff $x \Vdash \Box^n \bot$, where $\Box A := \Box_1 A \land \ldots \land \Box_N A$.

**Definition 1.10.5** A world $u$ in a frame is called a dead end (respectively, maximal; quasi-maximal) if $d(u) = 0$ (respectively, $dd(u) = 0$; $sd(u) = 0$). Dead ends in a tree are also called leaves (as well as maximal worlds in reflexive or transitive trees).

A maximal cluster in a K$\S_4$-frame $F$ is a maximal point in $F^\sim$.

So quasi-maximal points in a K$\S_4$-frame are exactly the points of the maximal clusters.
1.10. TREES AND UNRAVELLING

Definition 1.10.6 A frame \( F = (W, R_1, \ldots, R_N) \) with a root \( u_0 \) is called a tree (or an \( N \)-tree) if

\[
\forall i \forall y \quad \neg y R_i u_0,
\]

\[
\text{for any } x \neq u_0 \text{ there exists a unique pair } (i, y) \text{ such that } y R_i x.
\]

Hence we readily have

Lemma 1.10.7 A frame \( F \) is a tree with a root \( u \) iff for any \( x \in F \) there exists a unique path from \( u \) to \( x \) in \( F \).

Definition 1.10.8 The height of a world \( x \) in a tree \( F \) (notation: \( \text{ht}_F(x) \), or \( \text{ht}(x) \)) is the length of the (unique) path from the root to \( x \).

Definition 1.10.9 A successor of a world \( u \) in a poset\(^{15} \) \( F = (W, R) \) is a minimal element in the set \( R_-(u) \) of all strictly accessible worlds. \( \beta_F(u) \) (or \( \beta(u) \)) denotes the set of all successors of \( u \) in \( F \). \( F \) is called successive if for any \( u \), \( R_-(u) = R(\beta_F(u)) \). The branching at \( u \) in \( F \) is \( |\beta_F(u)| \).

Every finite p.o. set is clearly successive. For rooted successive p.o. sets we can also define the height of \( x \) as the minimal length of a 'successor-path' from the root to \( x \).

Definition 1.10.10 Let \( T_\omega = \omega^\omega \) be the set of all finite sequences of natural numbers. The 1-modal universal tree is the frame

\[
F_1 T_\omega := (T_\omega, \sqsubseteq),
\]

where

\[
u \sqsubseteq v := \exists k \in \omega \ v = uk.
\]

The \( N \)-modal universal tree is

\[
F_N T_\omega := (T_\omega, \sqsubseteq_{1N}, \ldots, \sqsubseteq_{NN}),
\]

where

\[
u \sqsubseteq_{iN} v := \exists k \in \omega \ (v = uk \& k \equiv i - 1 \ (\text{mod } N)).
\]

Sometimes we drop \( N \) in \( F_N T_\omega \) and \( \sqsubseteq_{iN} \) if it is clear from the context.

The intuitionistic universal tree is the p.o. set\(^{16} \) \( I T_\omega := (T_\omega, \leq) \), where

\[
u \preceq v := \exists w \ (v = uw).
\]

So \( \preceq \) is the reflexive and transitive closure of \( \sqsubseteq \), and the set of immediate successors of \( u \) in \( IT_\omega \) is just \( \beta(u) = \sqsubseteq(u) \). From the definitions it follows easily that \( F_N T_\omega \) is really a tree with the root \( \lambda \).

\(^{15}\)More generally, we may assume that \( F \) is transitive antisymmetric.

\(^{16}\)This is a p.o. tree according to Definition 1.11.10.
Definition 1.10.11 A subframe $F \subseteq F_N T_\omega$ or $F \subseteq IT_\omega$, that is stable under $\geq$ (or equivalently, under all $\supseteq_i$, $1 \leq i \leq N$), is called a standard tree (or a standard subtree of the corresponding frame).

Let us fix the following standard trees:

- the universal $n$-ary trees, or the universal trees of branching $n$, for finite $n > 0$:
  
  \[ F_{NT_n} := F_N T_\omega \upharpoonright T_{Nn}, \quad IT_n := IT_\omega \upharpoonright T_n, \]

  where $T_n$ is the set of all $\{0, 1, \ldots, n-1\}$-sequences;

- universal trees of depth (or height) $k$:
  
  \[ F_{NT_n^k} := F_N T_n \upharpoonright T_\omega^k, \quad IT_n^k := IT_n \upharpoonright T_\omega^k, \]

  where

  \[ T_\omega^k := \{ u \in T_\omega \mid l(u) \leq k \}, \]

  $l(u)$ is the length of sequence $u$, $n \leq \omega$. In particular, $T_\omega^0$ contains only the root $\top$. Obviously, $l(u) = ht(u)$ in $F_N T_\omega$.

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (root) at (0,0) {$\top$};
    \node (00) at (-1,1) {00};
    \node (01) at (0,1) {01};
    \node (10) at (1,1) {10};
    \node (11) at (2,1) {11};
    \node (1) at (-1,2) {1};
    \node (2) at (0,2) {2};
    \node (3) at (1,2) {1};
    \node (4) at (2,2) {2};
    \node (5) at (-1,3) {0};
    \node (6) at (1,3) {1};
    \node (7) at (2,3) {2};
    \draw (root) -- (00);
    \draw (root) -- (01);
    \draw (root) -- (10);
    \draw (root) -- (11);
    \draw (00) -- (1);
    \draw (01) -- (3);
    \draw (10) -- (4);
    \draw (11) -- (7);
    \draw (1) -- (5);
    \draw (3) -- (5);
    \draw (4) -- (6);
\end{tikzpicture}
\caption{$F_2 T_1^1$}
\end{figure}

Sometimes we also use the notation $T_n^\omega$ rather than $T_n$, for $0 < n \leq \omega$.

Definition 1.10.12 Let $F$ be a successive poset. A subframe $G \subseteq F$ is called greedy if for any $u \in G$ either $\beta_F(u) \subseteq G$ or $\beta_F(u) \cap G = \emptyset$. A greedy standard tree is a greedy standard subtree of $F_N T_\omega$ or $IT_\omega$.

Definition 1.10.13 Let $F = (W, R_1, \ldots, R_N)$ be a frame with a root $u_0$. The unravelling of $F$ is the frame $F^\# = (W^\#, R_1^\#, \ldots, R_N^\#)$, where $W^\#$ is the set of all paths in $F$ starting at $u_0$ and

\[ \alpha R_j^\# \beta \iff \exists v \beta = (\alpha, j, v). \]
1.10. TREES AND UNRAVELLING

Unravelling was first introduced probably in [Sahlqvist, 1975]; later it was used by many authors, cf. [Gabbay, 1976], [van Benthem, 1983], [Gabbay and Shehtman, 1998].

The proof of the following lemma is straightforward.

**Lemma 1.10.14**

1. $F^\#$ is a tree.
2. The map $\pi : W^\# \to W$ such that $\pi(u_0, \ldots, u_m) = u_m$, is a p-morphism from $F^\#$ onto $F$.

Now let us show how to increase branching in a standard way.

**Definition 1.10.15** Let $F = (W, R_1, \ldots, R_N)$ be a frame with root $u_0$. Put

\[
W_1 := \begin{cases} 
W & \text{if there exists a path of positive length} \\
W - \{u_0\} & \text{otherwise}
\end{cases}
\]

\[
W^\upsilon := (W_1 \times \omega) \cup \{(u_0, -1)\}.
\]

Let $R_1^\upsilon, \ldots, R_N^\upsilon$ be the relations on $W^\upsilon$ such that

\[(a, n)R_i^\upsilon(b, m) \iff aR_ib \& m \neq -1.\]

The frame $F^\upsilon := (W^\upsilon, R_1^\upsilon, \ldots, R_N^\upsilon)$ is called the $(\omega-)$ thickening of $F$.

Informally speaking, we make $\omega$ copies of every world of $W_1$ and connect all the copies of two worlds if the original worlds are connected in $F$; we also add the root $(u_0, -1)$.

From the definition we readily obtain

**Lemma 1.10.16**

1. For any $x \in W^\upsilon, R_i^\upsilon(x)$ is either empty or denumerable.
2. The map $\gamma : (a, n) \mapsto a$ is a p-morphism from $F^\upsilon$ onto $F$.

**Lemma 1.10.17** Let $F = (W, R_1, \ldots, R_N)$ be a tree, in which every set $R_i(x)$ is either empty or denumerable. Then $F$ is isomorphic to a greedy standard tree.

**Proof** Let $F^k$ be the restriction of $F$ to the worlds of height $\leq k$. We define the embeddings $f^k : F^k \to F_NT^k_u$ by induction.

If $u_0$ is the root of $F$, we put

\[f^0(u_0) := \alpha.\]

\[17\{Gabbay and Shehtman, 1998\].\]
If $f^k$ is already defined, we extend it to $f^{k+1}$ as follows. If $ht_F(y) = k$ and $R_i(y) \neq \emptyset$, then $R_i(y)$ is denumerable, by our assumption. So let $e_{y,i}: \omega \to R_i(y)$ be the corresponding bijective enumeration. Then put

$$f^{k+1}(e_{y,i}(j)) := (f^k(y), jN + i - 1).$$

Since $F$ is a tree, every world of height $(k + 1)$ is $e_{y,i}(j)$ for a unique triple $(y,i,j)$, so $f^{k+1}$ is well-defined. Since $e_{y,i}$ is a bijection, it is clear that $f^{k+1}$ is an embedding.

Its image $\text{rng}(f^{k+1})$ is greedy. In fact, if $u = f^k(y)$ and $R_i(y) \neq \emptyset$, then by our construction, $f^{k+1}$ gives rise to a bijection between $R_i(y)$ and

$$\Box_i (u) = \{(u, jN + i - 1) \mid j \geq 0\}.$$

Eventually, the required isomorphism is $f := \bigcup_{k \geq 0} f^k$. 

**Proposition 1.10.18**

1. Every countable rooted frame is a $p$-morphic image of a greedy standard tree.

2. Every serial countable rooted $N$-frame is a $p$-morphic image of $F_{NT_\omega}$.

**Proof**

1. Let $F$ be the original frame. By 1.10.14 and 1.10.16 we have $p$-morphisms

$$(F^\bullet)^d \to F^\bullet \to F.$$

By 1.10.17, $(F^\bullet)^d$ is a tree.

Since by 1.10.16, every $R_i^\bullet(x)$ is either empty or denumerable, the same property holds for $(F^\bullet)^d$, and thus it is isomorphic to a greedy standard tree, by 1.10.17.

2. If $F$ is serial and $f : G \to F$, then $G$ is also serial, due to the lift property. By definition, every serial greedy standard subtree of $F_{NT_\omega}$ is $F_{NT_\omega}$ itself.

**1.11 PTC-logics and Horn closures**

**Definition 1.11.1** A modal propositional formula is called closed (or constant) if it is a 0-formula, i.e. if it does not contain proposition letters.

**Lemma 1.11.2** If $F, F'$ are $N$-modal frames and $f : F \to F'$, then $F \Vdash A \iff F' \Vdash A$ for any closed $N$-modal formula $A$. 

Proof Let $M, M'$ be arbitrary Kripke models respectively over $F, F'$. By induction it easily follows that $M, x \models A$ iff $M', f(x) \models A$ for any $x \in M$ and closed $A$. Hence $F, x \models A$ iff $F', f(x) \models A$. This implies $F \models A \iff F' \models A$ since $f$ is surjective.

Proposition 1.11.3 Every logic axiomatisable by closed formulas is canonical.

Proof By Theorem 1.7.3, every axiom of the logic is true in the canonical model. For a closed formula, this is equivalent to validity in the canonical frame.

Definition 1.11.4 A pseudotransitive $N$-modal formula has the form $\Diamond_\alpha \Box_k p \supset \Box_\beta p$, where $p \in PL$, $\alpha, \beta \in I_N^\infty$. This formula is called one-way if $\alpha = \lambda$. A PTC-formula is a formula which is either pseudotransitive or closed. A PTC-logic is a modal logic axiomatised by a set of PTC-formulas. One-way PTC-formulas and logics are defined similarly.

For example, the following formulas are pseudotransitive:

\[
\Box p \supset p, \quad \Box p \supset \Box \Box p, \quad \Diamond \Box p \supset p, \quad \Diamond_1 \Box_2 p \supset p.
\]

Thus many well-known logics are PTC, e.g. D, K4, D4, S4, T, B, K.t, K4.t, S5.

Since every pseudotransitive formula is obviously Sahlqvist, from the Sahlqvist theorem and Proposition 1.11.3 we obtain

Proposition 1.11.5

1. A pseudotransitive $N$-modal formula $A = \Diamond_\alpha \Box_k p \supset \Box_\beta p$ corresponds to the $\mathcal{L}_{1N}$-formula\(^\text{18}\)

\[
A^k := \forall x, v (\exists u (uR_\alpha x \land uR_\beta v) \supset xR_kv),
\]

2. Every PTC-logic is canonical and $\Delta$-elementary.

Later to the well-known transitive closure, there exists a 'closure' under pseudotransitivity. This is a rather simple fact from classical logic, but still let us recall its proof briefly.

Further on arbitrary lists of individual variables are denoted by $x, y, \text{ etc.}$ and arbitrary lists of individuals by $a, b, \text{ etc.}$ As usual, we write $\varphi(x, y, z)$ to indicate that all parameters of the formula $\varphi$ are among $x, y, z$.

Definition 1.11.6 An $\mathcal{L}_{1N}$-sentence of the form

\[
\psi = \forall x \forall y \forall z (\varphi(x, y, z) \rightarrow R_k(x, y))
\]

is a called a universal strict Horn clause, if $\varphi(x, y, z)$ is a (non-empty) conjunction of atomic formulas.

\(^{18}\)Strictly speaking, this becomes an $\mathcal{L}_{1N}$-formula after writing $uR_\alpha x$ and $uR_\beta v$ in terms of the basic predicates $R_1, \ldots, R_N$.\]
Proposition 1.11.7 Let $F = (W, \varrho_1, \ldots, \varrho_N)$ be an $N$-modal frame, $\Gamma = \{\psi_k \mid k \in I\}$ a set of universal (strict) Horn clauses

$$\psi_k = \forall x \forall y \forall z_k (\varphi_k(x, y, z_k) \rightarrow R_k(x, y)).$$

Then there exists a frame $F^+_\Gamma$ such that\(^{19}\)

1. $F \subseteq F^+_\Gamma$;
2. $F^+_\Gamma \vDash \Gamma$;
3. If $G \vDash \Gamma$ and $f : F \rightarrow G$ is monotonic, then $f : F^+_\Gamma \rightarrow G$ is also monotonic.

We can say that $F^+_\Gamma$ is the 'smallest' full weak extension of $F$ satisfying $\Gamma$.

**Proof** We construct $F^+_\Gamma$ as the union of a sequence of frames $F_m = (W, \varrho_{m1}, \ldots, \varrho_{mN})$, beginning with $F_0 = F$. Namely, let:

$$\varrho_{(m+1)j} = \varrho_{mj} \cup \bigcup_{i_k = j} \{(a, b) \mid F_m \vDash \exists z_k \varphi_k(a, b, z_k)\},$$

$$\varrho^+_j = \bigcup_m \varrho_{mj}, \quad F^+_\Gamma = (W, \varrho^+_1, \ldots, \varrho^+_N).$$

Then $F_0 \subseteq \ldots F_m \subseteq F_{m+1} \ldots \subseteq F^+_\Gamma$, and thus (1) holds. Furthermore, we have:

4. if $\varphi(z)$ is a conjunction of atomic formulas, then the following conditions are equivalent:
   (i) $F^+_\Gamma \vDash \varphi(a)$;
   (ii) $\exists m \forall k \geq m F_k \vDash \varphi(a)$;
   (iii) $\exists m F_m \vDash \varphi(a)$.

In fact, (ii)$\Rightarrow$(iii) is trivial. (iii)$\Rightarrow$(i) holds, since $F_m \subseteq F^+_\Gamma$, and the truth of positive formulas is preserved by monotonic maps. (i)$\Rightarrow$(ii) easily follows from $\rho_{mj} \subseteq \rho^+_{(m+1)j}$ and $\rho^+_j = \bigcup_m \rho_{mj}$.

Now let us prove (2). Assume $F^+_\Gamma \vDash \varphi_k(a, b, c_k), i_k = j$. Then by (4), $F_m \vDash \varphi_k(a, b, c_k)$ for some $m$, and thus $(a, b) \in \varrho_{(m+1)j}$ by definition. Hence $F^+_\Gamma \vDash \varphi_k$. Therefore $F^+_\Gamma \vDash \psi_k$.

To check (3), we assume that $G \vDash \Gamma$, $f : F \rightarrow G$ is monotonic and show by the induction that $f : F_m \rightarrow G$ is monotonic for any $m$.

To make the step, assume $a \varrho_{(m+1)j} b$. Then there are two cases.

If $a \varrho_{mj} b$, then $G \vDash R_j(f(a), f(b))$ by induction hypothesis.

If $F_m \vDash \exists z_k \varphi_k(a, b, z_k), i_k = j$, then $G \vDash \exists z_k \varphi_k(f(a), f(b), z_k)$, since monotonic $f : F_m \rightarrow G$ preserves the truth of positive formulas. Now $G \vDash \psi_k$ implies $G \vDash R_j(f(a), f(b))$. \(\blacksquare\)

\(^{19}\)Recall that $\subseteq$ denotes a full weak subframe, cf. 1.3.17.
By Proposition 1.11.5 every pseudotransitive modal formula corresponds to a universal strict Horn formula. So Proposition 1.11.7 yields

Corollary 1.11.8 Let $\Gamma$ be a set of pseudotransitive $N$-modal formulas, $F$ an $N$-modal frame. Then there exists an $N$-modal frame $F^+_\Gamma$ such that

1. $F \subseteq F^+_\Gamma$;
2. $F^+_\Gamma \models \Gamma$;
3. for any $N$-modal frame $G$, if $G \models \Gamma$ and $f : F \rightarrow G$ is monotonic, then $f : F^+_\Gamma \rightarrow G$ is monotonic.

Definition 1.11.9 The frame $F^+_\Gamma$ described in Corollary 1.11.8 is called the pseudotransitive closure of $F$ (under $\Gamma$), or the $\Gamma$-closure of $F$. If $\Lambda = K_N + \Gamma + \Delta$ is a PTC-logic with the set of pseudotransitive axioms $\Gamma$ and a set of closed axioms $\Delta$, then $F^+_\Gamma$ is also called the $\Lambda$-closure if it validates $\Delta$.

Definition 1.11.10 If $\Lambda$ is a PTC-logic, then the $\Lambda$-closure of a tree is called a $\Lambda$-tree. $K_4$-trees (respectively, $S_4$-trees) are also called transitive trees (respectively, p.o. trees). In the particular case when $\Lambda = D_N + \Gamma$, with a set of pseudotransitive axioms $\Gamma$, the $\Gamma$-closure of $F_NT_\omega$ is called the full standard $\Lambda$-tree.

Proposition 1.11.11

1. Let $\Lambda$ be a PTC-logic. Then every countable rooted $\Lambda$-frame is a $p$-morphic image of a greedy standard $\Lambda$-tree.
2. If $\Lambda = D_N + \Gamma$, with a set of pseudotransitive axioms $\Gamma$, then every serial countable rooted $\Lambda$-frame is a $p$-morphic image of the full standard $\Lambda$-tree.
3. Every countable rooted $S_4$-frame is a $p$-morphic image of $IT_\omega$.

Proof

1. Let $F$ be such a frame. By Proposition 1.10.18, there exists $f : \Phi \rightarrow F$ for some greedy standard tree $\Phi$. By Proposition 1.11.7, the map $f : \Phi^+_\Gamma \rightarrow F$ is monotonic, where $\Gamma$ is the set of pseudotransitive axioms of $\Lambda$. The lift property obviously holds for this map, since it holds for $f : \Phi \rightarrow F$. So $F$ is a $p$-morphic image of $\Phi^+_\Gamma$. Finally note that all closed axioms of $\Lambda$ are valid in $\Phi^+_\Gamma$, by Lemma 1.11.2.
2. By Proposition 1.10.18(2), in this case we can take $\Phi = F_NT_\omega$.
3. In fact, $IT_\omega$ is the full standard $S_4$-tree.

Corollary 1.11.12

1. Every PTC-logic $\Lambda$ is determined by the class of greedy standard $\Lambda$-trees.
2. Every serial PTC-logic $\Lambda$ is determined by the full standard $\Lambda$-tree.
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(3) $S_4 = ML(IT_\omega)$, $H = IL(IT_\omega)$.

**Proof**

(1) $\Lambda$ is complete and $\Delta$-elementary, so it has the c.f.p by Proposition 1.8.5. Then by Proposition 1.11.11(1) and the morphism lemma, every $A \not\in \Lambda$ is refuted in a greedy standard $\Lambda$-tree.

(2) By the same argument using 1.11.11(2).

(3) By (2), since $IT_\omega$ is the full standard $S_4$-tree. The claim about $H$ follows by 1.5.7, since $H = T_{S_4}$.

**Proposition 1.11.13** $IT_\omega$ is a p-morphic image of $IT_2$.

**Proof** Let $\tau : \omega \to \omega$ be a map such that every set $\tau^{-1}(n)$ is infinite (such a map obviously exists, since there is a bijection between $\omega \times \omega$ and $\omega$). Next, every $\alpha \in T_2$ can be uniquely presented in the form $0^{n_1}1 \ldots 0^{n_k}10^{n_{k+1}}$, where $n_1, \ldots, n_{k+1} \geq 0$, and $0^n$ means $\underbrace{0 \ldots 0}_n$. Now put

$$f(\alpha) := \tau(n_1)\tau(n_2) \ldots \tau(n_k).$$

We claim that $f$ is a required p-morphism from $IT_\omega$ onto $IT_2$.

In fact $IT_\omega$ is a Horn closure of $F, T_\omega$, so by 1.11.7, it suffices to check monotonicity of $f$ w.r.t. $\sqsubseteq$. But this is the case, since $f(\alpha 0) = f(\alpha)$, and $f(\alpha 1) = f(\alpha)\tau(n_{k+1})$.

For the lift property, suppose $f(\alpha) \sqsubseteq u$; then $u = f(\alpha) m$ for some $m$. By the construction of $\tau$, there exists $n \geq n_{k+1}$ such that $\tau(n) = m$. So we can take

$$\beta = 0^{n_1}1 \ldots 0^{n_k}10^n,$$

then $f(\beta) = \tau(n_1) \ldots \tau(n_k)\tau(n) = f(\alpha) m = u$, and obviously, $\alpha \leq \beta$.

Since $\leq$ is the transitive closure of $\sqsubseteq$, the lift property now follows easily.

It remains to note that $\tau$ sends the root to the root, which implies the surjectivity of $f$.

**Corollary 1.11.14** Every countable rooted $S_4$-frame is a p-morphic image of $IT_2$.

**Proof** By Propositions 1.11.13 and 1.11.11.

**Definition 1.11.15** A standard 1-tree is called strongly standard if together with any sequence $\alpha n$ it contains all the sequences $\alpha m$ for $m < n$.

**Lemma 1.11.16** Every countable tree is isomorphic to a strongly standard tree.

**Proof** Given a tree $F$, we construct a required isomorphism $f$. $f(x)$ is defined by induction on $ht(x)$. If $x$ is the root, put $f(x) := \lambda$. If $f(x)$ is defined and $\beta(x) = \{y_0, \ldots, y_n\}$, put $f(y_i) := f(x)i$; similarly if $\beta(x)$ is countable. (Of course this definition depends on the chosen ordering of $\beta(x)$.)
Definition 1.11.17 An $S_4$-tree is called branchy if for any $u$ $|\beta(u)| \leq 1$.

Every branchy tree is clearly infinite.

Lemma 1.11.18 If $F$ is a strongly standard branchy tree, then $F \rightarrow IT_2$.

Proof We define the $p$-morphism $g$ as follows:

$$g(n_1 \ldots n_k) := \overline{n_1} \ldots \overline{n_k},$$

where $\overline{n}$ is the remainder of $n$ modulo 2. $g$ is obviously monotonic. The lift property follows, since $g(x)0 = g(x0)$, $g(x)1 = g(x1)$ and a branchy standard tree always contains $x0$, $x1$ together with $x$.

Lemma 1.11.19 If every cone in a countable $S_4$-tree $F$ is nonlinear, then $F \rightarrow IT_2$.

Proof By 1.11.18 and 1.11.16, it is sufficient to $p$-morphically map $F$ onto a branchy tree.

Let $F = (W, R)$. For $x \in W$ let $h(x)$ be the least element in $\{y \in R(x) \mid |\beta(y)| > 1\}$. $h(x)$ clearly exists, since this set has a minimal element, which must be unique.

Now $h$ is a morphism, i.e. a $p$-morphism onto its image. In fact, the monotonicity is obvious. The lift property follows since

$$h(x)R_\rightarrow h(y) \Rightarrow xR_\rightarrow y.$$ 

To show the latter, suppose $h(x)R_\rightarrow h(y)$. Since $yRh(y)$ and $xRh(x)Rh(y)$, it follows that $x$, $y$ are $R$-comparable (remember that $F$ is an $S_4$ tree). If $yRx$, we obtain $yRh(x)R_\rightarrow h(y)$, and $h(x)$ has at least two successors, which contradicts the choice of $h(y)$. Thus $xR_\rightarrow y$.

The image $G$ of $h$ is a branchy tree, since $h(x) = h(h(x))$, and thus $\beta_G(h(x)) = h[\beta_F(h(x))]$.

Definition 1.11.20 A p.o. tree is called effuse if it has a cone without linear subcones.

Proposition 1.11.21 If $F$ is an effuse countable tree, then $ML(F) = S_4$ and thus $IL(F) = H$.

Proof If a cone $F \uparrow u$ does not contain linear subcones, then by Lemma 1.11.19 and Proposition 1.11.13, $F \uparrow u \rightarrow IT_2 \rightarrow IT_\omega$, and thus $ML(F) \subseteq ML(IT_\omega) = S_4$ by the generation lemma, the morphism lemma and Corollary 1.11.12. The converse inclusion is trivial by soundness.
1.12 Subframe and cofinal subframe logics

Definition 1.12.1 A modal or intermediate propositional logic is called subframe if its Kripke frame variety is closed under taking subframes.

Definition 1.12.2 For a transitive Kripke frame $F = (W, R)$ a subset $V \subseteq W$ and the subframe $F' = F \upharpoonright V$ are called cofinal if $R(V) \subseteq R^{-1}(V)$.

Definition 1.12.3 A transitive modal or an intermediate propositional logic $\Delta$ is called cofinal subframe if its Kripke frame variety $V(\Delta)$ or $V^\sim(\Delta)$ is closed under taking cofinal subframes.

Definition 1.12.4 Let $F, G$ be Kripke frames of the same kind. A subreduction from $F$ to $G$ is a $p$-morphism from a subframe of $F$ onto $G$. A reduction is a subreduction defined on a cone. A subreduction of transitive frames is called cofinal if its domain is cofinal.

If $G$ is rooted and there exists a reduction (respectively, a subreduction, a cofinal subreduction) from $F$ to $G$, we say that $F$ is reducible (respectively, subreducible, cofinally subreducible) to $G$.

If $G$ is arbitrary we also say that $F$ is reducible to $G$ if it is reducible to every cone in $G$.

By the morphism and generation lemmas every variety $V(\Delta)$ or $V^\sim(\Delta)$ is closed under reductions. Thus for a subframe (respectively, cofinal subframe) $\Delta$ it is closed under subreductions (respectively, cofinal subreductions).

Definition 1.12.5 A class of first-order structures is called universal, if it is a class of models of a set of universal first-order sentences.

Definition 1.12.6 A modal or intermediate propositional logic $\Delta$ is universal, if $V(\Delta)$ (or $V^\sim(\Delta)$) is universal.

The well known Tarski–Los theorem states that a class is universal iff it is closed under substructures. So a $\Delta$-elementary propositional logic is universal iff it is subframe.

The following result yields a criterion of universality for subframe logics; for the proof see [Chagrov and Zakharyaschev, 1997], Theorem 11.31, or [Wolter, 1997].

Definition 1.12.7 A modal or intermediate propositional logic $\Delta$ has the finite embedding property if for any Kripke frame $F$, $F$ validates $\Delta$ whenever every finite subframe of $F$ validates $\Delta$.

Theorem 1.12.8 (Wolter) For any subframe modal logic $\Delta$ the following properties are equivalent

(1) $\Delta$ is universal and Kripke-complete;

(2) $\Delta$ is quasi-$\Delta$-elementary;
(3) \( A \) is d-persistent;

(4) \( A \) is r-persistent;

(5) \( A \) has the finite embedding property and is Kripke-complete.

Later on we will need the implication (1) \( \Rightarrow \) (4) from this theorem, so let us give some comments on its proof. (1) \( \Rightarrow \) (2) is obvious, (2) \( \Rightarrow \) (3) is the Fine–van Benthem theorem. For (3) \( \Rightarrow \) (4), note that every refined \( \Lambda \)-frame \((F, W)\) has a descriptive extension \((F', W')\) also validating \( \Lambda \) (the so-called ‘ultrafilter extension’). By d-persistence it follows that \( F' \vDash \Lambda \), and thus \( F \vDash \Lambda \), since \( \Lambda \) is subframe.

However an example from [Chagrov and Zakharyaschev, 1997] shows that subframe monomodal logics may be Kripke-incomplete.

On the other hand, subframe intermediate and \( K_4 \)-logics enjoy better properties. Their main new feature is axiomatisability by special ‘subframe formulas’ defined below. In this definition we assume that \( p_a \) are different proposition letters corresponding to worlds \( a \) of a finite Kripke frame \( F \).

**Definition 1.12.9** For a transitive frame \( F = (W, R) \) with root 0 put

\[
SM^{-}(F) := p_0 \land \bigwedge_{aRb} [\Box(p_a \lor \diamond p_b) \land \bigwedge_{a \neq b} (\Box(p_a \land \neg p_b) \land \bigwedge_{0Ra} \neg p_a),
\]

\[
XM^{-}(F) := SM^{-}(F) \land \left( \bigvee_{a \in W} p_a \right),
\]

\[
CSM^{-}(F) := SM^{-}(F) \land \left( \bigvee_{a \in W} p_a \right),
\]

\[
SM(F) := \neg SM^{-}(F), \ CSM(F) := \neg CSM^{-}(F), \ XM(F) := \neg XM^{-}(F).
\]

\( SM(F), CSM(F), XM(F) \) are respectively called the (modal) subframe, the cofinal subframe, and the frame formula of \( F \).

Obviously, the conjunct \( \bigwedge_{a \neq 0} \neg p_a \) is redundant if the underlying logic is S4 and all frames are reflexive.

These formulas as well as the next theorem originate from [Fine, 1974], [Fine, 1985], [Zakharyaschev, 1989]. They are particular kinds of Zakharyaschev canonical formulas, see [Zakharyaschev, 1989], [Chagrov and Zakharyaschev, 1997].

**Theorem 1.12.10** Let \( F \) be a finite rooted transitive Kripke 1-frame. Then for any transitive Kripke 1-frame \( G \)

(1) \( G \not\models XM(F) \) iff \( G \) is reducible to \( F \),

(2) \( G \not\models SM(F) \) iff \( G \) is subreducible to \( F \).

\(^{20}\) XM(F) is also called the Jankov–Fine, or the characteristic formula.
(3) \( G \not \models CSM(F) \) iff \( G \) is cofinally subreducible to \( F \).

Let us recall the idea of the proof. For example, if \( M = (G, \theta) \not \models CSM(F) \), then we obtain a cofinal subreduction from \( G \) to \( F \) by putting \( f(x) = a \) iff \( M, x \models p_a \). The other way round, if \( f \) is such a subreduction, we construct a countermodel \( M \) for \( CSM(F) \) by putting \( M, x \models p_a \) iff \( f(x) = a \).

The next two theorems from [Zakharyaschev, 1989] are also specific for the transitive case:

**Theorem 1.12.11** A transitive 1-modal logic is subframe (respectively, cofinal subframe) iff it is axiomatisable by subframe (respectively, cofinal subframe) formulas above \( K4 \).

**Theorem 1.12.12** Every cofinal subframe modal logic has the f.m.p.

**Corollary 1.12.13** Let \( \Lambda_0 \) be a subframe \( K4 \)-logic. Then for any 1-modal logic \( \Lambda \supseteq \Lambda_0 \), \( \Lambda \) is subframe (respectively, cofinal subframe) iff it is axiomatisable by subframe (respectively, cofinal subframe) formulas of \( \Lambda_0 \)-frames above \( \Lambda_0 \).

**Proof** 'If' easily follows from 1.12.11. To prove 'only if', suppose \( \Lambda \) is subframe; the case of cofinal subframe logics is quite similar. By 1.12.11, we have

\[
\Lambda = K4 + \{SM(F) \mid (F) \in \Lambda \}, \quad \Lambda_0 = K4 + \{SM(F) \mid SM(F) \in \Lambda_0 \}
\]

hence

\[
\Lambda = \Lambda_0 + \{SM(F) \mid SM(F) \in \Lambda - \Lambda_0 \}.
\]

Now \( SM(F) \not \in \Lambda_0 \) implies \( F \models \Lambda_0 \). In fact, since \( \Lambda_0 \) is Kripke-complete by 1.12.12, there exists a \( \Lambda_0 \)-frame \( G \) such that \( G \not \models SM(F) \). Then \( G \) is subreducible to \( F \) by 1.12.10, and since \( \Lambda_0 \) is a subframe logic, it follows that \( F \models \Lambda_0 \).

We shall use this corollary especially for the case \( \Lambda_0 = S4 \).

**Example 1.12.14** \( S4.1 = S4 + \Box \diamond p \supset \diamond \Box p \) is a cofinal subframe logic (McKinsey property is obviously preserved for cofinal subframes). It can be presented as \( S4 + CSM(FC_2) \), where \( FC_2 \) is a 2-element cluster — one can check that an \( S4 \)-frame has McKinsey property iff it is not cofinally subreducible to \( FC_2 \). Similarly the logic \( K4.1 := K4 + \Box \bot \vee \Box \bot \) is cofinal subframe; it is presented as \( K4 + CSM(FC_1) \), where \( FC_1 \) is a reflexive singleton and characterised by the (first-order) condition

\[
\forall x (R(x) = \emptyset \lor \exists y (xRy \& R(y) = \emptyset)).
\]

Thanks to completeness stated in 1.12.12, Theorem 1.12.8 has the following transitive version:

---

\(^{21}\)For subframe transitive logics this fact was first proved in [Fine, 1985].
Theorem 1.12.15 (Zakharyaschev) For any subframe modal logic $\Lambda \supseteq K4$ the following properties are equivalent:

1. $\Lambda$ is universal;
2. $\Lambda$ is quasi-$\Delta$-elementary;
3. $\Lambda$ is $d$-persistent;
4. $\Lambda$ is $r$-persistent;
5. $\Lambda$ has the finite embedding property.

By applying 1.12.10 to subframe logics $\Lambda = \Lambda_0 + \{SM(F_i) \mid i \in I\}$ described in 1.12.13, we can reformulate the finite embedding property as follows:

For any $\Lambda_0$-frame $G$, if $G$ is subreducible to some $F_i$ ($i \in I$), then some finite subframe of $G$ is subreducible to some $F_j$ ($j \in I$).

Hence we obtain a sufficient condition for elementarity of subframe logics above $K4$ or $S4$.

Proposition 1.12.16

1. A subframe $K4$-logic is $\Delta$-elementary if above $K4$ it is axiomatisable by subframe formulas of irreflexive transitive frames.
2. A subframe $S4$-logic is $\Delta$-elementary if above $S4$ it is axiomatisable by subframe formulas of posets.

Proof We prove only (1); the proof of (2) is similar. By 1.12.15, it is sufficient to check the finite embedding property for $K4 + \{SM(F_i) \mid i \in I\}$, where $F_i$ are irreflexive $K4$-frames. So for a $K4$-frame $G$ subreducible to some $F_i$, we find a finite subframe subreducible to $F_i$. Given a subreduction $f : G' \rightarrow F_i$, for $G' \subseteq G$, it is sufficient to construct a finite $G'' \subseteq G'$ such that $f \upharpoonright G'' : G'' \rightarrow F_i$.

This is done by induction on $|F_i|$. If $F_i$ is an irreflexive singleton, everything is trivial. Otherwise, let $u$ be the root of $F_i$, and let $f(a) = u$. Obviously, $a$ is irreflexive.

For every $v \in \beta(u)$ there exists a cone $G'_v \subseteq G' \upharpoonright a$ such that $f \upharpoonright G'_v : G'_v \rightarrow F_i \upharpoonright v$ — this follows from 1.3.32(3). Then by the induction hypothesis, there is a finite $G''_v \subseteq G'_v$ such that $f \upharpoonright G''_v : G''_v \rightarrow F_i \upharpoonright v$.

Finally put $G'' := \{a\} \cup \bigcup_{v \in \beta(u)} G''_v$ (as a subframe of $G'$). This $G''$ is the required one; in fact, monotonicity is preserved by restricted maps, and the lift property easily follows from the construction.

For subframe logics axiomatisable by a single subframe formula the converse also holds:

Proposition 1.12.17
(1) A subframe logic $\textbf{K4} + SM(F)$ is elementary iff $F$ is irreflexive.

(2) A subframe logic $\textbf{S4} + SM(F)$ is elementary iff $F$ is a poset.

Proof

(1) If $F$ contains reflexive points, we can replace each of them by the chain $(\omega, <)$. The resulting frame $F'$ is reducible to $F$, since a reflexive singleton is a $p$-morphic image of $(\omega, <)$. However, finite subframes of $F'$ are not reducible to $F$, since a $p$-morphic image of a finite irreflexive $\textbf{K4}$-frame is always irreflexive (e.g. because finite irreflexive $\textbf{K4}$-frames are exactly finite $\textbf{GL}$-frames). Thus $\textbf{K4} + SM(F)$ does not have the finite embedding property in this case.

(2) Similarly, if $F$ contains nontrivial clusters, we can replace each of them by $(\omega, \leq)$. Then we obtain a frame $F'$ reducible to $F$. Every finite subframe of $F'$ is a poset, so it is not reducible to $F$ (e.g. because it is a $\textbf{Grz}$-frame).

Remark 1.12.18 In general, the converse to 1.12.16 is not true. For example, the trivial logic $\textbf{S4} + p \supset \Box p$ is obviously subframe and elementary, but it cannot be axiomatised by subframe formulas of posets. In fact, every formula $SM(F)$ for a nontrivial $F$, is valid in every $\textbf{S5}$-frame $G$ (which is a cluster), since $G$ is not subreducible to $F$. Moreover, there is a conjecture that elementarity of a logic axiomatisable by a finite set of subframe formulas is undecidable.\(^{22}\)

Theorem 1.12.15 has an analogue for cofinal subframe logics.

Definition 1.12.19 A world in a transitive frame is called inner if its cluster is not maximal. The restriction of a frame $F$ to inner worlds is denoted by $F^-$. 

Definition 1.12.20 Let $F, G$ be $\textbf{K4}$-frames and suppose that $G$ is finite. A cofinal subreduction $f$ from $F$ to $G$ is called a cofinal quasi-embedding iff $f^{-1}(x)$ is a singleton for any inner $x$. If such a subreduction exists, we say that $G$ is a finite cofinal quasi-subframe of $F$.

Definition 1.12.21 A modal or intermediate propositional logic $\Lambda$ has the finite cofinal quasi-embedding property if for any Kripke frame $F$, $F$ validates $\Lambda$ whenever every its finite cofinal quasi-subframe validates $\Lambda$.

Theorem 1.12.22 (Zakharyaschev) For any cofinal subframe modal logic the following properties are equivalent:

(1) $\Lambda$ is elementary;

(2) $\Lambda$ is quasi-$\Delta$-elementary;

\(^{22}\)M. Zakharyaschev, personal communication.
1.12. SUBFRAME AND COFINAL SUBFRAME LOGICS

(3) $\Lambda$ is $d$-persistent;

(4) $\Lambda$ has the finite cofinal quasi-embedding property.

Note that unlike the previous theorem, 1.12.22 does not include r-persistence.

Let us now consider the intuitionistic case. Now we assume that $q_a$ are different proposition letters indexed by worlds of a finite poset $F$.

**Definition 1.12.23** For a poset $F = (W, \leq)$ with root $0$ put

$$SI^{-}(F) := \bigwedge_{a < b} (q_b \supset q_a) \wedge \bigwedge_{a \in W} \left( \bigwedge_{a \leq b} q_a \right),$$

$$CSI^{-}(F) := SI^{-}(F) \wedge \bigwedge_{a \in W} q_a,$$

$$XI^{-}(F) := CSI^{-}(F) \wedge \bigwedge_{a \in W} \left( \bigwedge_{b < a} q_b \supset \bigvee_{a \leq b} q_a \right),$$

$$SI(F) := SI^{-}(F) \supset q_0, \quad CSI(F) := CSI^{-}(F) \supset q_0, \quad XI(F) := XI^{-}(F) \supset q_0.$$ 

$SI(F)$, $CSI(F)$, $XI(F)$ are respectively called the (intuitionistic) subframe, cofinal subframe, and frame formula of $F$.\(^{23}\)

Note that $SI(F)$ is an implicative formula and $CSI(F)$ is built from proposition letters and $\supset, \perp$.

Then similarly to the modal case, we have (cf. [Zakharyaschev, 1989], [Chagrov and Zakharyaschev, 1997]):

**Theorem 1.12.24** Let $F$ be a finite rooted poset. Then for any poset $G$

(1) $G \not\models XI(F)$ iff $G$ is reducible to $F$;

(2) $G \not\models SI(F)$ iff $G$ is subreducible to $F$;

(3) $G \not\models CSI(F)$ iff $G$ is cofinally subreducible to $F$.

The idea of the proof is quite similar to 1.12.10. E.g. if $M = (G, \theta) \not\models XI(F)$, we obtain a reduction $f$ from $G$ to $F$ by putting

$$f(x) = a := (M, x \models \bigwedge_{a \leq b} q_b \& M, x \not\models q_a).$$

The formulas from 1.12.23 can be simplified. For example, in $CSI^{-}(F)$ we can replace the conjunct $\bigwedge_{a \leq b} q_a$ with $\bigwedge_{a \in \text{max}(F)} q_a$, where $\text{max}(F)$ is the set of maximal points of $F$; also $\bigwedge_{b < a} q_b$ occurring in the second conjunct of $XI^{-}(F)$ can be replaced with $\bigwedge_{a \in \beta(b)} q_b$.

Other versions of these formulas are described in [Chagrov and Zakharyaschev, 1997] and [Shimura, 1993].

The next result is also due to [Zakharyaschev, 1989], cf. [Chagrov and Zakharyaschev, 1997].\(^{23}\)

$XI(F)$ is also called the Jankov, or the characteristic formula of $F$.\(^{23}\)
Theorem 1.12.25 For an intermediate propositional logic $\Lambda$ the following properties are equivalent:

1. $\Lambda$ is a subframe (respectively, cofinal subframe) logic;
2. $\Lambda$ is axiomatisable by subframe (respectively, cofinal subframe) formulas;
3. $\Lambda$ is axiomatisable by implicative formulas (respectively, $(\supset, \bot)$-formulas).

Let us now formulate analogues of Theorems 1.12.12, 1.12.15 and 1.12.22 for intermediate logics.

Theorem 1.12.26 Every cofinal subframe intermediate logic has the finite model property.

This result follows from [McKay, 1968] and 1.12.25; also see [Chagrov and Zakharyaschev, 1997].

Theorem 1.12.27

1. An intermediate logic is subframe iff it is universal.
2. Every subframe intermediate logic is $r$-persistent.

Theorem 1.12.28 Every cofinal subframe intermediate logic is $\Delta$-elementary and $d$-persistent.

To formulate a modal analogue of this result, we need

Definition 1.12.29 A transitive frame is called almost irreflexive if all its non-maximal clusters are degenerate. An S4-frame is called blossom if all its non-maximal clusters are trivial.

Theorem 1.12.30

1. A cofinal subframe S4-logic is $\Delta$-elementary if above S4.1 it is axiomatisable by cofinal subframe formulas of blossom frames.
2. A cofinal subframe K4-logic is $\Delta$-elementary if above K4.1$^-$ it is axiomatisable by cofinal subframe formulas of almost irreflexive transitive frames.

Proof (1) By 1.12.22, it suffices to check the finite cofinal quasi-embedding property for our logic $\Lambda$. So suppose $G \not\models \Lambda$. If $G \not\models S4.1$, the further proof is easy, so let $G \models S4.1$. Then there is a finite rooted blossom frame $F$ such that $CSM(F) \in \Lambda$ and a cofinal subreduction $f$ from $G$ to $F$. Let us show that then there exists a subframe $G_0 \subseteq G$ such that $f \upharpoonright G_0$ is a cofinal subreduction to $F$ and $f$ is injective on $f^{-1}(F^-)$. Moreover, we can choose $G_0$ so that $f[G_0] \subseteq F^-$, and thus $G_0$ is finite.

The proof is by induction on the cardinality of $F$. Let $u$ be an element of $G$ with $f(u) = 0_F$, where $0_F$ is the root of $F$. For every $a \in \beta(0_F)$, choose an
arbitrary \( v \in G \) such that \( u \leq_G v \) and \( f(v) = a \). Note that if \( a \) is maximal in \( F \) and \( G \) has McKinsey property, we can choose a maximal \( v \), since \( f \) is cofinal. Then \( f_a := f \upharpoonright (G \upharpoonright v) \), is a cofinal subreduction from \( G \upharpoonright v \) to \( F \upharpoonright a \). Hence by the induction hypothesis, there exists a finite subframe \( G_a \) of \( G \) such that \( f_a \upharpoonright G_a \) is a cofinal subreduction from \( G \) to \( F \upharpoonright a \). Put

\[
G_0 := \{u\} \cup \bigcup_{a \in \beta(0, p)} G_a.
\]

Then \( G_0 \) obviously has the required property.

(2) is proved similarly.

\subsection{1.13 Splittings}

The following result is a modal version of [Jankov, 1969].

\textbf{Theorem 1.13.1} Let \( F \) be a finite rooted Kripke frame. Then for any modal algebra \( \Omega \) the following properties are equivalent:

1. \( \Omega \not\models XM(F) \);
2. \( MA(F) \) is a subalgebra of a homomorphic image of \( \Omega \).

\textbf{Lemma 1.13.2} For any \( 1 \)-modal logic \( \Lambda \supseteq K4 \) and a finite rooted transitive frame \( F \)

\[ XM(F) \in \Lambda \iff \Lambda \not\subseteq ML(F); \]

in particular, for any \( 1 \)-modal formula \( A \)

\[ K4 + A \vdash XM(F) \iff A \not\in ML(F). \]

If \( F \) is reflexive, then the latter is also equivalent to \( S4 + A \vdash XM(F) \).

\textbf{Remark} This lemma shows that the pair of logics \((K4 + XM(F), ML(F))\) 'splits' the set of extensions of \( K4 \): every \( \Lambda \supseteq K4 \) is either below \( ML(F) \) or above \( K4 + XM(F) \).

The original Jankov formula [Jankov, 1969] is defined as follows.

\textbf{Definition 1.13.3} Let \( F = (W, \leq) \) be a finite p.o. set, \( \Omega = HA(F) \). Then

\[
XI(F) := \left( \bigwedge_{a, b \in \Omega} (p_{a \land b} \equiv p_a \land p_b) \land \bigwedge_{a, b \in \Omega} (p_{a \lor b} \equiv p_a \lor p_b) \land \bigwedge_{a, b \in \Omega} (p_{a \rightarrow b} \equiv (p_a \supset p_b)) \land \neg p_0 \right) \supset p_\omega,
\]

where \( \{p_a \mid a \in \Omega\} \) is a set of distinct propositional variables corresponding to the elements of \( \Omega \) and \( \omega \) is the subgreatest element of \( \Omega \).
Theorem 1.13.4 Let $F$ be a finite rooted Kripke frame. Then for any Heyting algebra $\Omega'$ the following properties are equivalent:

1. $\Omega' \not\models XI(F);$  
2. $HA(F)$ is a subalgebra of a homomorphic image of $\Omega'.$

Corollary 1.13.5 Let $F$ be a finite rooted p.o. set. Then for any p.o. set $G$ the following properties are equivalent:

1. $G \not\models XI(F);$  
2. there exists $u \in G$ such that $G^u \rightarrow F.$

Lemma 1.13.6 For any superintuitionistic logic $\Lambda$

$$XI(F) \in \Lambda \text{ iff } \Lambda \not\subseteq \mathbf{IL}(F),$$

in particular, for any intuitionistic formula $A$

$$XI(F) \in (H + A) \text{ iff } A \not\subseteq \mathbf{IL}(F).$$

Hence for example, we obtain

Lemma 1.13.7 Let $A$ be a propositional formula. Then:

1. $AJ \in (H + A)$ iff $A \not\subseteq \mathbf{IL}(F_2)$ (Fig. 1.2);  
2. $AJ^- \in (H + A)$ iff $A \not\subseteq \mathbf{IL}(F_3)$ and $A \not\subseteq \mathbf{IL}(F_4)$ (Figs. 1.3, 1.4).

Proof Note that

$$H + AJ = H + XI(F_2), \ H + AJ^- = H + XI(F_3) \land XI(F_4).$$

Then apply Lemma 1.13.6
1.14 Tabularity

Definition 1.14.1 A modal or intermediate propositional logic is called tabular if it is determined by a finite modal or Heyting algebra, or equivalently, by a finite Kripke frame.

Lemma 1.14.2 In the notation of 1.10.2, if \( F \) has a root \( u_0 \), \( dd(u_0) \) is finite and \( R(x) \) is finite for any \( x \in F \), then \( F \) is finite.

Proof By induction on \( k = dd(u_0) \). The statement is trivial for \( k = 0 \). For the step note that

\[
W = R^*(u_0) = \{u_0\} \cup R(u_0) \cup \bigcup \{W \uparrow u \mid u_0 R u, u_0 \neq u\}.
\]

By the induction hypothesis, every \( W \uparrow u \) for \( u_0 R u, u_0 \neq u \) is finite, since \( dd(u) < dd(u_0) \).

Hence \( W \) is finite.

Consider the following \( N \)-modal \( k \)-formulas

\[p^1_k := \neg p_i \land \bigwedge_{i \neq j} p_j;\]
\[Alt_k := \neg \bigwedge_{i=1}^k \lozenge p^i_k;\]
\[B^i_k := p^i_k;\]
\[B^{i-1}_k := p^i_k \land \lozenge B^{i-1}_k \text{ for } 1 < i \leq k;\]
\[AH_k := \neg B^k_k;\]

where

\[\lozenge A := \lozenge_1 A \lor \cdots \lor \lozenge_N A.\]

These formulas have the following first-order characterisations.

Lemma 1.14.3 For a frame \( F = (W, R_1, \ldots, R_N) \) put \( R := R_1 \cup \cdots \cup R_N \). Then

(1) \( F, x \models Alt_k \iff |R(x)| < k; \)

(2) \( F, x \models AH_k \iff dd_F(x) < k. \)

For intermediate logics we have the same characterisation with different formulas.

Lemma 1.14.4 For a poset \( F = (W, R) \)

\( F, x \models IG_k \iff |R(x)| < k. \)

Hence we obtain a characterisation of tabular modal logics similar to that in [Chagrov and Zakharyaschev, 1997].

Proposition 1.14.5
(1) An \(N\)-modal propositional logic \(\Lambda\) is tabular iff \(\Lambda \vdash \text{Alt}_k \land \text{AH}_k\) for some \(k\).

(2) An intermediate propositional logic \(\Lambda\) is tabular iff \(\Lambda \vdash \text{IG}_k\) for some \(k\).

**Proof** The claim ‘only if’ follows from 1.14.3 and 1.14.4.

To show ‘if’, use the canonicity of \(\text{AH}_k\), \(\text{Alt}_k\), \(\text{IG}_k\) and the observation that every rooted frame validating \(\text{Alt}_k \land \text{AH}_k\) (or \(\text{IG}_k\)) is finite.

**Lemma 1.14.6** If \(\Phi = (F, W)\) is a finite refined frame, then \(\text{ML}(\Phi) = \text{ML}(F)\) (or \(\text{IL}(\Phi) = \text{IL}(F)\) in the intuitionistic case).

**Proof** (Modal case) For \(x \neq y\) let \(U_{xy}\) be an interior set such that \(x \in U_{xy}, y \not\in U_{xy}\). Since \(F\) is finite, it follows that

\[\{x\} = \bigcap_{y \neq x} U_{xy} \in W.\]

Thus all subsets of \(F\) are interior, so the valuations in \(\Phi\) and \(F\) are the same. Therefore valid formulas are the same.

(Intuitionistic case.) Suppose \(F = (W, R)\) is an intuitionistic frame and for \(y \not\in R(x)\) let \(V_{xy} \in W\) be such that \(x \in V_{xy}, y \not\in V_{xy}\). Then

\[R(x) = \bigcap_{y \not\in R(x)} V_{xy} \in W.\]

So for every \(R\)-stable \(U\) we have

\[U = \bigcup_{x \in U} R(x) \in W,\]

and again it follows that \(\Phi, F\) have the same valuations.

**Proposition 1.14.7** Every tabular modal or intermediate propositional logic is \(r\)-persistent.

**Proof** (Modal case) Consider a tabular logic \(\Lambda\) and a refined frame \(\Phi = (F, W)\) validating \(\Lambda\). By 1.14.5, \(\Lambda \vdash \text{Alt}_k \land \text{AH}_k\) for some \(k\), hence \(\Phi \models \text{Alt}_k \land \text{AH}_k\). According to the characterisation given in 1.14.3, the classes \(\text{V}(\text{Alt}_k), \text{V}(\text{AH}_k)\) are universal, so \(\text{Alt}_k \land \text{AH}_k\) is \(r\)-persistent by Theorem 1.12.8 and thus valid in \(F\) and therefore in every cone \(F \uparrow u\). So by 1.14.5, \(\Phi \uparrow u\) is finite. By 1.7.8, \(\Phi \uparrow u\) is refined and by 1.6.11, \(\Phi \uparrow u \vdash \Lambda\). Therefore, \(F \uparrow u \vdash \Lambda\) by 1.14.6. Since \(u\) is arbitrary, by Lemma 1.3.26, it follows that \(F \vdash \Lambda\).

In the intuitionistic case use the same argument and 1.14.4 instead of 1.14.5.

**Proposition 1.14.8** For finite modal frames \(G, F\), \(\text{ML}(G) \subseteq \text{ML}(F)\) iff \(G\) is reducible to \(F\); similarly for the intuitionistic case.

**Theorem 1.14.9** Every tabular modal or intermediate logic is finitely axiomatisable. Moreover, a transitive modal or intermediate logic is finitely axiomatisable by frame formulas.
1.15 Transitive logics of finite depth

The results of this section are based on [Hosoi, 1967], [Hosoi, 1969].

Lemma 1.15.1 For a poset $F$
\[ F \vdash AP_n \iff d(F) \leq n. \]

Proposition 1.15.2 Let $Z_n$ be an $n$-element chain. Then
\[ IL(Z_n) = LC + AP_n, \quad LC = \bigcap_n IL(Z_n) = IL(F) \]
for every infinite chain $F$. In particular, $IL(Z_1) = H + AP_1$ is a classical logic and $H + AP_n \subset IL(Z_n)$ for $n \geq 2$.

Proposition 1.15.3 $H + AP_n$ is determined by the class of finite posets of depth $n$.

Proof As we know, $AP_n$-frames are $S4$-frames of depth $\leq n$, so $H + AP_n$ is a subframe logic. Thus it is determined by finite posets of depth $\leq n$. It remains to note that every poset of depth $< n$ is a generated subframe in a poset of depth $n$.

Hence by unravelling we obtain

Corollary 1.15.4 $H + AP_n = IL(IT^n_\omega)$, where $IT^n_\omega$ is the standard tree of depth $n$ and branching $\omega$.

Hence by induction on $n$ we obtain

Corollary 1.15.5 For every $m, n \in \omega$ there exists $k \in \omega$ such that
\[ (H + AP_n)[m = IL(IT^m_k)][m, \]
where $IL(IT^m_k)$ is the tree of depth $n$ and branching $k$.

So to say, $m$ letters can distinguish only finitely many successors of any point.

Definition 1.15.6 A logic $L$ is called locally tabular if all its finitely generated Lindenbaum algebras $Lind(L[m])$ are finite; in other words, if for every $m$ there exist finitely many non-$L$-equivalent formulas in $m$ propositional letters.

Corollary 1.15.7 Every logic $H + AP_n$ is locally tabular.

Proposition 1.15.8 Every locally tabular logic has the f.m.p.

Lemma 1.15.9 $H + XI(Z_{n+1}) = H + AP_n$ for $n \geq 0$. 
1.15. **TRANSITIVE LOGICS OF FINITE DEPTH**

**Proof** A frame $F$ is of depth $\leq n$ iff $F$ is not reducible to $Z_{n+1}$ iff $F \models X I(Z_{n+1})$. Thus $H + AP_n$ and $H + X I(Z_{n+1})$ have the same finite frames, and it remains to show that the latter logic has the f.m.p.

But this logic is subframe, since the class of frames of depth $\leq n$ is closed under subframes. So we can apply Fine's theorem.

Hence by Lemma 1.13.6 we obtain

**Proposition 1.15.10** For a superintuitionistic logic $L$

$$L \not\subseteq IL(Z_{n+1}) \text{ iff } H + AP_n \subseteq L.$$ 

This proposition shows that the structure of superintuitionistic logics splits into an $(\omega + 1)$-sequence of slices.

**Definition 1.15.11** For a finite $n$, the $n$th slice is the set of superintuitionistic logics

$$S_n := \{L \in S \mid H + AP_n \subseteq L \subseteq IL(Z_n)\};$$

the $\omega$th slice is

$$S_\omega := \{L \mid H \subseteq L \subseteq LC\}.$$ 

So

$$S_0 = \{H + AP_0\} = \{H + \perp\},$$

i.e. $S_0$ contains only the logic of the empty 'chain' $Z_0$.

Let $S_n$ be the set of logics of slice $n$ (for $0 \leq n \leq \omega$). Proposition 1.15.10 implies

**Proposition 1.15.12**

1. $S = \bigcup_{n \leq \omega} S_n$,
2. $S_n \cap S_m = \emptyset$ for $n \neq m$,
3. if $L$ is a superintuitionistic logic, then $L \in S_n$, where

$$n = \begin{cases} \omega & \text{if } \forall n \ L \in IL(Z_n), \\ \max\{n \in \omega \mid L \subseteq IL(Z_n)\} & \text{otherwise} \end{cases}$$

4. the slices $S_1$ is one-element: $S_1 = \{H + AP_1\} = CL$,
5. the slice $S_2$ is a decreasing $(\omega + 1)$-chain,
6. for $n \geq 3$ all slices $S_n$ are of cardinality $2^{\aleph_0}$.

**Proposition 1.15.13** For a finite $n$

$$IL(F) \in S_n \iff d(F) = n.$$
**Proposition 1.15.14** $S_n = \{\text{IL}(F) \mid F$ is a poset of depth $n\}$. Every logic of slice $S_n$ is determined by a disjoint union of finite posets of depth $n$.

**Proof** Local tabularity is clearly inherited by extensions, so it holds for all logics of finite slices. Therefore all logics of finite slices have the f.m.p., and hence the proposition follows.

As we mentioned, $\text{LC} = \bigcap_{n \in \omega} \text{IL}(Z_n)$. The set

$$\{L \mid \text{LC} \subset L\} = \{\text{IL}(Z_n) \mid n \in \omega\}$$

is a decreasing $\omega$-chain; this readily follows from Proposition 1.15.14, since $\text{IL}(Z_n) = \text{LC} + \text{AP}_n$ is a simple extension of $\text{LC}$ in the $n$th slice. Also $\text{H} = \bigcap_{n \in \omega} (\text{H} + \text{AP}_n)$, since $\text{H}$ has the f.m.p. and every finite poset is of finite depth.

**Proposition 1.15.15** $S_n$ are sublattices of the lattice of superintuitionistic logics. Every finite slice $S_m$ is embeddable in all slices $S_n$ for $m \leq n \leq \omega$ by maps $\lambda_{mn}: L \rightarrow L \cap \text{IL}(Z_m)$ (and $\text{IL}(Z_m) = \text{LC}$).

**Proof** Note that $L_1 \subseteq L_2$ iff $\lambda_{mn}(L_1) \subseteq \lambda_{mn}(L_2)$, since $L = \lambda_{mn}(L) + \text{AP}_m = \lambda_{mn}(L) + (\text{H} + \text{AP}_m)$, for $L \in S_m$; moreover, $\lambda_{mn}(L) = \lambda_{mk}(L) + \text{AP}_n = \lambda_{mn}(L) + (\text{H} + \text{AP}_n)$ for $k \geq n$, by distributivity. Also

$$\lambda_{mn}(L \cap L') = \lambda_{mn}(L) \cap \lambda_{mn}(L'),$$

and $\lambda_{mn}(L + L') = \lambda_{mn}(L) + \lambda_{mn}(L')$, again by distributivity.

### 1.16 $\Delta$-operation

In this section we study an embedding of $S_n$ in $S_{n+1}$ ($\Delta$-operation) introduced in [Hosoi, 1969].

**Definition 1.16.1** For $A \in IF$ put $\delta A := p \lor (p \supset A)$, where $p$ is a proposition letter that does not occur in $A$. For a superintuitionistic logic $L$ put $\Delta L := \text{H} + \{\delta A \mid A \in L\}$.

The original definition [Hosoi, 1969] is $\Delta L := \{\delta'A \mid A \in L\}$, where $\delta'A := ((p \supset A) \supset p) \supset p$ (the 'A-version' of Peirce's law). Let us show that both definitions are equivalent.

**Lemma 1.16.2** $\text{H} + \delta A = \text{H} + \delta'A$ for any formula $A$.

24 Obviously, the definition of $\Delta L$ does not depend on the choice of $p$; but we fix $p$ to make $\delta A$ unique.
1.16. \( \Delta \)-OPERATION

**Proof**  On the one hand, obviously \( \mathbf{QH} \vdash \delta A \supset \delta'A \). On the other hand, let us prove

\[(0) \quad \mathbf{H} + \delta'A \vdash \delta A.\]

In fact, obviously

\[(1) \quad p \vdash_{\mathbf{H}} p \lor (p \supset A),\]

due to the deduction theorem 1.1.4

\[(2) \quad p, p \lor (p \supset A) \supset A \vdash_{\mathbf{H}} A,\]

and thus by the deduction theorem of \( \delta A \supset A \),

\[(3) \quad \delta A \supset A \vdash_{\mathbf{H}} p \supset A.\]

This implies

\[(4) \quad \delta A \supset A \vdash_{\mathbf{H}} \delta A,\]

due to the deduction theorem. Now since

\[(5) \quad \mathbf{H} \vdash (\delta A \supset A) \supset \delta A,\]

by the deduction theorem. Now since

\[((\delta A \supset A) \supset \delta A) \supset \delta A\]

is a substitution instance of \( \delta'A \), (5) implies (0).

The next lemma shows the semantical meaning of \( \delta A \).

**Lemma 1.16.3** Let \( F \) be a rooted poset with root \( 0_F \). Then

\[F \vdash \delta A \text{ iff } \forall u \neq 0_F \ F \uparrow u \vdash A.\]

**Proof** (Only if.) Suppose \( F \uparrow u \not\models A \), \( u \neq 0_F \), so \( M \not\models A \) for some Kripke model \( M = (F \uparrow u, \theta) \). By truth preservation it follows that \( M, u \not\models A \). Then consider \( M' = (F, \xi) \) such that

\[\xi(q) : = \begin{cases} \theta(q) & \text{if } q \neq p, \\ F - \{0_F\} & \text{if } q = p. \end{cases}\]

Obviously \( M' \) is intuitionistic. Also \( M', u \not\models A \), since \( p \) does not occur in \( A \). Now from \( M', u \vdash p \) and \( M', 0_F \not\models p \) it follows that \( M', 0_F \not\models \delta A \); thus \( F \not\models \delta A \).

(If.) Suppose \( F \not\models \delta A \), i.e. \( M \not\models \delta A \) for some Kripke model \( M \) over \( F \). Then by truth preservation \( M, 0_F \not\models \delta A \); hence \( M, 0_F \not\models p \), and \( M, u \vdash p, M, u \not\models A \) for some \( u \neq 0_F \). By the generation lemma it follows that \( M \uparrow u, u \not\models A \); thus \( F \uparrow u \not\models A \).
Hence we obtain

**Proposition 1.16.4** If $F \models L$, then $1 + F \models \Delta L$, where $1 + F$ is obtained by adding a root below $F$.

**Proof** In fact, if $1 + F \not\models \delta A$, then by 1.16.3, $F \upharpoonright u \not\models A$ for some $u \in F$; thus $F \not\models A$ by the generation lemma.

**Lemma 1.16.5** For any superintuitionistic logic $L$, $\delta A \in \Delta L$ iff $A \in L$.

**Proof** We consider a particular case, when $L$ is Kripke-complete. The claim easily follows from Lemma 1.16.3 (and Proposition 1.16.4). In fact, if $A \not\in L = \text{IL}(F)$ for a poset $F$, then $\Delta L \subseteq \text{IL}(1 + F)$ and $\delta A \not\in \text{IL}(1 + F)$.

A predicate analogue of 1.16.5 will be discussed later on in Section 2.13.

**Lemma 1.16.6**

1. $H \vdash A \supset \delta A$.

2. $H \vdash \delta(A_1 \supset A_2) \supset (\delta A_1 \supset \delta A_2)$.

3. $H \vdash \delta \left( \bigwedge_{i=1}^{n} A_i \right) \equiv \bigwedge_{i=1}^{n} \delta A_i$.

**Proof** An easy exercise; also see Chapter 2.

**Proposition 1.16.7** For propositional superintuitionistic logics $L_1, L_2$.

1. $\Delta L \subseteq L$;

2. $L_1 \subseteq L_2$ iff $\Delta L_1 \subseteq \Delta L_2$;

3. $L_1 = L_2$ iff $\Delta L_1 = \Delta L_2$.

**Proof** (1) $\Delta L \subseteq L$ follows from $H \vdash A \supset \delta A$.

(2) 'Only if' is obvious. To show 'if', suppose $L_1 \not\subseteq L_2$ and $A \in L_1 - L_2$; then $\delta A \in \Delta L_1 - \Delta L_2$ by 1.16.5, and thus $\Delta L_1 \not\subseteq \Delta L_2$.

(3) A trivial consequence of (2).

Note that (2) means that $\Delta$ is a monotonic embedding $S \rightarrow S$. By (2), $L_1 \subseteq L_2$ also implies $\Delta^n L_1 \subseteq \Delta^n L_2$ for any $n$.

Now the deduction theorem implies

**Lemma 1.16.8** $\Delta(H + \Gamma) = H + \delta \Gamma$, where $\delta \Gamma := \{ \delta A \mid A \in \Gamma \}$. Hence $\Delta^n(H + \Gamma) = H + \delta^n \Gamma$, where $\delta^n \Gamma := \{ \delta^n A \mid A \in \Gamma \}$.

**Proof** In fact, if $L = H + \Gamma \vdash A$ then $H \vdash \bigwedge_{i=1}^{m} B_i \supset A$ for some $B_1, \ldots, B_m \in \text{Sub}(\Gamma)$. Then $\delta B_i \in \text{Sub}(\delta \Gamma)$, and so $H + \delta \Gamma \vdash \delta A$. Thus $\Delta L = H + \delta L \subseteq H + \delta \Gamma$. 

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Obviously $AP_{m+1} = \delta AP_m$, so $\delta^n AP_m = AP_{m+n}$, in particular $AP_m = \delta^n \bot$. Hence we conclude that $\Delta^n (H + AP_m) = H + AP_{m+n}$, and thus for any $n$ $\Delta(H + AP_n) = H + AP_{n+1}$; and $H + AP_n = \Delta^n (H + \bot) = \Delta^{n-1} (CL)$ for $n > 0$.

Also by Proposition 1.16.7,

$$\Delta (IL(Z_n)) \subset IL(Z_{n+1}) \text{ for } n > 0;$$

The inclusion is proper, since $\Delta L \subset \Delta CL = H + AP_2$ for $L \subseteq CL$, and thus $AZ = (p \supset q) \lor (q \supset p) \not\in \Delta L$.

So we obtain

**Proposition 1.16.9** $\Delta L \in S_{n+1}$ for $L \in S_n$, $n \in \omega$. Thus $\Delta$ embeds $S_n$ in $S_{n+1}$ and $S_\omega$ in itself.

Note that $\Delta$ is not a lattice embedding; more precisely, it preserves joins not meets.

**Lemma 1.16.10** If $A_1 \in (L_1 - L_2)$ and $A_2 \in (L_2 - L_1)$, then $\delta A_1 \lor \delta A_2 \in \Delta L_1 \cap \Delta L_2 - \Delta (L_1 \cap L_2)$.

**Proof** For Kripke-complete $L_1$ and $L_2$ (in particular, for all logics of finite slices) this readily follows from Lemma 1.16.3. Namely, if $A_1 \not\in L_2 = IL(F_2)$ and $A_2 \not\in L_1 = IL(F_1)$, then the frame $1 + F_1 \cup F_2$ separates $\delta A_1 \lor \delta A_2$ from $\Delta (L_1 \cap L_2)$.

The lemma actually holds for arbitrary logics; its analogue for predicate logics will be discussed in Section 2.13.

**Proposition 1.16.11** For superintuitionistic logics $L_1, L_2$

(1) $\Delta (L_1 + L_2) = \Delta L_1 + \Delta L_2$;

(2) $\Delta (L_1 \cap L_2) = \Delta L_1 \cap \Delta L_2$ iff $L_1$ and $L_2$ are $\subseteq$-comparable.

**Proof** (1) follows from Lemma 1.16.5: $L_1 + L_2 = H + L_1 \cup L_2$, so $\Delta (L_1 + L_2) = H + \delta L_1 \cup \delta L_2 = (H + \delta L_1) + (H + \delta L_2) = \Delta L_1 + \Delta L_2$.

(2) follows from 1.16.10.

**Proposition 1.16.12** $\Delta L = L$ iff $L = H$. So $\Delta L \subset L$ for any $L \neq H$.

**Proof** If $\Delta L = L$, then $L \subset H + \bot$ implies $L \subset \Delta^n (H + \bot) = H + AP_n$ for all $n \in \omega$. Hence $L \subset \bigcap_n (H + AP_n) = H$.

**Remark 1.16.13** A similar argument shows that for any superintuitionistic logic $L$,

$$\bigcap_{n \in \omega} \Delta^n L = H$$  

in other words, the ‘$\omega$-iteration’ of $\Delta$ is trivial. (*) readily implies Proposition 1.16.12. In Volume 2 we will prove that 1.16.12 and (*) do not transfer to the predicate case.
1.17 Neighbourhood semantics

Neighbourhood semantics is a generalisation of Kripke semantics. In this case 'possible worlds' are regarded as points of an abstract 'space' or a 'neighbourhood frame'. In such a frame every world has a set of 'neighbourhoods', and $\Box A$ is true at $w$ iff $A$ is true in all worlds in some neighbourhood of $w$, that is, in all the worlds that are 'rather close' to $w$. So in neighbourhood semantics 'necessary' is interpreted as 'locally true'.

Here is a precise definition.

Definition 1.17.1 An n-modal (propositional) neighbourhood frame is an $(n+1)$-tuple $F = (W, \Box_1, \ldots, \Box_N)$, such that $W \neq \emptyset$, $\Box_i$ are unary operations in $2^W$ satisfying the identities:

$$\Box_i(a \cap b) = \Box_i a \cap \Box_i b,$$

$$\Box_i W = W.$$

As in Kripke frames, the elements of $W$ are called possible worlds, or points; $u \in \Box_i V$ is read as '$V$ is an $i$-neighbourhood of $u$'. The basic identities mean that the intersection of two $i$-neighbourhoods of $u$ is also an $i$-neighbourhood of $u$, every extension of an $i$-neighbourhood is again an $i$-neighbourhood and that $W$ is an $i$-neighbourhood of any $u$. However a neighbourhood of $u$ may not contain $u$, or may even be empty.

Obviously, an $N$-modal neighbourhood frame $F$ corresponds to the $N$-modal algebra $MA(F) := (2^W, \cup, \cap, -, \emptyset, W, \Box_1, \ldots, \Box_N)$.

The following is a trivial consequence of definitions, cf. Section 1.3.

Lemma 1.17.2 Every Kripke frame $F = (W, R_1, \ldots, R_N)$ corresponds to a neighbourhood frame $Nd(F) = (W, \Box_1, \ldots, \Box_N)$, such that $MA(Nd(F)) = MA(F)$.

Definition 1.17.3 A neighbourhood model over a neighbourhood frame $F$ is a pair $M = (F, \theta)$, in which $\theta : PL \rightarrow 2^W$ is a valuation. $\theta$ is extended to all formulas, according to Definition 1.2.8.

We use the same terminology and notation as in Kripke semantics. For a formula $A$, we write: $M, w \vDash A$ (or $w \vDash A$) instead of $w \in \theta(A)$, and say that $A$ is true at the world $w$ of $M$ (or that $w$ forces $A$).

So we have:

$$M, x \vDash \Box_i A \text{ iff } \theta(A) \text{ is an } i\text{-neighbourhood of } x.$$

Definition 1.17.4 A modal formula $A$ is true in a neighbourhood model $M$ (notation: $M \vDash A$) if it is true at every world of $M$; $A$ is valid in a neighbourhood frame $F$ (notation: $F \vDash A$) if it is true in every model over $F$. A set of formulas $\Gamma$ is valid in $F$ (notation: $F \vDash \Gamma$) if every $A \in \Gamma$ is valid.
Similarly to Lemmas 1.3.7 and 1.3.8 we obtain

**Lemma 1.17.5** For any modal formula $A$ and a neighbourhood frame $F$

$$F \models A \iff MA(F) \models A.$$ 

**Lemma 1.17.6**

1. For a neighbourhood frame $F$ the set

$$ML(F) := \{ A \mid F \models A \}$$

is a modal logic.

2. For a class $\mathcal{C}$ of $N$-modal neighbourhood frames the set

$$ML(\mathcal{C}) := \bigcap \{ ML(F) \mid F \in \mathcal{C} \}$$

is an $N$-modal logic.

**Definition 1.17.7** The logic $ML(F)$ (respectively, $ML(\mathcal{C})$) is called the modal logic of $F$ (respectively, of $\mathcal{C}$), or the modal logic determined by $F$ (by $\mathcal{C}$).

A modal logic is called neighbourhood complete if it is determined by some class of neighbourhood frames.

From Lemma 1.17.2 we have:

**Lemma 1.17.8** Every Kripke-complete propositional logic is neighbourhood complete.

The converse to the previous Lemma is false [Gabbay, 1975], [Gerson, 1975a], [Shehtman, 1980], [Shehtman, 2005]. There also exist examples of modal logics that are incomplete in neighbourhood semantics [Gerson, 1975b], [Shehtman, 1980], [Shehtman, 2005]. The question of whether all intermediate propositional logics are neighbourhood complete (Kuznetsov's problem [Kuznecov, 1974]), is still open.
Chapter 2

Basic predicate logic

2.1 Introduction

The main notion of this chapter is first-order logic. Similarly to the propositional case, we define a logic as a set of formulas that contains some basic axioms and is closed under some basic inference rules.

Here the crucial point is the substitution rule, which is important because we would like to distinguish between logics and theories.

On the one hand, every axiomatic logical calculus (postulated by a set of axioms and inference rules) generates a 'theory' — the set of all theorems. Usually theories are supposed to collect properties of a certain kind of objects. Many well known theories, such as Peano arithmetic, Tarski's elementary geometry, Zermelo–Fraenkel set theory, were developed for that purpose.

On the other hand, we may be interested in theories that do not depend on 'application domains' and express the basic 'logical laws'. For example, the proposition

Every human has a father and a mother

expressed by a formula

\[ A = \forall x (H(x) \supset \exists y F(y, x) \land \exists z M(z, x)), \]

is a specific property of humans that does not hold for all living creatures.

The formula

\[ B = \forall x (H(x) \supset \exists y M(y, x)) \]

also expresses a true property of humans, which does not hold in other cases.

But the implication

\[ A \supset B \]

(allowing us to deduce \(B\) from \(A\)) is a logical law — its truth does not depend on the meaning of the predicates \(H, F, M\).
So the laws of logic should sustain replacing of predicates by arbitrary formulas. We can regard them as schemata for producing theorems, and define a logic just as a substitution closed theory.\footnote{Of course this definition is rather conventional, and there exist examples of 'logics' that are not substitution closed.}

A standard example is classical first-order logic, the set of all theorems of classical predicate calculus. Numerous classical theories contain it as a fixed basic part.

In the nonclassical area there is a great variety of logics deserving special attention. Of course study of nonclassical theories (such as Heyting arithmetic or modal set theories) is also interesting and important, but due to the lack of time, we postpone it until Volume 2.

Study of nonclassical logics in this volume is closely related to study of different semantics. Unlike the classical case, there are many options here. From our viewpoint, a semantics \( S \) for a certain class of logics (say, \( \Sigma \)) should include the notions of a 'frame' and 'validity'. A semantics \( S \) is 'sound' for \( \Sigma \) if the set of all formulas valid in any \( S \)-frame is a logic from \( \Sigma \).\footnote{Some authors still use 'semantics' that are not sound in this sense [Kracht and Kutz, 2005], [Goldblatt and Manes, 2006]. This is not so convenient, because it may be difficult to describe all 'frames' characterising a given logic.} Thus to check soundness, it is necessary to prove that the substitution rule preserves 'validity' in a 'frame'.

In this respect there is a big difference between the classical and nonclassical cases. In classical logic we may not care about formula substitutions, and they are usually not discussed in textbooks and monographs.\footnote{With few exceptions, such as [Church, 1996], [Novikov, 1977].} Classical predicate calculus is traditionally formulated using axiom schemes rather than substitution rule, and soundness is proved without mentioning substitutions. But in our nonclassical studies we deal with rather exotic types of frames and the proof of soundness may be nontrivial.

Therefore let us first take a closer look at the syntactic notion of a logic, and especially at the substitution rule. Its intuitive meaning is clear: given a first-order formula \( A \) we can deduce every formula \([C/P(x_1,\ldots,x_n)]A\) obtained by substituting a formula \( C \) for an atomic formula \( P(x_1,\ldots,x_n) \). More exactly, to obtain \([C/P(x_1,\ldots,x_n)]A\), one should replace every occurrence of \( P(x_1,\ldots,x_n) \) with \( C \) and every occurrence of \( P(y_1,\ldots,y_n) \) (using other variables \( y_1,\ldots,y_n \)) with the corresponding version of \( C \), \([y_1,\ldots,y_n/x_1,\ldots,x_n]C\). In its turn \([y_1,\ldots,y_n/x_1,\ldots,x_n]C\) is obtained by a 'correct' replacement of parameters \( x_1,\ldots,x_n \) with \( y_1,\ldots,y_n \).

Thus the definition of a formula substitution \([C/P(x_1,\ldots,x_n)]\) relies on the definition of a correct variable substitution \([y_1,\ldots,y_n/x_1,\ldots,x_n]\). Our approach to variable substitutions is rather nonstandard and resembles [Bourbaki, 1968]. But it is convenient from the technical viewpoint, see Section 2.3 for further details.
2.2 Formulas

The expansion of a propositional language to a first-order language is defined in a standard way.

Let \( \mathit{Var} = \{v_1, v_2, \ldots \} \) and \( \mathit{PL}^n = \{P_i^n \mid i \geq 0\} \) be disjoint countable sets. The elements of \( \mathit{Var} \) and \( \mathit{PL}^n \) are respectively called (individual) variables, and \( n \)-ary predicate letters.\(^4\) An atomic formula without equality is either \( \perp \), or \( P_0^n \) (a proposition letter), or \( P_i^n(x_1, \ldots, x_n) \) for some \( n > 0 \), \( x_1, \ldots, x_n \in \mathit{Var} \). Atomic formulas with equality can also be of the form \( x = y \), where \( x, y \in \mathit{Var} \) and \( '=' \) is an extra binary predicate letter.\(^5\) Note that our basic language does not include constants or function letters; we will return to these matters in Volume 2. Also note that the language is countable, but we shall consider its uncountable expansions with constants.

Classical (or intuitionistic) predicate formulas (with or without equality) are built from atomic formulas using the propositional connectives \( \land, \lor, \rightarrow \), and the quantifiers \( \forall, \exists \); in \( N \)-modal predicate formulas\(^6\) the unary connectives \( \Box_i \), \( 1 \leq i \leq N \), can also be used. The abbreviations \( \neg A \), \( T \), \( A \equiv B \), \( \Diamond_i A \) have the same meaning as in the propositional case; \( x \neq y \) abbreviates \( \neg (x = y) \).

For a formula \( A \), a list of variables \( x = x_1 \ldots x_n \) and a quantifier \( \mathit{Q} \in \{\forall, \exists\} \), \( \mathit{Q}xA \) denotes \( \mathit{Q}x_1 \ldots \mathit{Q}x_n A \).

**Definition 2.2.1** The (modal) degree \( d(A) \) of a modal predicate formula \( A \) is defined by induction:

\[
\begin{align*}
  d(A) &= 0 & \text{for } A \text{ atomic;} \\
  d(A \land B) &= d(A \lor B) &= d(A \supset B) &= \max(d(A), d(B)); \\
  d(\forall x A) &= d(\exists x A) &= d(A); \\
  d(\Box_i A) &= d(A) + 1.
\end{align*}
\]

So \( d(A) = 0 \) iff \( A \) is a classical formula.

\( \mathit{AF} \), \( \mathit{IF} \), \( \mathit{MF}_N \) denote respectively the sets of atomic, intuitionistic, and \( N \)-modal formulas without equality; the corresponding sets of formulas with equality are denoted by \( \mathit{AF}^= \), \( \mathit{IF}^= \), \( \mathit{MF}_N^= \); we omit the subscript \( N \) if \( N = 1 \) or if \( N \) is clear from the context. Sometimes we write \( \mathit{IF}(=), \mathit{MF}_N(=) \), etc. in order to combine the cases with and without equality in a single statement. The set \( \mathit{MF}_N(=) \) is also denoted by \( \mathit{L}_N(=) \) and called the (basic) \( N \)-modal first-order language; \( \mathit{L}_0(=) \) is (basic) classical (or intuitionistic) first-order language.

We skip a routine proof of the following standard statement (cf. slightly different versions in [Shoenfield, 1967], [Bourbaki, 1968]).

**Lemma 2.2.2 (Parsing lemma)** Let \( A, B, A', B' \) be formulas, \( *, *' \in \{\lor, \land, \supset\} \) binary connectives such that \( A * B = A' *' B' \). Then \( A = A' \), \( *=*' \), and \( B = B' \).

---

\(^4\)We also use the symbols \( P, Q, R, \ldots \) (sometimes with subscripts) as names of predicate letters; \( p, q, r, \ldots \) as names of proposition letters and \( x, y, \ldots \) as names of variables.

\(^5\)'=' is also used as a metasymbol.

\(^6\)To avoid confusion, we use the letter \( N \) instead of \( n \), because in many cases \( n \) denotes the number of variables in a list.
The definition of free and bound occurrences of variables in formulas is well-known, but we shall now give it in a more formal way.

**Definition 2.2.3** An occurrence of a letter $c$ in a word $\alpha$ is a triple $(\alpha, i, c)$ such that $c$ is the $i$th letter of $\alpha$; an occurrence of a word $\beta$ in $\alpha$ is a triple $(\alpha, i, \beta)$ such that $\beta$ is a subword of $\alpha$ starting from the $i$th letter of $\alpha$.

Free and bound occurrences of variables in formulas are now defined by induction.

**Definition 2.2.4**

- All variable occurrences in atomic formulas are free.
- If $(A, i, x)$ is free (bound), then $(\Box_j A, i + 1, x)$ is free (bound).
- Let $C = (A \ast B)$, where $\ast$ is a binary connective, $|A| = |B|$. If $(A, i, x)$ is free (bound), then $(C, i + 1, x)$ is free (bound). If $(B, i, x)$ is free (bound), then $(C, i + l + 2, x)$ is free (bound).
- Let $B = \forall y A$ or $\exists y A$. If $(A, i, x)$ is free (bound) and $x \neq y$, then $(B, i + 2, x)$ is free (bound). All occurrences of $y$ in $B$ are bound.

The occurrence $(B, 2, y)$ is called strongly bound.

$FV(A)$ denotes the set of parameters (free variables) of a formula $A$, i.e. of all variables having free occurrences in $A$. A closed formula (or a sentence) is a formula without parameters. $M_{\Box}^{S_N}$ (respectively, $I_{\forall}^{S_N}$) denotes the set of all $N$-modal (respectively, intuitionistic) sentences.

$BV(A)$ denotes the set of bound variables of $A$ (i.e. variables having bound occurrences in $A$). We also use the notation

$$BV^-(A) := BV(A) - FV(A), \quad V(A) := BV(A) \cup FV(A).$$

A variable is called new for $A$ if it does not occur in $A$.

Recall that a universal closure of a formula is usually understood as the result of the universal quantification over all its parameters. But such a definition is a priori ambiguous. To fix a unique notation for the universal closure, one can take the parameters in a certain order, for example as follows.

**Definition 2.2.5** The standard list of parameters of a formula $A$ is the set of its parameters $FV(A)$ ordered in accordance with their first occurrences in $A$. The standard universal closure $\forall^A$ of a formula $A$ is the sentence $\forall x A$, where $x$ is the standard list of parameters of $A$. For any ordering $y = y_1 \ldots y_n$ of $FV(A)$ the sentence $\forall y A$ is called a universal closure of $A$.

The set of all universal closures of formulas from a set $\Gamma$ is denoted by $\overline{\Gamma}$.

Note that in all the logics considered in this book the universal closures of the same formula are always equivalent, so we can deal only with standard universal closures, i.e. with $\{ \forall^A \mid A \in \Gamma \}$ instead of $\overline{\Gamma}$. 
For a subformula $QxB$ of a formula $A$ (where $Q$ is a quantifier), every free occurrence of $x$ in $B$, as well as the first occurrence of $x$, is called referent to the first occurrence of $Q$. More precisely:

**Definition 2.2.6** Let $(A, i, QxB)$ be an occurrence of $QxB$ in $A$. If $(B, j, x)$ is a free occurrence of $x$ in $B$, then $(A, i + j + 2, x)$ is an occurrence of $x$ referent to $(A, i, Q)$; $(A, i + 1, x)$ is also referent to $(A, i, Q)$. All variable occurrences in $A$ referent to the same occurrence of a quantifier are called coreferent (or correlated). We also say that an occurrence of a quantifier binds all referent occurrences of variables.

Now the reference structure of a formula $A$ can be defined as a function sending every bound occurrence of a variable in $A$ to the occurrence of a binding quantifier. But the following definition is more convenient.

**Definition 2.2.7** Let $A$ be a formula of length $n$. The reference function of $A$ is a function $rf_A$ defined on the set

$$\{ i \mid 1 \leq i \leq n \& (A, i, x) \text{ is a bound variable occurrence for some } x \}$$

such that $rf_A(i) = j$ whenever $(A, i, x)$ is referent to $(A, j, Q)$ (for some variable $x$ and a quantifier $Q$).

So $rf_A$ sends positions of bound variables to positions of their binding quantifiers.

**Definition 2.2.8** Let $\bullet$ be a new symbol, regarded as an extra variable ('joker'). The stem of a formula $A$ is a formula $A^-$ obtained by replacing every bound occurrence of every variable in $A$ with $\bullet$. The scheme of $A$ is a pair $\mathcal{A} := (A^-, rf_A)$.

Thus $BV(A^-) \subseteq \{\bullet\}$, $FV(A^-) = FV(A)$.

Reference functions can be represented graphically, by connecting occurrences of quantifiers with referent variable occurrences. For example, for

$$A := \forall x(P(x) \lor \exists yQ(y, x))$$

the reference function is pictured as follows:

$$\forall x(P(x) \lor \exists yQ(y, x))$$

and the scheme as follows:

$$\forall \bullet (P(\bullet) \lor \exists \bullet Q(\bullet, \bullet))$$

(Note that the stem itself has a different reference function!)
**Remark 2.2.9** Instead of this graphic representation, one can number quantifier occurrences and add corresponding superscripts to referent variable occurrences, cf. [Kleene, 1967].

Alternatively, the scheme of a formula can be defined by induction as follows.

**Definition 2.2.10**

- \( \mathcal{A} := A \) for \( A \) atomic;
- \( (A \ast B) := (A \ast B) \) for \( \ast \in \{\lor, \land, \Rightarrow\} \);
- \( \Box \varepsilon \mathcal{A} := \Box \varepsilon \mathcal{A} \);
- \( \Box \varepsilon \mathcal{A} \) (for \( \varepsilon \in \{\forall, \exists\} \)) is obtained from \( \Box \varepsilon \mathcal{A} \) by replacing every occurrence of \( x \) with \( \varepsilon \) and connecting it with the first occurrence of \( \varepsilon \).

**Remark 2.2.11** Strictly speaking, this definition shows us how to reconstruct a graphic representation of \( \mathcal{A} \). In terms of functions, it means that \( rf_{(A \ast B)} \) is the union of \( rf_A \) and \( rf_B \) with some shifts, and so on. Such an explication does not seem useful however.

Schemes can also be defined by induction without appealing to formulas:

**Definition 2.2.12**

- Every atomic formula is a scheme.
- If \( S_1, S_2 \) are schemes, then \( (S_1 \ast S_2) \) is a scheme.
- If \( S \) is a scheme, then \( \Box \varepsilon S \) is a scheme.
- If \( S \) is a scheme, \( x \in \text{Var} \), then there is a scheme \( \Box \varepsilon S \) obtained from \( \Box \varepsilon S \) by replacing every occurrence of \( x \) with \( \varepsilon \) and connecting it with the first occurrence of \( \varepsilon \).

It is obvious (a strict proof is by induction) that for any scheme \( S \) in the sense of this definition, there is a formula \( A \) such that \( S = \mathcal{A} \). So in our syntax we can deal with schemes rather than formulas. In a systematic way this approach is developed in [Bourbaki, 1968]. One can even argue that schemes better correspond to human intuition about first-order logic. But in our book, we prefer to keep to the traditional notion of a formula and use schemes only incidentally, for technical purposes.

---

7 With minor differences — instead of quantifiers Bourbaki uses the \( \varepsilon \)-symbol (denoted by \( \tau \)) and defines \( \exists \varepsilon A \) as an abbreviation for \( rf_{(A/\varepsilon \varepsilon)} \).

8 Natural language does not use bound variables and hides them in the reference structure, cf. the sentence *Every triangle has at least two acute angles.*
2.3 Variable substitutions

It is well-known that a 'logically correct' variable substitution is not a simple replacement of variables, due to possible variable collisions. For example, if \( A = \exists y (x \neq y) \), and we want the formula \( \forall x A \supset [y/x]A \) to be (classically) valid; it is incorrect to define \([y/x]A\) as \( \exists y (y \neq y) \). Many authors consider such substitutions as 'bad' and simply do not allow them; formally, \([y/x]A\) is a 'good' substitution if free occurrences of \( x \) are not within the scope of any quantifier over \( y \); as they say, \( y \) is free for \( x \) in \( A \), cf. [Kleene, 1952], [Kleene, 1967], [Mendelson, 1997].

But this restriction does not help for defining formula substitutions, because e.g.
\[
[\exists y Q(x, y) / P(x)]P(y)
\]
should be
\[
[y/x] \exists y Q(x, y),
\]
and the latter substitution \([y/x]\) is 'bad'.

A well-known way to solve the problem is renaming of bound variables, cf. [Kleene, 1963]. For example, in the formula \( \exists y (x \neq y) \) we can rename the bound \( y \) by \( z \) and define \([y/x](\exists y (x \neq y))\) as \( \exists z (y \neq z) \). But this variable \( z \) can be chosen in many different ways — it can be arbitrary except for the original \( y \). If the result of a substitution should be unique, we have to fix one of these options. For example, we can always take the first variable allowed for renaming (in the list of all variables) [Kolmogorov and Dragalin, 2005]. But this definition is technically inconvenient and rather unnatural, because the alphabetical order of variables is not related to logic at all.

So we propose another approach. A substitution is considered as a relation not function; the result of a substitution is unique only up to congruence (correct renaming of bound variables).\(^9\) We regard \([y/x]A\) not as a single formula, but as a member of a certain class of formulas. This resembles the well-known mathematical notation \( \int f(x)dx \) of a primitive function, which is unique up to adding a constant. Using schemes instead of formulas simplifies all the details: two formulas are congruent if they have the same scheme.

A formula is called clean if all its quantifiers bind different variables and none of its bound variables is free. Every formula \( A \) can be transformed into an equivalent clean formula \( A^\circ \), without bound occurrences of \( x \) or \( y \). Then we define \([y/x]A\) as \([y/x]A^\circ\). Since there are no variable collisions, the latter substitution is made by a straightforward replacement.

Now we pass to the details. Let \( x = (x_1, \ldots, x_n) \) be a list of variables \((n \geq 0)\).\(^{10}\) Later on we use the following notation:

---

\(^9\)Congruence corresponds to \( \alpha \)-equivalence in \( \lambda \)-calculi.

\(^{10}\)If there is no confusion, we also denote the list \((x_1, \ldots, x_n)\) by \( x_1 \ldots x_n \).
CHAPTER 2. BASIC PREDICATE LOGIC

\( r(x) \) for the set \( \{x_1, \ldots, x_n\} \);
\( z \in x \) for 'z occurs in x', i.e. for \( z \in r(x) \);
\( xy \) for the concatenation of the lists \( x, y \);
\( x \cap S \) for 'the sublist of \( x \) containing the elements of \( S \)';
\( x - S \) for 'the sublist of \( x \) obtained by removing the elements of \( S \)', etc.

A list is said to be distinct if all its members are different.

If \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \) are lists of variables and all \( x_1, \ldots, x_n \) are distinct, we define the variable substitution \( [y/x] \) as the following function \( \text{Var} \rightarrow \text{Var} \):

\[
[y/x](z) = \begin{cases} 
y_i & \text{if } z = x_i; 
z & \text{if } z \notin x.
\end{cases}
\]

Note that the same definition can be given in the case when some of the \( x_i \) are equal, but \( x \) sub \( y \).

Of course a variable substitution is nothing but a function \( \text{Var} \rightarrow \text{Var} \) that changes only finitely many variables. So a composition of substitutions is a substitution.

For a list of variables \( z = z_1 \ldots z_m \), put

\[
[y/x]z := [y/x](z_1) \ldots [y/x](z_m).
\]

We also use the dummy substitution \( [] \), which is the identity function on \( \text{Var} \).

**Definition 2.3.1** A variable transformation of a (distinct) list \( x \) to \( y \) is the finite function (the set of pairs) \( \{(x_1, y_1), \ldots, (x_n, y_n)\} \); this function is denoted by \( [x \mapsto y] \).

If a transformation \( [x \mapsto y] \) is bijective (i.e. \( y \) is distinct), it is called a variable renaming. If also \( V(A) \subseteq r(x) \) for a certain formula \( A \), \( [x \mapsto y] \) is called a variable renaming in \( A \). A bound variable renaming in \( A \) is a variable renaming in \( A \) fixing all parameters of \( A \).

Note that \( [x \mapsto y] \) is a variable renaming iff the corresponding substitution \( [y/x] \) is a permutation of \( \text{Var} \).

For a set of variables \( S \), \( [x \mapsto y]_S \) denotes the restriction of \( [x \mapsto y] \) to \( r(x) \cap S \). We also use the abbreviations

\[
[x \mapsto y]_{FV(A)} \quad \text{and} \quad [x \mapsto y]_{r(x) \setminus \{z\}}.
\]

A transformation \( [x \mapsto y] \) is called proper if \( x_i \neq y_i \) for every \( i \). We say that a transformation \( [x \mapsto y] \) represents the substitution \( [y/x] \).\(^{12}\) Obviously, every substitution is represented by infinitely many transformations, but only one of them is proper.

It is easy to describe how a transformation \( [x \mapsto y] \) acts on (modal) predicate formulas. Let \( A[x \mapsto y] \) be the result of simultaneous replacement of all (both

\(^{11}\)This notation is only occasional.

\(^{12}\)Sometimes transformations are also called 'substitutions', but we avoid this terminology.
2.3. VARIABLE SUBSTITUTIONS

free and bound) occurrences of $x_i$ in $A$ with $y_i$ (for $i = 1, \ldots, n$). We can say that $A[x \leftrightarrow y]$ is obtained from $A$ by a 'straightforward' renaming of variables.\footnote{The notation $[y/x]A$ is reserved for a 'correct' variable substitution with renaming of bound variables, see below.}

**Lemma 2.3.2** Let $[x \leftrightarrow y']$, $[y \leftrightarrow z]$ be variable transformations such that $y' = y \cdot \sigma$ for some surjection $\sigma$. Then

1. $[x \leftrightarrow y'] \circ [y \leftrightarrow z] = [x \leftrightarrow z \cdot \sigma]$;
2. for any predicate formula $A$ such that $V(A) \subseteq r(x)$

\[
(A[x \leftrightarrow y'])[y \leftrightarrow z] = A[x \leftrightarrow z \cdot \sigma].
\]

In particular, $(A[x \leftrightarrow y])[y \leftrightarrow x] = A$ if $[x \leftrightarrow y]$ is a variable renaming in $A$.

**Proof**

1. Trivial: $[x \leftrightarrow y']$ sends $x_i$ to $y_{\sigma(i)}$, and next $[y \leftrightarrow z]$ sends $y_{\sigma(i)}$ to $z_{\sigma(i)}$.
2. Also obvious, formally, one should argue by induction on the length of $A$.

In a more general situation the following holds (the proof is similar).

**Lemma 2.3.3** Let $[x \leftrightarrow y']$, $[y \leftrightarrow z]$ variable transformations such that $r(y') \subseteq r(y)$. Then for any predicate formula $A$,

\[
(A[x \leftrightarrow y'])[y \leftrightarrow z] = A([x \leftrightarrow y'] \circ [y \leftrightarrow z]).
\]

The formulas $A$ and $A[x \leftrightarrow y]$ may be not logically equivalent in classical logic. For example, this is the case for

\[
A = \exists x_1 (P(x_1) \land \neg P(y_1)) \text{ and } A[x_1 \leftrightarrow y_1] = \exists y_1 (P(y_1) \land \neg P(y_1)).
\]

Later on we will describe 'admissible' transformations that do not affect the truth values of formulas, cf. Lemma 2.3.19.

**Lemma 2.3.4** Let $A$ be a predicate formula, $[x \leftrightarrow y]$ a variable renaming in $A$. Then

1. free (respectively, bound) occurrences of variables in $A$ correspond to free (bound) occurrences of variables in $A[x \leftrightarrow y]$:

\[
(A, i, x_j) \text{ is free (bound) iff } (A[x \leftrightarrow y], i, y_j) \text{ is free (bound)};
\]
2. $rf_A = rf_{A[x \leftrightarrow y]}$.

**Proof** By induction. We denote $A[x \leftrightarrow y]$ by $A'$.

- If $A$ is atomic, then $A'$ is atomic, so all variable occurrences are free, and the claim is trivial.
• If \( A = (B \ast C) \), then \( A' = (B' \ast C') \). Variable occurrences in \( A \) coming from \( B \) are of the form \((A, i + 1, x_j)\) where \((B, i, x_j)\) is an occurrence in \( B \). Then

\[
(A, i + 1, x_j) \text{ is free (bound) iff } \quad (B, i, x_j) \text{ is free (bound) (by Definition 2.2.4) iff } \quad (B', i, y_j) \text{ is free (bound) (by induction hypothesis) iff } \quad (A', i + 1, y_j) \text{ is free (bound) (by 2.2.4)}.
\]

Moreover, a bound occurrence \((A, i + 1, x_j)\) is referent to a quantifier occurrence \((A, k + 1, Q)\) iff \((B, i, x_j)\) is referent to \((B, k, Q)\):

\[
A = (\ldots Qx_j \quad \overbrace{\ldots x_j \quad \ldots}^{D} \quad * \quad C)
\]

In fact, according to 2.2.5, this happens iff either \( i = k + 1 \) or this occurrence of \( x_j \) in \( A \) comes from a free occurrence in a subformula \( D \) starting at position \( k + 3 \) (more precisely, \( A, k + 1, Qx_jD \) is an occurrence in \( A \) and \( (D, i - k - 1, x_j) \) is a free occurrence in \( D \)).

So

\[
rf_A(i + 1) = k + 1 \iff rf_B(i) = k.
\]

By the same reason,

\[
rf_{A'}(i + 1) = k + 1 \iff rf_{B'}(i) = k.
\]

Since \( rf_B = rf_{B'} \) by induction hypothesis, it follows that

\[
rf_A(i + 1) = rf_A'(i + 1).
\]

A similar argument applies to variable occurrences in \( C \); we leave formal details to the reader.

• The case \( A = \Box_k B \) is also left to the reader.

• If \( A = Qx_kB \), then \( A' = Qy_kB' \).

Now if \( j \neq k \), then as above, we obtain:

\[
(A, i + 2, x_j) \text{ is free (bound) iff } (B, i, x_j) \text{ is free (bound) iff } (B', i, y_j) \text{ is free (bound) iff } (A', i + 2, y_j) \text{ is free (bound)}.
\]

Note that \( y_j \neq y_k \), since \([x \mapsto y]\) is a bijection.

A bound occurrence \((A, i + 2, x_j)\) is referent to \((A, k + 2, Q_1)\) iff \((B, i, x_j)\) is referent \((B, k, Q_1)\):

\[
A = Qx_j \quad \overbrace{\ldots Q_1x_j \quad \ldots x_j \quad \ldots}^{D} \quad * \quad C
\]
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So
\[ rf_{A}(i + 2) = k + 2 \iff rf_{B}(i) = k, \]
and similarly,
\[ rf_{A'}(i + 2) = k + 2 \iff rf_{B'}(i) = k. \]
Hence
\[ rf_{A}(i + 2) = rf_{A'}(i + 2). \]
And if \((A, i, x_{k})\) is an arbitrary occurrence, it is bound, as well as \((A', i, y_{k})\). Then
\[ rf_{A}(i) = 1 \iff rf_{A'}(i) = 1. \]

Definition 2.3.5 Two predicate formulas \(A, B\) are called congruent (notation: \(A \cong B\)) if they have the same scheme.

Definition 2.3.6 Let \(A\) be a predicate formula. A bound variable renaming in \(A\) is a variable renaming in \(A\) fixing all parameters of \(A\).

So this is a bijective transformation \([x \mapsto y]\) such that \(V(A) \subseteq r(x)\) and \(x_{i} = y_{i}\) for \(x_{i} \in FV(A)\).

Definition 2.3.7 We say that \(B\) is strongly congruent to \(A\) (notation: \(A \equiv B\)) if there is a bound variable renaming \([x \mapsto y]\) in \(A\) such that \(B = A[x \mapsto y]\).

Lemma 2.3.2 shows that \(\equiv\) is an equivalence relation on formulas.

Lemma 2.3.8 If \([x \mapsto y]\) is a bound variable renaming in \(A\), then \(A \equiv A[x \mapsto y]\). Thus strong congruence implies congruence.

Proof By Lemma 2.3.4 the reference functions coincide. As for the stems, \(A\) and \(A'\) may differ only in bound variables. Since \(x_{i} = y_{i}\) for \(x_{i} \in FV(A)\), \(A\) and \(A' = A[x \mapsto y]\) also differ only in bound variables.

Now consider bound variable occurrences. We have:
- Occurs in \(A'\) at position \(j\) iff some \(x_{i}\) occurs in \(A\) at position \(j\) iff some \(y_{i}\) occurs in \(A'\) at position \(j\) (by 2.3.4) iff
- Occurs in \((A')^{-}\) at position \(j\).

Thus \(A^{-} = (A')^{-}\), and eventually \(A \equiv A'\).

Lemma 2.3.9

(1) If \(A = (B * C) \equiv A'\), then \(A' = (B' * C')\) for some \(B' \equiv B\), \(C' \equiv C\).

(2) If \(A = QyB \equiv A'\) and \(y \notin BV(B)\), then for some variable \(z \notin FV(B)\) and formula \(B'\), \(A = QzB' \text{ and } B' \equiv B[y \mapsto z]\).
Proof

(1) Let \( A = (B \ast C) \cong A' \). Since \( A' \) begins with \( (, \) it must have the form \((B' \ast C')\). Then \( A_\ast = (B_\ast \ast C) \), and \( A'_\ast = (B'_\ast \ast C') \), hence \( B_\ast = B' \), \* = \*, \( C_\ast = C' \) by 2.2.2 (or strictly speaking, 2.2.2 implies the coincidence of the corresponding stems; the reference functions coincide, because they are inherited from \( A \) and \( A' \)).

(2) Assume \( A = \mathcal{Q}yB \cong A' \) and \( y \notin BV(B) \). By definition, \( A \) is obtained from \( \mathcal{Q}yB \) by replacing \( y \) with \( \bullet \) and adding connections to the first \( \mathcal{Q} \). If \( A = A' \), it follows that \( A' \) has the form \( \mathcal{Q}zB' \) (otherwise \( A' \) does not begin with \( \mathcal{Q} \)) and

\[
B'[y \mapsto z],
\]

where \( B'[y \mapsto z] \) means \( B^-[y \mapsto z], r_fB \), and also \( z \notin FV(B) \). Since \( y \notin BV(B) \), \( r_fB = r_fB[y \mapsto z] \). It is also clear that \( B^-[y \mapsto z] = (B[y \mapsto z])^-[ \) (a formal proof is routine by induction); therefore

\[
B'[y \mapsto z] = B[y \mapsto z].
\]

From \((\#)\), \((\#\#)\) we have

\[
B' \cong B[y \mapsto z].
\]

Definition 2.3.10 A formula \( A \) is called clean if there are no variables both free and bound in \( A \) and the number of occurrences of quantifiers equals the number of bound variables, i.e. different occurrences of quantifiers in \( A \) bind different variables in \( A \).

This is equivalent to the following inductive definition:

Definition 2.3.11

- Every atomic formula is clean.
- If formulas \( A, B \) are clean, \( \ast \in \{\land, \lor, \supset\} \), and \( BV(A) \cap V(B) = BV(B) \cap V(A) = \emptyset \), then \( (A \ast B) \) is clean.
- If \( A \) is clean, \( x \in Var \), and \( x \notin BV(A) \), then \( \mathcal{Q}xA \) is clean.

Lemma 2.3.12 If \([x \mapsto y]\) is variable renaming in a clean formula \( A \), then \( A[x \mapsto y] \) is clean.

Proof  By induction. Again we denote \( A[x \mapsto y] \) by \( A' \). If \( A \) is atomic, then \( A' \) is atomic. In this case, if \([x \mapsto y]\) is bound for \( A \), then \( A' = A \).

Suppose \( A = (B \ast C) \), \( B, C \) are clean,

\[
BV(B) \cap V(C) = BV(C) \cap V(B) = \emptyset.
\]
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$B'$, $C'$ are clean by induction hypothesis. By Lemma 2.3.4,

$$BV(B') = \{ u_j \mid x_j \in BV(B) \}, \quad V(C') = \{ y_j \mid \in V(C) \}.$$  

Hence

$$BV(B') \cap V(C') = \emptyset.$$  

Similarly we obtain

$$BV(C') \cap V(B') = \emptyset,$$

and thus $A' = (B' \ast C')$ is clean.

Suppose $A = Qx_i B$, $B$ is clean, $x_i \notin BV(B)$. Then $A' = Qy_i B'$, and by induction hypothesis, $B'$ is clean. By Lemma 2.3.4, $y_i \notin BV(B')$. Thus $A'$ is clean by 2.3.11.

Lemma 2.3.13 Let $A$ be a clean formula, $[x \mapsto y]$ a variable transformation such that $BV(A) \cap r(xy) = \emptyset$. Then $A[x \mapsto y]$ is clean.

Proof By induction, similar to the previous lemma. We leave this as an exercise for the reader.

Lemma 2.3.14 If $A \equiv A'$ and $A, A'$ are clean, then $A \equiv A'$. 

Proof By induction on the complexity of $A$, we prove the claim for any $A'$.

- If $A$ is atomic and $A \equiv A'$, then $A' = A$, and the claim is trivial.

- If $A = (B \ast C)$, then by 2.3.9(1), $A' = (B' \ast C')$ for $B \equiv B'$, $C \equiv C'$. So $B \equiv B'$ and $C \equiv C'$ by the induction hypothesis, i.e.

$$B' = B[x \mapsto y], \quad C' = C[z \mapsto t]$$

for some bound variable renamings $[x \mapsto y]$, $[z \mapsto t]$. Obviously, we may assume that $r(x) = V(B)$, $r(z) = V(C)$. Then $[x \mapsto y] \cup [z \mapsto t]$ is a function. In fact, the lists $x, z$ do not have common bound variables, since $A$ is clean. Both functions $[x \mapsto y], [z \mapsto t]$ accord on free variables as they fix them.

The same argument shows that $[y \mapsto x] \cup [t \mapsto z]$ is a function, and thus $[x \mapsto y] \cup [z \mapsto t]$ is bijective. So we obtain a bound variable renaming that transforms $A$ into $A'$.

- If $A = QyB$ is clean, $y$ does not have bound occurrences in $B$. Then by Lemma 2.3.9(2), $A' = QzB'$, for $B' \equiv B[y \mapsto z]$. Note that $B'$ is clean as a subformula of $A'$, $B$ is clean as a subformula of $A$, $B[y \mapsto z]$ is clean by 2.3.12 $([y \mapsto z]$ becomes a variable renaming if we prolong it in a trivial way — as the identity function on $V(B) - \{y\}$). So by the induction hypothesis

$$B' \equiv B[y \mapsto z],$$
for some bound variable renaming \([x \mapsto u]\) in \(B'\). Since \(A'\) is clean, \(z \not\in BV(B')\), so \([x \mapsto u]\) fixes \(z\).

Thus

\[(QzB')[x \mapsto u] = Qz(B[y \mapsto z]).\]  

(1)

Hence by applying \([z \mapsto y]\) we obtain

\[(QzB')([x \mapsto u] \circ [z \mapsto y]) = QyB.\]  

(2)

Note that

\([x \mapsto u] \circ [z \mapsto y] = [(x - z)z \mapsto (u - z)y]\)

is a bound variable renaming in \(B'\), so (2) implies

\[A' = QzB' \equiv QyB = A.\]

- If \(A = \Box_i B \equiv A'\), then \(A' = \Box_i B'\), and \(B \equiv B'\). By induction hypothesis, \(B \equiv B'\), which easily implies \(A \equiv A'\).

\[\text{Proposition 2.3.15} \quad \text{For any clean formula } A,\]

\[\equiv (A) = \equiv (A) \cap \text{(clean formulas)}.\]

\[\text{Proof} \quad \text{Immediate from 2.3.12, 2.3.14, 2.3.8.}\]

\[\text{Proposition 2.3.16} \quad \text{Every predicate formula } A \text{ is congruent to some clean formula (called a clean version of } A).\]

\[\text{Proof} \quad \text{By induction on the complexity of } A.\]

- If \(A\) is atomic, it is already clean.

- If \(A = \Box_i B\) and \(B \equiv B_0\) for a clean \(B_0\), then obviously \(A \equiv \Box_i B_0\) and \(\Box_i B_0\) is clean.

- If \(A = (B * C)\) and \(B \equiv B_0\), \(C \equiv C_0\) for clean \(B_0, C_0\), then \(A \equiv (B_0 * C_0)\). \((B_0 * C_0)\) may be not clean, but we can make it clean by an appropriate bound variable renaming.

In fact, let \(BV(B_0) = r(x)\), and let \([x \mapsto y]\) be a bijection such that \(r(y) \cap V(C_0) = \emptyset\). then \(B_1 := B_0[x \mapsto y]\) is clean by Lemma 2.3.12, and

\[BV(B_1) \cap V(C) = \emptyset.\]

Similarly there exists \(C_1 \equiv C_0\) such that

\[BV(C_1) \cap V(B_1) = \emptyset.\]
Since
\[ BV(B_1) \cap FV(C_1) = \tau(y) \cap FV(C_0) = \emptyset, \]
by 2.3.11, it follows that \( (B_1 * C_1) \) is clean.

\[ B_1 \is B_0 \] and \( C_1 \is C_0 \) implies \( (B_1 * C_1) \is (B_0 * C_0) \is A. \)

- If \( A = QxB \) and \( B \is B_0 \) for a clean \( B_0 \), there are two cases.
  
  If \( x \in FV(B) \), then also \( x \in FV(B_0) \) (since free variable occurrences in a formula are at the same positions as in its stem, and \( B^- = B_0^- \)). Thus \( x \notin BV(B_0) \), so \( QxB_0 \) is clean. Now

\[ A = QxB \is QxB_0 = QxB_0, \]

so \( A \is QxB_0 \).

Finally, if \( x \notin FV(B) \), then also \( x \notin FV(B_0) \). However, it may be that \( x \in BV(B_0) \).

So we rename \( x \) into a new variable \( y \notin V(B_0) \). Then \([x \mapsto y]\) can be prolonged to a bound variable renaming in \( QxB_0 \); thus \( QxB_0 \is Qy(B_0[x \mapsto y]) \) by 2.3.8, and \( B_0[x \mapsto y] \) is clean by 2.3.12. Hence the formula \( Qy(B_0[x \mapsto y]) \) is clean by 2.3.11, and we have proved that it is congruent to \( A \).

The next proposition gives us a convenient inductive definition of congruence that does not appeal to schemes.

**Proposition 2.3.17** Congruence is the smallest equivalence relation \( \sim \) between \( N \)-modal predicate formulas with the following properties:

1. \( QxA \sim Qy(A[x \mapsto y]) \) for \( x \notin BV(A), \ y \notin V(A), \ Q \in \{ \forall, \exists \}; \)

2. \( A \sim B \Rightarrow QxA \sim QxB \) for \( Q \in \{ \forall, \exists \}; \)

3. \( A \sim A' \land B \sim B' \Rightarrow (A \ast B) \sim (A' \ast B') \) for \( \ast \in \{ \lor, \land, \lor \}; \)

4. \( A \sim B \Rightarrow \Box_i A \sim \Box_i B \) for \( i \in I_N \).

**Proof** First note that congruence really has properties (1)-(4). (2), (3), (4) follow from the inductive definition of schemes 2.2.12. (1) follows from 2.3.8. In fact, let \( z \) be the list of all other variables occurring in \( A \), then \([xz \mapsto yz]\) is a bound variable renaming in \( A \), and it transforms \( QxA \) into \( Qy(A[x \mapsto y]) \). So these formulas are congruent.

Now consider an arbitrary equivalence relation \( \sim \) satisfying (1)-(3) and show that congruent formulas are \( \sim \)-related.

So we prove that for any \( B \)

5. \( A \is B \Rightarrow A \sim B \)
by induction on \(|A|\).

If \(A\) is atomic, then obviously \(A \cong B\) implies \(A = B\), so the claim is trivial.

If \(A = (A_1 \ast A_2)\), then as we saw in the proof of 2.3.14, for some \(B_1 \cong A_1, B_2 \cong A_2\) we have \(B = (B_1 \ast B_2)\).

Then by induction hypothesis, \(A_i \sim B_i\), and thus \(A \sim B\) by (3).

We skip a similar simple case when \(A = \Box_i A'\).

Finally, suppose \(A = Qx C \cong B\). Since \(A = B\), it follows that \(B = Qy D\) for some \(D, y\).

By Proposition 2.3.16, \(C\) is congruent to a clean formula \(C_0\). Next, by a bound variable renaming (Lemmas 2.3.8, 2.3.12) we transform \(C_0\) into a congruent clean formula \(C_1\) such that \(x, y \not\in BV(C_1)\). Since \(|C| < |A|\), we have \(C \sim C_1\) by the induction hypothesis, and hence \(A = QxC \sim QxC_1\) by (2).

We also have
\[
QyD \cong QxC \cong QxC_1,
\]
where the latter readily follows from \(C \cong C_1\).

Since congruent formulas have the same occurrences of parameters, we obtain that \(y \not\in FV(QxC_1)\), and thus \(y \not\in V(QxC)\) by the choice of \(C_1\). Hence
\[
QyD \cong QxC_1 \cong Qy(C_1[x \mapsto y]),
\]
which implies
\[
D \cong C_1[x \mapsto y].
\]

By induction hypothesis,
\[
D \sim C_1[x \mapsto y],
\]
So
\[
B := QyD \sim QyC_1[x \mapsto y]
\]
by (2).

On the other hand, by (1),
\[
QxC_1 \sim QyC_1[x \mapsto y],
\]
and we already know that \(A \sim QxC_1\). Therefore \(A \sim B\).

The characterization given in Proposition 2.3.17 resembles the definition of \(\alpha\)-equivalence in \(\lambda\)-calculi, cf. ... It is worth noting that there exists an equivalent description of congruence (or \(\alpha\)-equivalence) via variable swaps, cf. [Gabbay and Pitts, 2002]:

Lemma 2.3.18 Congruence is the smallest equivalence relation \(\sim\) between \(N\)-modal predicate formulas with the following properties:

(1') \(A \sim B\) iff \(B\) is obtained from \(A\) by replacing some of its subformula \(QxC\) with \(D = Qy(C[xy \mapsto yx])\), where \(y \not\in FV(C)\), \(Q \in \{\forall, \exists\}\),

(2)-(4) from 2.3.17.
Proof  Congruence satisfies (1'), since for \( y \notin FV(C) \),

\[
(QxC)[xyz \mapsto yxz] = Qy(C[xy \mapsto yx]),
\]

where \( z \) is a list of all other variables from \( V(C) \), and thus

\[
QxC = Qy(C[xy \mapsto yx])
\]

by 2.3.12. It also satisfies (2)-(4) as noted in 2.3.17.

The other way round, (1') obviously implies (1) from 2.3.17, since \( A[xy \mapsto yx] = A[x \mapsto y] \) for \( y \notin V(A) \). Thus every equivalence relation satisfying (1'), (2)-(4) contains \( \cong \) by 2.3.17.

To complete the whole picture, let us also give a description of a congruence class of a clean formula in terms of transformations. However this description will not be used in further studies.

Lemma 2.3.19 Let \( A \) be a clean formula. Then every formula congruent to \( A \) can be obtained by a transformation \([x \mapsto y]\) such that

- \( r(x) = BV(A) \);
- for any subformula of \( A \) of the form \( Qx_iB \), where \( Q \) is a quantifier, the following holds:
  1. if \( x_j \in FV(B) \) and \( j \neq i \) then \( y_j \neq y_i \);
  2. if \( y_i \in FV(B) \) then \( y_i \in r(x) \).

Proof  [Sketch] If \( A \cong C \), then all parameters of these formulas are at the same positions.

Since \( A \) is clean, all occurrences of every bound variable \( x_i \) in \( A \) are coreferent and since \( r_{f_A} = r_{f_C} \), in \( C \) they are replaced with the same bound variable \( y_i \). So \( C = A[x \mapsto y] \) for a list \( x \) of bound variables and some \( y \). Now suppose \( Qx_iB \) occurs in \( A \). If \( x_j \) has a free occurrence in \( B \) and \( j \neq i \), this occurrence of \( x_i \) corresponds to an occurrence of \( y_j \) in the formula \( Qy_i(B[x \mapsto y]) \) occurring in \( C \). So \( y_j \neq y_i \), since otherwise \( r_{f_C} \neq r_{f_A} \). Thus (1) holds.

If \( y_i \) has a free occurrence in \( B \) and \( y_i \notin r(x) \), this occurrence remains free in \( B[x \mapsto y] \), and thus referent to the first occurrence of \( Q \) in \( Qy_i(B[x \mapsto y]) \), so \( r_{f_C} \neq r_{f_A} \). Thus (2) holds.

On the other hand, transformations described in the statement of the lemma, preserve the scheme of \( A \).

In fact, any occurrence of \( x_i \) free in a subformula \( B \) of \( Qx_iB \) occurring in \( A \) becomes an occurrence of \( y_i \) in \( B[x \mapsto y] \). This occurrence of \( y_i \) is also free, since otherwise it is referent to another occurrence of a quantifier \( Q' \) over \( y_i \) in \( C \):

\[
C = \ldots Qy_i \overbrace{\ldots Q'y_i \ldots y_i \ldots}^{B[x \mapsto y]} \ldots
\]
But this $Q' y_i$ comes from some occurrence of $Q' x_j$ in $A$, and $j \neq i$, since $A$ is clean. Then

$$A = \ldots Qx_i \ldots Q' x_j x_i \ldots \ldots$$

Now $x_i$ is free in a subformula $D$ of $Q' x_j D$ occurring in $B$, while $y_i = y_j$. This contradicts the condition (1).

Also every occurrence of $y_i$ free in a subformula $E[x \rightarrow y]$ of $Q y_i E[x \rightarrow y]$ occurring in $C$ comes from a free occurrence of $x_i$ in $E$ from a subformula $Q x_i E$ of $A$. In fact, otherwise $y_i \in FV(E)$.

$$C = \ldots Qy_i y_i \ldots \ldots$$

(since $A$ is clean, $x_i \notin \text{BF}(E)$), so $y_i \in r(x)$ by (2).

Then $y_i = x_j$ for $j \neq i$ (since $y_i \in FV(E)$), and $y_i = y_j$, since $[x \rightarrow y]$ fixes $y_i$. This again contradicts (1).

Therefore $A \models A[x \rightarrow y]$.

Now we can define how variable substitutions act on formulas.

**Definition 2.3.20** For a variable transformation $[x \rightarrow y]$ and a scheme $S$, we define $S[x \rightarrow y]$ as the result of replacing every occurrence of $x_i$ in $S$ by $y_i$.

Thus

$$(A[x \rightarrow y] = (A^-[x \rightarrow y], r f_A)).$$

**Lemma 2.3.21** Let $A$ be a formula, $[x \rightarrow y]$ a variable transformation such that $r(xy) \cap BV(A) = \emptyset$. Then

$$A[x \rightarrow y] = A[x \rightarrow y].$$

**Proof** Easy by induction on $|A|$. We consider only the case $A = Q z B$.

$$Q z B[x \rightarrow y] = Q z B[x \rightarrow y] = Q z B[x \rightarrow y],$$

since $z$ does not occur in $x$.

Now by induction hypothesis

$$B[x \rightarrow y] = B[x \rightarrow y],$$

thus

$$Q z B[x \rightarrow y] = Q z B[x \rightarrow y] = Q z (B[x \rightarrow y]).$$

But

$$Q z (B[x \rightarrow y]) = A[x \rightarrow y],$$

again since $z$ does not occur in $x$. Hence the claim follows.
2.3. VARIABLE SUBSTITUTIONS

Definition 2.3.22 Let $A$ be a predicate formula, $[y/x]$ a variable substitution. Then we define $[y/x]A$ as an arbitrary formula $B$ such that $A[x \mapsto y] = B$.

Let us show soundness of this definition.

Lemma 2.3.23 Every predicate formula $A$ has a clean version $A^\circ$ such that $r(xy) \cap BV(A^\circ) = \emptyset$.

Then $[y/x]A \equiv A^\circ[x \mapsto y]$.

Proof To obtain $A^\circ$, take an arbitrary clean version and make an appropriate bound variable renaming. Then

$$A[x \mapsto y] = A^\circ[x \mapsto y],$$

by Lemma 2.3.21.

From the definition we readily obtain

Lemma 2.3.24 $A \equiv A' \Rightarrow [y/x]A \equiv [y/x]A'$.

Note that $[y/x]$ and $[x \mapsto y]$ do not depend on the ordering of $x$; in precise terms, if $|x| = n$, $\sigma \in \Upsilon_n$, then $[x \cdot \sigma \mapsto y \cdot \sigma] = [x \mapsto y]$.

Therefore congruent formulas have the same substitution instances under every variable substitution.

The next lemma contains some simple properties of variable substitutions.

Lemma 2.3.25 For any predicate formula $A$, substitutions $[y/x], [y'/x'], [y''/x''], [x/u], [z/u]$ and quantifier $Q$

1. $FV([y/x]A) = (FV(A) - r(x)) \cup \text{rng}[x \mapsto y]A$;
2. $[x/x]A \equiv A$;
3. $Q_y[y/x]A \equiv Qx A$ if $y \not\in FV(A)$ or $y = x$;
4. $[y''/x''][y'/x']A \equiv [y/x]A$, where $[y/x] = [y''/x''][y'/x']$ (the composition of functions on $\text{Var}$);
5. $[y/x][z/u]A \equiv [y/x]z/u A$ if $FV(A) \cap r(x) \subseteq r(u)$;
6. $[y/x][x/u]A \equiv [y/u]A$ if $FV(A) \cap r(x) \subseteq r(u)$;
7. $[v/u][y/x]A \equiv [v/u]y/x A$ if $u \not\in FV(A) - r(x)$;
8. $[y/x]Qz A \equiv Qz[y/x] A$ if $z \not\in r(x)$;
9. $[y/x](A * B) \equiv ([y/x]A * [y/x]B)$ for $* \in \{\lor, \land, \Rightarrow\}$;
10. $[y''/x''][y'/x']A \equiv [y''/x''x']A$ if $r(x'') \cap r(x'y') = \emptyset$;
(11) \[ [y/x]A \triangleq [y_n/z_n] \ldots [y_1/z_1][z_n/x_n] \ldots [z_1/x_1]A \]
for distinct variables \( z_1, \ldots, z_n \not\in FV(A) \cup r(xy) \); so every variable substitution in a formula can be presented as a composition of substitutions of the form \([y/x]\) (simple substitutions);

(12) \( [y/x]QzA \triangleq Qz[y/x]A \) if \( r(z \cap r(x)) = \emptyset \);

(13) \( Qy[y/x]A \triangleq QxA \) if \( y \) is distinct, \( r(y) \cap FV(A) = r(y) \cap r(x) = \emptyset \).

**Proof**

(1) Since \( [y/x]A = A[x \leftarrow y] \), it follows that \( FV([y/x]A) = V(A[x \leftarrow y]) \) (or, to be more precise, \( FV(A[x \leftarrow y]) = \emptyset \)). So we should take the set \( V(A) = FV(A) \) and replace every \( x_i \) occurring in this set with the corresponding \( y_i \); this gives us exactly

\[
(FV(A) - r(x)) \cup \text{rng}[x \mapsto y]_A.
\]

(2) Trivial.

(3) The case \( y = x \) is trivial, so suppose \( y \not\in FV(A) \). Let \( A^\circ \) be a clean version of \( A \) such that \( BV(A^\circ) \cap r(xy) = \emptyset \). As we know

\[
[y/x]A = A^\circ[x \leftarrow y],
\]

hence by 2.3.17,

\[
(\#1) \ Qy[y/x]A \triangleq Qy(A^\circ[x \leftarrow y]).
\]

Since \( y \not\in FV(A) \), by 2.3.17 we also have

\[
(\#2) \ Qy(A^\circ[x \leftarrow y]) \triangleq QxA^\circ.
\]

Now obviously, \( A \triangleq A^\circ \) implies

\[
(\#3) \ QxA^\circ \triangleq QxA
\]

(by 2.3.17 or just by the definition of a scheme). So from (\#1), (\#2), (\#3) we obtain

\[
Qy[y/x]A \triangleq QxA.
\]

(4) Let \( A^\circ \) be a clean version of \( A \) such that \( BV(A^\circ) \cap r(xx'x''yy'y'') = \emptyset \). Then

\[
[y'/x']A \triangleq A^\circ[x' \leftarrow y'].
\]

By Lemma 2.3.13, the latter formula is clean, and obviously its bound variables are the same as in \( A^\circ \). So

\[
[y''/x''][y'/x]A \triangleq [y''/x''](A^\circ[x' \leftarrow y']) \triangleq (A^\circ[x' \leftarrow y'])(x'' \leftarrow y'').
\]
Since \([y'/x'] = [y't/x't]\), we can always add variables to both \(x'\) and \(y'\), so we may assume that \(r(x') \supseteq FV(A)(= FV(A^o))\),

We can also write

\[ A^o[x' \mapsto y'] = A^o[u \mapsto v], \]

where

\[ [u \mapsto v] := [x' \mapsto y]_A, \]

since \(x'\) does not contain bound variables of \(A\). Then \(r(u) = FV(A^o)\), so \(r(v) = FV(A^o[u \mapsto v])\).

Similarly we have

\[ (A^o[x' \mapsto y'])[x'' \mapsto y''] = (A^o[u \mapsto v])[x'' \mapsto y''] = (A^o[u \mapsto v])[w \mapsto z], \]

where

\[ [w \mapsto z] := [x'' \mapsto y'']_A[u \mapsto v]. \]

So \(r(w) = r(v)\), and thus \(v = w \cdot \sigma\) for some surjective \(\sigma \in \Sigma_{|w|,|v|}\). Hence

\[ (A^o[u \mapsto v])[w \mapsto z] = A^o[u \mapsto z \cdot \sigma] \]

by Lemma 2.3.2.

On the other hand, for \([y/x] = [y'/x'] \circ [y''/x'']\) we may assume that \(r(x) \supseteq FV(A)\). Then we have

\[ [y/x]A = A^o[x \mapsto y] \supseteq A^o[x \mapsto y]_A, \]

and it remains to show that

\[ [u \mapsto z \cdot \sigma] = [x \mapsto y]_A. \]

In fact, suppose \(x_j = u_i \in FV(A)\). Then

\[ y_j = [y''/x''](y'_j) = [y''/x'](v_i) = [y''/x''](w_{\sigma(i)}) = z_{\sigma(i)}, \]

by the choice of \([u \mapsto v], [w \mapsto z]\).

(5) By (4), it suffices to check that \([y/x] \cdot [z/u]\) and \([y/x][z/u]\) coincide on parameters of \(A\). In fact, the first substitution sends every \(u_i\) to \(z_i\) and next to \([y/x]z_i\) and every \(x_j \notin r(u)\) to \(y_j\). So if \(r(x) \cap FV(A) \subseteq r(u)\), all parameters of \(A\) beyond \(r(u)\) remain fixed. The second substitution sends \(u_i\) directly to \([y/x]z_i\) and also fixes other parameters; thus the claim holds.

(6) This is a particular case of (5) when \(x = z\). Then \([y/x]z = y\).

(7) Apply (5) to the case \(y := v, x := u, z := u, u := x\). Note that

\[ FV(A) \cap \{u\} \subseteq r(x) \text{ iff } \{u\} \subseteq -FV(A) \cup r(x). \]
(8) Consider a clean version $A^o$ of $A$ such that $r(xyz) \cap BV(A^o) = \emptyset$. Then

$$QzA \triangleq QzA^o$$

and $QzA^o$ is clean by 2.3.11.

So

\[(8.1) \ [y/x]QzA \triangleq (QzA^o)[x \mapsto y] = Qz(A^o[x \mapsto y]).\]

On the other hand,

$$A^o[x \mapsto y] \triangleq [y/x]A,$$

hence

\[(8.2) \ Qz(A^o[x \mapsto y]) \triangleq Qz[y/x]A \]

by 2.3.16. Now (8) follows from (8.1) and (8.2).

(9) Let $A^o, B^o$ be clean versions of $A$ and $B$ such that

$$BV(A^o) \setminus FV(A) = BV(B^o) \setminus FV(B) = \emptyset.$$ 

Then $(A^o \ast B^o)$ is a clean version of $(A \ast B)$ and by 2.3.21 we have:

$$[y/x](A \ast B) \triangleq (A^o \ast B^o)[x \mapsto y] = (A^o[x \mapsto y] \ast B^o[x \mapsto y])$$

$$\triangleq ([y/x]A \ast [y/x]B).$$

(10) Note that in this case

$$[y''/x''] \cdot [y'/x'] = [y'''/x''x']$$

and apply (4)

(11) Since $z_i \not\in FV(A)$, from (6) we obtain

\[(\#) \ [y/z][z/x]A \triangleq [y/x][A \]

By induction from (10) we also have

\[(###) \ [z/x]A \triangleq [z_n/x_n] \ldots [z_1/x_1]A,$$

since $r(x) \cap r(z) = \emptyset$ and $z$ is distinct.

In the same way from (10) we obtain

\[(####) \ [y/z][z/x]A \triangleq [y_n/z_n] \ldots [y_1/z_1][z/x]A,$$

since $r(z) \cap r(y) = \emptyset$ and $z$ is distinct.

Now (11) follows from (\#), (###), (####) and 2.3.24.
(12) Follows from (8) by induction on $|z|$. For the step, suppose $z = z_1z'$ and

$$[y/x]Qz'A \simeq Qz'[y/x]A;$$

then by 2.3.17(2),

$$Qz_1[y/x]Qz'A \simeq Qz[y/x]A.$$

On the other hand, by (8)

$$[y/x]QzA \simeq Qz_1[y/x]Qz'A;$$

hence (12) follows.

(13) Apply induction on $|x| = |y|$. The base follows from (3).

For the step, suppose $y = y_1y'$, $x = x_1x'$ and

$$Qy'[y'/x']A \simeq Qx'A.$$

Then by 2.3.17(2),

$$Qx_1Qy'[y'/x']A \simeq QxA.$$  \[\text{(*)1}\]

On the other hand, by (10),

$$[y/x]A \simeq [y_1/x_1][y'/x']A;$$

hence by 2.3.17(2),

$$Qy[y/x]A \simeq Qy[y_1/x_1][y'/x']A.$$  \[\text{(*)2}\]

By (12),

$$Qy'[y_1/x_1][y'/x']A \simeq [y_1/x_1]Qy'[y'/x']A,$$

hence by 2.3.17(2) and (3)

$$Qy[y_1/x_1][y'/x']A \simeq Qy[y_1/x_1](Qy'[y'/x']A) \simeq Qx_1Qy'[y'/x']A.$$  \[\text{(*)3}\]

Now (13) follows from (\text{*2}), (\text{*3}), and (\text{*1}).

Exercise 2.3.26 Describe the composition of substitutions (or of corresponding transformations) explicitly.
2.4 Formulas with constants

Although our basic languages do not contain individual constants, we will need auxiliary languages with constants.

So let \( D \) be a non-empty set; we assume that \( D \cap \text{Var} = \emptyset \). Let \( \mathcal{L}_N(D) \) be the language \( \mathcal{L}_N \) expanded by individual constants from the set \( D \). Formulas of the language \( \mathcal{L}_N(D) \) (respectively, \( \mathcal{L}_0(D) \)) are called \( N \)-modal (respectively, intuitionistic) \( D \)-formulas; the set of all these formulas is denoted by \( MF_N(=)(D) \) (respectively, \( IF(=)(D) \)). Obviously, every predicate formula (in the ordinary sense, i.e. without extra constants) is a \( D \)-formula. A \( D \)-sentence is a \( D \)-formula without parameters; \( MS_N(=)(D) \) and \( IS(=)(D) \) denote the sets of \( D \)-sentences of corresponding types.

Definition 2.4.1 Let \( x = (x_1, \ldots, x_n) \) be a list of distinct variables, \( a = (a_1, \ldots, a_n) \) a list of constants (individuals) from \( D \) (not necessarily distinct). Then the \( D \)-transformation \([x \leftarrow a]\) is a finite function \( \{(x_1, a_1), \ldots, (x_n, a_n)\} \) sending every \( x_i \) to \( a_i \), \( i = 1, \ldots, n \). The \( D \)-instance \([a/x]A\) of a \( D \)-formula \( A \) under \([x \leftarrow a]\) is obtained by simultaneous replacement of all free occurrences of \( x_1, \ldots, x_n \) in \( A \) respectively with \( a_1, \ldots, a_n \).

Strictly speaking, \([a/x]A\) is defined by induction on \( |A| \):

- \([a/x]P(y) := P([a/x]y)\), where \([a/x]y\) is a tuple \( z \) such that \( |z| = |y| \) and for any \( j \),

\[
z_j = \left\{ \begin{array}{ll}
a_i & \text{if } y_j = x_i, \\
y_j & \text{if } y_j \notin r(x).
\end{array} \right.
\]

- \([a/x]P := P \text{ if } P \in PL^0\),
- \([a/x](B \ast C) := ([a/x]B \ast [a/x]C) \text{ if } \ast \in \{\lor, \land, \Rightarrow\}\),
- \([a/x]\bot := \bot\),
- \([a/x]\Box_i B := \Box_i[a/x]B\),
- \([a/x]QzB := Qz[a/x]B \text{ if } z \notin r(x)\),
- \([a/x]Qx_i B := Qx_i[\hat{a}_i/\hat{x}_i]B.\)

So we can also denote \([a/x]A\) by \( A[x \leftrightarrow a] \) if \( r(x) \cap BV(A) = \emptyset \). Normally we use the notation \([a/x]A\) in the case when both \( A \) is a usual formula and \([a/x]A\) is a \( D \)-sentence (which is equivalent to \( FV(A) \subseteq r(x) \)). A formula \( A \) is called a generator of every \( D \)-sentence \([a/x]A\).

For \( D \)-formulas we define schemes, clean versions and congruence in the natural way.

\[\[\text{Or briefly, by } MF(=)(D).\]
\[\[\text{Recall that } \hat{x}_i \text{ is obtained by eliminating } x_i \text{ from } x; \text{ similarly for } \hat{a}_i.\]
Lemma 2.4.2  

(1) \( A \equiv B \Rightarrow [a/x]A \equiv [a/x]B \) 
for any \( D \)-transformation \([x \mapsto a]\) and \( D \)-formulas \( A, B \). 

(2) If \( x \) is a distinct list of variables \( |x| = n, a \in D^n \), then for any predicate 
formula \( A \), for any \( \sigma \in \Sigma_n \) 
\[ ([a \cdot \sigma]/x) A = [a/(x \cdot \sigma^{-1})] A. \]

(3) For any predicate formula \( A \), for any distinct list \( xy \) 
\[ [a/y][y/x] A \equiv [a/x] A. \]

(4) Let \( x, z \) be distinct lists of variables, \( |x| = n, |z| = m \leq n \), and let 
\( \sigma : I_n \rightarrow I_m \). Let \( A \) be a formula such that \( r(z \cdot \sigma) \cap BV(A) = \emptyset \). Then 
\[ [a/z][(z \cdot \sigma)/x] A \equiv [(a \cdot \sigma)/x] A. \]

Proof  

(1) It is clear that 
\[ [a/x]A = A[x \mapsto a] \]
(a strict proof is by induction). So \( A = B \) implies \([a/x]A = [a/x]B\). 

(2) Note that \([x \mapsto a \cdot \sigma] = [x \cdot \sigma^{-1} \mapsto a]\) — each of these maps sends \( x_i \) to \( a_{\sigma(i)} \), and \( x_{\sigma^{-1}(j)} \) to \( a_j \). Now the claim follows from 2.4.1. 

(3) As noted above, constant substitutions respect congruence. So we can 
prove the claim for a clean version \( A^\circ \) of \( A \), where \( BV(A^\circ) \cap r(xy) = \emptyset \). In 
this case it is equivalent to 
\[ (A^\circ[x \mapsto y])[y \mapsto a] = A^\circ[x \mapsto a]. \]
This holds, since obviously 
\[ [x \mapsto y] \circ [y \mapsto a] = [x \mapsto a]. \]

But again, a strict proof is by induction. 

(4) Similar to (2). Consider a clean version \( A^\circ \) of \( A \), with \( BV(A^\circ) \cap r(xz) = \emptyset \). Then the claim reduces to 
\[ (A^\circ[x \mapsto z \cdot \sigma])[z \mapsto a] = A^\circ[x \mapsto a \cdot \sigma], \]
which follows from 
\[ [x \mapsto z \cdot \sigma] \circ [z \mapsto a] = [x \mapsto a \cdot \sigma]. \]

We also use a somewhat ambiguous notation \( A(x) \) to indicate that \( FV(A) \subseteq r(x) \); in this case \([a/x]A\) is abbreviated to \( A(a) \). The abbreviation \( A(a) \) is 
convenient and rather common, but it leads to some confusion: it may happen 
that a \( D \)-sentence \( B \) can be presented as \([a_1, \ldots, a_n/x_1, \ldots, x_n]A\) for different 
formulas \( A \). For example, \( P(a, a) = [a/x]P(x, x) = [a, a/x, y]P(x, y) \). Such an 
ambiguity may be undesirable (cf. Section 5.1), so we will mainly use 'maximal' 
representations described as follows.
Definition 2.4.3 A formula $A$ is called a maximal generator of a $D$-formula $B$ if $B = [a/x]A$ for some bijective $D$-transformation $[x \mapsto a]$. 

Since $[a/x]A$ does not depend on the variables $x_i$ that are not parameters of $A$, in the above definition we may further assume that $r(x) \subseteq FV(A)$, and thus $a$ is the list of all constants occurring in $B$.

Lemma 2.4.4 Every $D$-formula has a maximal generator.

Proof Let $a = a_1 \ldots a_n$ be a list of all constants occurring in a $D$-sentence $B$, $x = x_1 \ldots x_n$ a list of different new variables for $B$. The formula $A := B[a \mapsto x]$ (obtained by replacing every occurrence of $a_i$ with $x_i$) is a maximal generator of $B$, since $x_i \notin BV(B)$, and thus $[a/x]A = A[x \mapsto a] = (B[a \mapsto x])[x \mapsto a] = B$. $
$

Lemma 2.4.5

(1) If $B = [a/x]A$ for a formula $A$ and a bijection $[x \mapsto a]$, then $A = B[a \mapsto x]$.

(2) If $A_1, A_2$ are maximal generators of $B$, then $A_2 = [y/x]A_1$ for some variable renaming $[x \mapsto y]$ (and of course, $A_1$ is obtained from $A_2$ in the same way).

More precisely, if $B = [a/x]A_1 = [a/y]A_2$ for bijections $[x \mapsto a], [y \mapsto a]$, then $A_2 = [y/x]A_1$.

(3) A maximal generator of a $D$-formula $B$ is a substitution instance of any generator of $B$ under some variable substitution.

Proof (1) We check that

$$([a/x]A)[a \mapsto x] = A$$

by induction on $|A|$. This is clear for atomic $A = P(y)$, when $[a/x]A = A[x \mapsto a]$ (cf. Lemma 2.3.2). All induction steps are routine; let us consider only the case $A = Qx_i B$ for a quantifier $Q$. By definition,

$$([a/x]A)[a \mapsto x] = (Qx_i[\hat{a}_i/\hat{x}_i]B)[a \mapsto x] = Qx_i(([\hat{a}_i/\hat{x}_i]B)[a \mapsto x])$$

Since $B$ is a usual formula and $[x \mapsto a]$ is a bijection, $a_i$ does not occur in $[\hat{a}_i/\hat{x}_i]B$; thus

$$([\hat{a}_i/\hat{x}_i]B)[a \mapsto x] = ([\hat{a}_i/\hat{x}_i]B)[\hat{a}_i \mapsto \hat{x}_i] = B$$

by the induction hypothesis. Therefore (1) holds for $A$.

(2) If $A_1$ is a maximal generator of $B$, then $B = [a/x]A_1$ for some bijection $[x \mapsto a]$. Similarly, $B = [a/y]A_2$ for a bijection $[y \mapsto a]$. Now let $A_1^\circ$ be a clean version of $A_1$ such that $BV(A_1^\circ) \cap r(xy) = \emptyset$. 


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By 2.4.2(1), \( A_1 \bowtie A_1^o \) implies

\[
B = [a/x]A_1 \bowtie [a/x]A_1^o = A_1^o[x \mapsto a].
\]

Hence

\[
B[a \mapsto x] \bowtie (A_1^o[x \mapsto a])[a \mapsto x] = A_1^o.
\]

Similarly, there exists a clean \( A_2^o \bowtie A_2 \) such that

\[
B[a \mapsto y] \bowtie A_2.
\]

Therefore

\[
A_2^o \bowtie B[a \mapsto y] \bowtie (A_1^o[x \mapsto a])[a \mapsto y] = A_1^o[x \mapsto y],
\]

and the latter formula is \([y/x]A_1 \) by 2.3.22.

(3) Let \( C \) be a generator of \( B \), thus \( B = [b/z]C \) for some \([z \mapsto b] \); and let \( A \) be a maximal generator of \( B \), with \( B = [a/x]A \), for a bijection \([x \mapsto a] \). We may assume that \( r(x) \subseteq FV(A) \), \( r(z) \subseteq FV(C) \) and \( r(b) = r(a) \) is the set of all constants occurring in \( B \).

Since \( a \) is distinct, every \( b_i \) equals to some \( a_j \), so \( b = a \cdot \tau \) for some surjective map \( \tau : I_n \to I_m \).

Thus

\[
B = [a \cdot \tau/z]C = [a/x]A,
\]

and we have by 2.4.2(4)

\[
[a \cdot \tau/z]C \triangleq [a/x][x \cdot \tau/z]C.
\]

Hence by (2), \( A \bowtie [x \cdot \tau/z]C \). ■

### 2.5 Formula substitutions

**Definition 2.5.1** A (simple) formula substitution is a pair \((C, P(x))\), where \( C \) is a predicate formula, \( P(x) \) is an atomic equality-free formula. The substitution \((C, P(x))\) is usually denoted by \([C/P(x)]\). More exactly, \([C/P(x)]\) is called an MF\(_N\)-, (MF\(_N^P\), IF-\(), IF\(_{-}\)-) substitution if the formula \( C \) is of the corresponding type.

**Definition 2.5.2** For a substitution \([C/P(x)]\),

\[
FV[C/P(x)] := FV(C) - r(x)
\]

is called the set of parameters,

\[
BV[C/P(x)] := r(x)
\]

the set of bound variables. A substitution \([C/P(x)]\) is called strict if \( FV[C/P(x)] = \emptyset \), i.e. if \( FV(C) \subseteq r(x) \).
Definition 2.5.3 Let $A$ be a clean predicate formula, $S = [C/P(x)]$ a formula substitution such that $BV(A) \cap FV(S) = \emptyset$. Let $B$ be a result of replacing all subformulas of $A$ of the form $P(y)$ with $[y/x]C$.

Every formula congruent to $B$ is denoted by $SA$ and is called a substitution instance of $A$ under $S$.

More precisely, $SA$ is defined by induction:

\[
SP(y) \triangleq [y/x]C, \\
SA \triangleq A \text{ if } A \text{ is atomic and does not contain } P, \\
S\Box_i A \triangleq \Box_i SA, \\
S(A \ast B) \triangleq (SA \ast SB) \text{ for } \ast \in \{\lor, \land, \rightarrow\}, \\
SQz A \triangleq QzSA \text{ for } Q \in \{\forall, \exists\}.
\]

A formula $SA$ is called a substitution instance of $A$, or more exactly, an $MF^\approx_N$- ($IF^\approx$-) substitution instance if $S$ is an $MF^\approx_N$- ($IF^\approx$-) substitution.

Due to the assumption $BV(A) \cap FV(S) = \emptyset$, in $SA$ the parameters of $S$ do not collide with the existing bound variables from $A$.

Note that applying $S$ to $A$ does not affect occurrences of equality in $A$, but may introduce new occurrences if $C$ contains equality.

Lemma 2.5.4 Let $A$ be a clean formula, $S = [C/P(x)]$ a formula substitution such that $FV(S) \cap BV(A) = \emptyset$. Then

\[
S(A[u \mapsto v]) \triangleq [v/u]SA.
\]

for any variables $u, v$ such that $v \notin V(A)$, $u \in FV(A)$, and $u, v \notin FV(S)$.

Proof Since $v \notin V(A)$, we can prolong $[u \mapsto v]$ to a variable renaming in $A$ by fixing all variables from $V(A) - \{u\}$. So $A[u \mapsto v]$ is clean by 2.3.11 with the same bound variables as $A$, and Definition 2.5.3 is applicable to this formula.

Now we argue by induction on $|A|$.

- If $A = P(y)$, then $A[u \mapsto v] = P([v/u]y)$, $SA \triangleq [y/x]C$, and
  \[
  S(A[u \mapsto v]) \triangleq [[v/u]y/x]C.
  \]

By assumption, $u \notin FV(S) = FV(C) - r(x)$, so we obtain

\[
S(A[u \mapsto v]) \triangleq [v/u]SA
\]

by applying 2.3.25 (7).

- Let $A = QzB$, then $SA \triangleq QzSB$. Hence
  \[
  [v/u]SA \triangleq [v/u]QzSB \triangleq Qz[v/u]SB
  \]

by 2.3.25 (8); note that $z \neq u$, since $u \in FV(A)$
By the induction hypothesis,

\[ [v/u]SB \cong S(B[u \mapsto v]), \]

hence

(2) \[ Qz[v/u]SB \cong QzS(B[u \mapsto v]) \]

by 2.3.6. Now by 2.5.3

(3) \[ QzS(B[u \mapsto v]) \cong SQz(B[u \mapsto v]) \]

so from (1), (2), (3) we have

\[ [v/u]SA \cong SQz(B[u \mapsto v]). \]

It remains to note that

\[ Qz(B[u \mapsto v]) = A[u \mapsto v], \]

since \( z \neq u \). Therefore the claim holds for \( A \).

- If \( A = (B \ast C) \), we can use 2.3.25 (9) and the distribution of \( S \) and \([u \mapsto v]\) over \( \ast \). Note that if \( u \) does not occur in \( B \) (or in \( C \)), the main statement trivially holds for \( B \) (or \( C \)), and the argument does not change. The details are left to the reader.

- The case \( A = \Box_i B \) is trivial.

\[ \square \]

**Lemma 2.5.5** Let \( A, B \) be congruent clean formulas, \( S \) a formula substitution such that \( BV(A) \cap FV(S) = BV(B) \cap FV(S) = \emptyset \). Then \( SA \cong SB \).

**Proof** By induction on \(|A| = |B|\).

- If \( A \) is atomic, then \( A = B \), and there is nothing to prove.

- If \( A = (A_1 \ast A_2) \), then by Lemma 2.3.9(1), \( B = (B_1 \ast B_2) \) for \( A_1 \cong B_1 \), \( A_2 \cong B_2 \). Hence

  \[ SA \cong (SA_1 \ast SA_2), \quad SB \cong (SB_1 \ast SB_2), \]

  and \( SA_i \cong SB_i \) by the induction hypothesis. Eventually \( SA \cong SB \) by 2.3.17.

- We skip the easy case when \( A = \Box_i A_1 \).
Suppose $A = QxA_1$ for a quantifier $Q$. Since $A$ is clean, $x \notin BV(A_1)$, so by Lemma 2.3.9(2), for some $y \notin FV(A_1), B_1$

$$B = QyB_1, \ B_1 \models A_1[x \mapsto y].$$

We may also assume that $y \notin BV(A_1)$. (Otherwise consider $A_2 \models A_1$ such that $y \notin BV(A_2)$, then

$$B_1 \models A_1[x \mapsto y] \models A_2[x \mapsto y],$$

so $A_2$ can be used instead of $A_1$.) Thus

$$SA = QxSA_1, \ SB = QySB_1,$$

and

$$SB_1 \models S(A_1[x \mapsto y]) \models [y/x]SA_1$$

by the induction hypothesis and Lemma 2.5.4 (which is applicable, since $y \notin V(A_1)$ and $x, y \notin FV(S)$ by the assumption of the lemma). Hence by 2.3.24(3)

$$SB = QySB_1 \models Qy[y/x]SA_1 \models QxSA_1 = SA.$$

Now we can define substitution instances of arbitrary formulas.

**Definition 2.5.6** A substitution instance $SA$ of a predicate formula $A$ under a simple substitution $S$ is an arbitrary formula congruent to $SA_0$, where $A^o$ is a clean version of $A$ such that $FV(S) \cap BV(A^o) = \emptyset$. A strict substitution instance is a substitution instance under a strict substitution.

Lemma 2.5.5 shows soundness of this definition, i.e. that the congruence class of $SA_0$ does not depend on the choice of $A^o$.

Note that according to the definition, for a trivial formula substitution $S = [P(x)/P(x)]$ and a formula $A$, $SA$ denotes an arbitrary formula congruent to $A$.

**Lemma 2.5.7** Let $[C_1/P(x)], \ [C_2/P(x)]$ be formula substitutions such that $C_1 \equiv C_2$. Then for any predicate formula $A$, $[C_1/P(x)]A \equiv [C_2/P(x)]A$.

**Proof** We denote $[C_i/P(x)]$ by $S_i$. Let $A^o$ be a clean version of $A$ such that $FV(S_i) \cap BV(A^o) = \emptyset$ for $i = 1, 2$. Obviously we can construct such $A^o$ by an appropriate bound variable renaming from an arbitrary clean version. Now $S_iA \models S_iA^o$, so we show $S_1A^o \models S_2A^o$ by induction on $|A^o|$. To simplify notation, put $B := A^o$.

If $B = P(y)$, then $S_iB = [y/x]C_i$, so $S_1B \models S_2B$ follows from 2.3.24.

If $B$ is atomic and does not contain $P$, the claim is trivial.
The induction step easily follows from the distribution of $S_i$ over all connectives and quantifiers. E.g. suppose $B = QyB_1$; then $y \notin FV(S_i)$, so

$$S_i QyB_1 \doteq QyS_i B_1$$

by 2.3.17. By induction hypothesis,

$$S_1 B_1 \doteq S_2 B_1,$$

hence

$$QyS_1 B_1 \doteq QyS_2 B_1$$

by 2.3.17, and therefore

$$S_1 B \doteq S_2 B.$$  

All the remaining cases are left to the reader.

Now let us consider complex substitutions.

**Definition 2.5.8** For atomic equality-free formulas $P_1(x_1), \ldots, P_k(x_k)$ (with different predicate letters $P_1, \ldots, P_k$ and distinct lists $x_1, \ldots, x_k$) and formulas $C_1, \ldots, C_k$ we define the complex formula substitution

$$[C_1, \ldots, C_k / P_1(x_1), \ldots, P_k(x_k)]$$

as the tuple $(C_1, \ldots, C_k, P_1(x_1), \ldots, P_k(x_k))$.

The set of its parameters and bound variables are respectively

$$FV[C_1, \ldots, C_k / P_1(x_1), \ldots, P_k(x_k)] := \bigcup_{i=1}^{k} FV[P_i(x_i)] = \bigcup_{i=1}^{k} (FV(C_i) \cap \tau(x_i))$$

and

$$BV[C_1, \ldots, C_k / P_1(x_1), \ldots, P_k(x_k)] := \tau(x_1 \ldots x_k).$$

A substitution without parameters is called strict.

Now we have an analogue of Definition 2.5.3.

**Definition 2.5.9** For a substitution $S = [C_1, \ldots, C_k / P_1(x_1), \ldots, P_k(x_k)]$ and a clean formula $A$ such that $FV(S) \cap BV(A) = \emptyset$, a substitution instance $SA$ is defined up to congruence by induction:

$$SP_i(y) \doteq [y/x_i]C_i,$$

$$SA \doteq A \text{ if } A \text{ is atomic and does not contain } P_1, \ldots, P_k,$$

$$S\Box_i A \doteq \Box_i SA,$$

$$S(A*B) \doteq (SA*SB) \text{ for } \ast \in \{\lor, \land, \supset\},$$

$$S\forall z A \doteq \forall z SA \text{ for } \forall \in \{\forall, \exists\}.$$  

We also have an analogue of Lemma 2.5.5.
Lemma 2.5.10 If $A, B$ are clean formulas, $A \equiv B$, $S$ is a complex formula substitution and

$$BV(A) \cap FV(S) = BV(B) \cap FV(S) = \emptyset,$$

then $SA \equiv SB$.

Proof The same as in 2.5.5 (including an analogue of 2.5.4). $\blacksquare$

So the following definition is sound.

Definition 2.5.11 For an arbitrary formula $A$ and a formula substitution $S$, we define $SA$ as $SA^\circ$, for a clean version $A^\circ$ of $A$ such that $FV(S) \cap BV(A^\circ) = \emptyset$.

Hence we readily obtain

Lemma 2.5.12 For any predicate formulas $A, B$ and a formula substitution $S$,

$$A \equiv B \Rightarrow SA \equiv SB.$$ 

The inductive definition 2.5.9 now extends to arbitrary formulas:

Lemma 2.5.13 Let $S$ be a formula substitution. Then for any formulas $A, B$

1. $S \Box_i A \equiv \Box_i SA$,

2. $S(A \star B) \equiv (SA \star SB)$ for $\star \in \{\lor, \land, \rightarrow\}$,

3. $SQzA \equiv QzSA$ for $Q \in \{\lor, \exists\}, z \not\in FV(S)$.

Proof

1. Let $A^\circ$ be a clean version of $A$ such that $FV(S) \cap BV(A^\circ) = \emptyset$. Then $\Box_i A^\circ$ is a clean version of $\Box_i A$, so

$$S \Box_i A \equiv S \Box_i A^\circ \equiv \Box_i SA^\circ.$$ 

By definition, $SA^\circ \equiv SA$, hence

$$\Box_i SA^\circ \equiv \Box_i SA,$$

therefore (1) holds.

2. An exercise for the reader.

3. Let $A^\circ$ be a clean version of $A$ such that $FV(S) \cap BV(A^\circ) = \emptyset$, $z \not\in BV(A^\circ)$. Then $QzA^\circ$ is a clean version of $QzA$ and

$$FV(S) \cap BV(QzA^\circ) = \emptyset.$$ 

By definition,

$$SQzA \equiv SQzA^\circ \equiv QzSA^\circ, SA^\circ \equiv SA;$$ 

hence

$$QzSA^\circ \equiv QzSA.$$
This implies (3).

The next lemma shows that the result of applying a substitution does not really depend on the names of its bound variables. We prove this only for simple substitutions, leaving the general case to the reader.

**Lemma 2.5.14** Let \([C/P(x)]\) be a formula substitution, \([x \mapsto y]\) a variable renaming such that \(r(y) \cap \text{FV}(C) = \emptyset\). Then for any formula \(A\),

\[[C/P(x)]A \equiv [[y/x]C/P(y)]A.\]

**Proof**

If \(A = P(z)\), we have

\[[C/P(x)]A \equiv [z/x]C, \quad [[y/x]C/P(y)]A \equiv [z/y][y/x]C,\]

while

\[[z/x]C \equiv [z/y][y/x]C\]

by 2.3.25(6).

If \(A\) is atomic and does not contain \(P\), the claim is trivial.

Now we can argue by induction on \(|A|\). Put

\(S_1 := [C/P(x)], \quad S_2 := [[y/x]C/P(y)].\)

If \(A = QuB\), we may assume that \(u \not\in \text{FV}(S_1)(= \text{FV}(S_2))\) — otherwise consider \(A' \equiv A\) of the form \(Qu'B'\), where \(u' \not\in \text{FV}(S_1)\).

Suppose \(S_1B \equiv S_2B\), then \(S_1A \equiv QuS_1B\) by 2.5.3; hence \(S_1A \equiv S_2A\) by 2.3.17.

Other cases are also based on 2.5.3 and 2.3.17; we leave them to the reader.

**Lemma 2.5.15** Let \(S = [C/P(u)]\) be a simple formula substitution, \([y/x]\) a variable substitution such that \(\text{FV}(C) \cap r(x) = \emptyset\). Then for any predicate formula \(A\), \(S[y/x]A \equiv [y/x]SA\).

**Proof** Since \(S\) respects congruence by Lemma 2.5.12, \([y/x]\) respects congruence by 2.3.24 and both \(S\) and \([y/x]\) distribute over all connectives and quantifiers in an appropriate clean version of \(A\) (Definition 2.5.3, Lemma 2.3.25), it suffices to consider only the case when \(A\) is atomic. The nontrivial option is \(A = P(z)\). Then

\(S[y/x]A \equiv SP([y/x]z) \equiv [[y/x]z/u]C,\)

\([y/x]SA \equiv [y/x][z/u]C.\)

Now the claim follows by 2.3.25(5).

Lemma 2.5.14 shows that in some cases variable substitutions commute with formula substitutions. The next lemma considers situations where formula substitutions 'absorb' variable substitutions.
Lemma 2.5.16 Let \([C/P(x)]\) be a simple formula substitution, \(A\) a predicate formula, \([y/z]\) a variable substitution such that \(r(z) \cap \text{FV}(A) = r(z) \cap r(x) = r(y) \cap r(x) = \emptyset\). Then

\[[y/z][C/P(x)]A \triangleq [y/z]C/P(x)A.\]

Note that \([y/z]C\) is defined up to congruence, but the congruence class of \([y/z]C/P(x)A\) does not depend on the choice of a congruent version of \([y/z]C\), thanks to Lemma 2.5.7.

**Proof** The same idea as in 2.5.14 shows that it is sufficient to consider only the case when \(A = P(u)\) is atomic (and by the assumption, \(r(z) \cap r(u) = \emptyset\)).

In this case the claim becomes

\((*)\)

\[[y/z][u/x]C \triangleq [u/x][y/z]C.\]

The latter congruence follows from 2.3.24. In fact, by 2.3.24,

\[[y/z][u/x]C \triangleq ([y/z] \cdot [u/x])C.\]

from \(r(z) \cap r(ux) = \emptyset\), by 2.3.25(10) we have

\[[y/z][u/x]C \triangleq [yu/zx]C,\]

and similarly from \(r(x) \cap r(yz) = \emptyset\),

\[[u/x][y/z]C \triangleq [uy/xz]C.\]

Since \([yu/zx] = [uy/xz]\), this implies \((*)\).

The previous lemma easily transfers to complex substitutions:

**Lemma 2.5.17** Let

\[S = [C_1, \ldots, C_k/P_1(x_1), \ldots, P_k(x_k)]\]

be a formula substitution, \(A\) a predicate formula, \([y/x]\) a variable substitution such that \(r(z) \cap \text{FV}(A) = \emptyset\) and \(r(yz) \cap r(x_1, \ldots, x_k) = \emptyset\).

Then

\[[y/z]SA \triangleq S_0A,\]

where \(S_0 = [[y/z]C_1, \ldots, [y/z]C_k/P_1(x_1), \ldots, P_k(x_k)]\). Note that \(r(z) \cap \text{FV}(S_0) = \emptyset\).

**Proof** Again everything reduces to the case of atomic \(A\). But in this case \(S\) acts as a simple substitution, so we can apply 2.5.16.

**Lemma 2.5.18** \([c/x]B/q\) \(A \triangleq [c/x][B/q]A\) for a propositional formula \(A\), a list of proposition letters \(q\), a list of constants \(c\), a distinct list of variables \(x\), a list of predicate formulas \(B\), \(r(x) \cap \text{FV}(A) = \emptyset\).
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Proof The same argument as above reduces everything to the case when $A$ is atomic, i.e. a proposition letter. Then the claim is trivial.

Lemma 2.5.19 Every complex substitution acts on formulas as a composition of simple substitutions. More precisely, if $S = [C_1, \ldots, C_k/P_1(x_1), \ldots, P_k(x_k)]$ is a complex substitution, $P'_i$ is of the same arity of $P_i$ and $P'_i$ does not occur in $C_1, \ldots, C_k$ for $i = 1, \ldots, k$, then for any formula $A$

$$SA \triangleq [C_1/P'_1(x_1)] \ldots [C_k/P'_k(x_k)]/[P'_1(x_1)/P_1(x_1)] \ldots [P'_k(x_k)/P_k(x_k)]A.$$ 

Proof Since substitutions respect congruence and distribute over all connectives and quantifiers over non-parametric variables (by Lemma 2.5.13), we may prove the claim for a congruent version of $A$, in which the parameters of $S$ are not bound. In this case it suffices to check the claim for an atomic $A$. If $P_1, \ldots, P_k$ do not occur in $A$, there is nothing to prove. So let $A = P_i(y)$. Then by definition

$$SA \triangleq [y/x_i]C_i,$$

while

$$[C_1/P'_1(x_1)] \ldots [P'_k(x_k)/P_k(x_k)]A \triangleq [C_1/P'_1(x_1)]/[P'_1(x_1)/P_1(x_1)] [P'_i(x_i)/P_i(x_i)] P_i(y) \triangleq [y/x_i]C_i.$$

So the claim holds.

The composition of substitutions reduces to a single (complex) substitution as the following lemma shows.

Lemma 2.5.20 Let $S_0 = [C_0/P_i(x_0)]$, $S_1 = [C_1, \ldots, C_k/P_1(x_1), \ldots, P_k(x_k)]$ be formula substitutions. Then for any formula $A$

$$S_0S_1A \triangleq S_2A,$$

where $S_2 = [S_0C_1, \ldots, S_0C_k/P_1(x_1), \ldots, P_k(x_k)].$

Proof Similarly to the previous lemma, it suffices to check this only for $A = P_i(y)$. In this case we have

$$S_0S_1A \triangleq S_0[y/x_i]C_i.$$

Lemma 2.5.14 shows that a formula substitution acts in the same way after renaming bound variables. So we may assume that $r(x_i) \cap FV(C_0) = \emptyset$. Then by Lemma 2.5.15

$$S_0[y/x_i]C_i \triangleq [y/x_i]S_0C_i \triangleq S_2P_i(y).$$

This completes the proof.

Lemma 2.5.21 For any formula substitutions $S_0, S_1$, there exists a formula substitution $S$ such that for any formula $A$

$$S_0S_1A \triangleq SA.$$
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Proof  By Lemma 2.5.15

\[ S_0S_1A \overset{\omega}{=} S_2 \ldots S_nS_1A \]

for some simple formula substitutions \( S_2, \ldots, S_n \). Then we can use induction on \( n \) and Lemma 2.5.16.

Now let us consider parameters of substitution instances. We begin with a simple remark that a strict substitution instance of a formula \( A \) may be not a sentence if \( A \) is not a sentence.

Intuitively it is clear that free variable occurrences in a substitution instance \([C/P(x)]A\) may be of three kinds:

1. those derived from original free occurrences in \( A \) if they occur in atoms not containing \( P \) (and thus not affected by the substitution);
2. members of \( y \) in subformulas of the form \([y/x]C\) replacing occurrences of \( P(y) \) in \( A \);
3. those produced by parameters of the substitution wherever \( P(y) \) is replaced with \([y/x]C\).

Parameters of the first two types are called essential. Here is a precise definition for an arbitrary substitution.

Definition 2.5.22 A parameter \( z \in FV(A) \) is called essential for a formula substitution \([C/P(x)]\) if one of the following conditions holds:

1. there exists a free occurrence of \( z \) in \( A \) within an atomic subformula that does not contain \( P \);
2. there exists a free occurrence of \( z \) in \( A \) as some \( y_j \) within an occurrence of \( P(y) \), where \( y = y_1 \ldots y_n \) and \( x_j \in FV(C) \).

The set of all essential parameters of \( A \) for \( S \) is denoted by \( FVe(S, A) \). Now let us prove the above observation on parameters of \( SA \) in more detail.

Lemma 2.5.23 Let \( A \) be a formula, \( S = [C/P(x)] \) a simple formula substitution. Then

\[ FV(SA) = FV(S) \cup FVe(S, A) \text{ if } P \text{ occurs in } A \]
\[ FV(SA) = FVe(S, A) \text{ otherwise.} \]

Proof  The second claim obviously follows from 2.5.22(1). To prove the first, we argue by induction. We may assume that \( A \) is clean, \( BV(A) \cap FV(S) = \emptyset \).

- For atomic \( A \) there are two cases.

1. \( A = P(y) \). Then \( SA \overset{\omega}{=} [y/x]C \) and by Lemma 2.3.25(1), \( FV(SA) = FV(S) \cup \text{rng}[x \mapsto y]_C \). By definition, \( \text{rng}[x \mapsto y]_C = FVe(S, A) \) in this case, cf. 2.5.22(2).
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(2) $A$ does not contain $P$. Then $FV(A) = FVe(S, A)$.

- For $A = \Box_i B$ we have $SA = \Box_i SB$ and thus $FV(SA) = FV(SB)$. Since $P$ occurs in $A$ iff it occurs in $B$ and $FVe(S, A) = FVe(S, B)$, the claim follow readily.

- For $A = B \ast D$, where $\ast$ is a binary connective, the proof is similar to the previous case; note that $FVe(S, B \ast D) = FVe(S, B) \cup FVe(S, D)$.

- For $A = QuB$ we have $FV(SA) = FV(SB) - \{u\}$. By induction hypothesis, $FV(SB) = FV(S) \cup FVe(S, B)$, since $P$ occurs in $B$. So it remains to show that $FVe(S, A) = FVe(S, B) - \{u\}$.

In fact,

(1) $z$ has a free occurrence in $QuB$ within an atom that does not contain $P$ iff $z \neq u$ and $z$ has the same kind of occurrence in $B$;

(2) $z$ has a free occurrence in $QuB$ within $P(y)$ as described in 2.5.22 (2) iff $z \neq u$ and $z$ has the same kind of occurrence in $B$.

From the previous lemma we obtain

**Proposition 2.5.24** Let $A$ be a formula, $S = [C/P(x)]$ a simple formula substitution such that $P$ occurs in $A$. Then

1. $FV(S) \subseteq FV(SA) \subseteq FV(S) \cup FV(A)$,
2. $FV(SA) = FV(A)$ if $S$ is strict,
3. for any subformula $B$ of $A$, $FV(SB) \subseteq FV(SA) \cup BV(A)$.

**Proof**

1. Note that $FVe(S, A) \subseteq FV(A)$.

2. Follows from (1).

3. By 2.5.23,

<table>
<thead>
<tr>
<th>$FV(SB)$</th>
<th>$FV(S) \cup FVe(S, B)$</th>
<th>$FV(SA) \cup (FVe(S, B) - FVe(S, A))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FV(S) \cup FVe(S, A) \cup (FVe(S, B) - FVe(S, A))$</td>
<td>$FV(SA) \cup (FVe(S, B) - FVe(S, A))$</td>
<td></td>
</tr>
</tbody>
</table>
Now note that according to Definition 2.5.22, the set $FVe(S, B) - FVe(S, A)$ contains only variables that are free in $B$, but not free in $A$, so this set is contained in $BV(A)$. Hence (3) follows.

**Remark 2.5.25** The reader can try to prove this proposition directly without using Lemma 2.5.23. This does not seem easier.

**Definition 2.5.26** For a set of formulas $\Gamma \subseteq MF_N^{(=)}$ (respectively, $IF^{(=)}$), its substitution closure is the set of all their substitution instances of the corresponding kind:

$$\text{Sub}(\Gamma) := \{ SA \mid A \in \Gamma, S \text{ is an } MF_N^{(=)}- (IF^{(=)}-) \text{ formula substitution} \}.$$ 

The universal substitution closure of $\Gamma$ is the set $\overline{\text{Sub}}(\Gamma)$ of all universal closures\(^\dagger\) of formulas from $\text{Sub}(\Gamma)$.

Since every $N$-modal formula is also $N'$-modal for $N' > N$, there is some ambiguity in this definition. But usually it is clear from the context, what kind of formulas we consider.

**Lemma 2.5.27**

(1) $\text{Sub}(\text{Sub}(\Gamma)) = \text{Sub}(\Gamma)$

(2) $\overline{\text{Sub}}(\text{Sub}(\Gamma)) \equiv \overline{\text{Sub}}(\Gamma)$ for a set of sentences $\Gamma$ (where $\equiv$ means that these sets are the same up to congruence).

**Proof**

(1) Every $B \in \text{Sub}(\Gamma)$ has the form $SA$ for some $A \in \Gamma$ and formula substitution $S$. Then for any formula substitution $S_1$, $S_1SA \in \text{Sub}(\Gamma)$ by Lemma 2.5.21.

(2) Let us show that for any $B \in \overline{\text{Sub}}(\Gamma)$ and for any substitution $S_1$, $\overline{\forall S_1B}$ is congruent to a formula from $\overline{\text{Sub}}(\Gamma)$. We have

$$B = \forall z SA$$

for some $A \in \Gamma$, substitution $S$ and $r(z) = FV(SA)$. We may also assume that $FV(S) \cap BV(A) = \emptyset$ (otherwise we replace $A$ with a congruent formula). Since $A$ is a sentence, we have $FV(SA) = FV(S)$ by 2.5.23.

Now let $y$ be a distinct list of new variables such that $|y| = |z|$ and $r(y) \cap FV(S_1) = \emptyset$. Then

$$\forall z SA \equiv \forall y[y/z]SA$$

by 2.3.25(13), and so by 2.5.12

(\star) \quad S_1B = S_1\forall z SA \equiv S_1\forall y[y/z]SA.$$

\(^\dagger\)Cf. Definition 2.2.5.
Now by 2.5.16
\[ [y/z]SA \triangleq S_2A \]
for some formula substitution \( S_2 \) (note that the condition \( r(z) \cap BV(S) = \emptyset \) holds, since \( r(z) = FV(S) \)).

Hence by 2.3.17(2) and 2.5.12
\[(**) \quad S_1 \forall y[y/z]SA \triangleq S_1 \forall y S_2 A. \]

Since \( r(y) \cap FV(S_1) = \emptyset \), from 2.5.13(3) it follows that
\[(***) \quad S_1 \forall y S_2 A \triangleq \forall y S_1 S_2 A. \]

Eventually, by (*), (**), (***) we obtain
\[ \forall S_1 B \triangleq \forall \forall y S_1 S_2 A, \]
and the latter formula is in \( \overline{\text{Sub}}(\Gamma) \), by 2.5.21.

Now let us define ‘minimal’ non-strict substitution instances of predicate formulas.

Let \( P_1, \ldots, P_k \) be all predicate letters (besides equality) occurring in a formula \( A \), \( P_i \in PL_{n_i} \), and put
\[ P := P_1(x_1) \ldots P_k(x_k), \]
where every \( x_i \) is a distinct list of variables of length \( n_i \). Next, let \( m \geq 0 \), and let \( P'_i \) be different \((m + n_i)\)-ary predicate letters \((i = 1, \ldots, k)\), \( z = z_1 \ldots z_m \) a distinct list of new variables for \( A \). Then we call \( P'_i \) the \( m \)-shift of \( P_i \); an \( m \)-shift of the formula \( A \) is \( A^m \triangleq [C/P]A \), where
\[ C = P'_1(x_1, z) \ldots P'_k(x_k, z). \]

We also put \( A^0 := A \). Obviously \( FV(A^m) = FV(A) \cup r(z) \) if \( A \) is not purely equational, i.e. it contains some predicate letters other than ‘=’; for purely equational \( A \), \( A^m = A \). Note that \( A^m \) is a substitution instance of any \( A^n \); this substitution instance is strict iff \( m \leq n \).

**Lemma 2.5.28** Let \( S = [C/P(x)] \) be a formula substitution. Then for any formula \( A \) and \( m \geq 0 \)
\[ (SA)^m \triangleq [C^m/P'(x, z)]A^m, \]
where we assume that the list of extra parameters \( z \) of \( A^m \), \( C_i^m \) is disjoint with \( x \).
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Proof By definition, for a certain substitution $S_1$,

$$(SA)^m \models S_1 SA, \; A^m \models S_1 A,$$

and $FV(S_1) = r(z)$. So, as before, we have to check the claim only for atomic $A$ (without equality).

If $A = P_i(y)$, then (for some $z$)

$$(SA)^m \models ([y/x]C_i)^m \models S_1[y/x]C_i,$$

$$[C^m/P'(x,z)]A^m \models [C^m/P'(x,z)]P_i'(y,z) \models [yz/xz]C_i^m \models [y/x]S_1 C_i.$$

By our assumption, $FV(S_1) = r(z)$ is disjoint with $x$. So the claim follows by Lemma 2.5.15.

\[\blacksquare\]

Exercise 2.5.29 Deduce 2.5.28 from 2.5.20.

Lemma 2.5.30 Every substitution instance $SA$ of a formula $A$ is obtained by a variable renaming from a strict substitution instance of $A^m$ for some $m \geq 0$, e.g. for $m = |FV(S)|$.

Proof Let us first show this for a simple substitution $S = [C(x,y)/P(x)]$ and a formula $A$ containing $P$. Let $P_1, \ldots, P_k$ be a list of all other predicate letters occurring in $A$, and let $P_1', \ldots, P_k'$ be their $m$-shifts, where $m = |y|$. Next, let $z$ be distinct list of new variables. Then

$$[C(x,z)/P(x)]A \models \left[P_1(x_1) \ldots P_k(x_k)/P_1'(x_1, z) \ldots P_k'(x_k, z)\right][C(x,z)/P'(x,z)]A^m,$$

where every $P_i(x_i)$ is an atomic formula with distinct $x_i$.

In fact, by Lemma 2.5.20

$$[C(x,z)/P'(x,z)]A^m \models [C(x,z)/P'(x,z)][P'(x,z), P_1'(x_1, z), \ldots, P_k'(x_k, z)]/P'(x), P_1(x_1), \ldots, P_k(x_k)]A \models [C(x,z), P_1'(x_1, z), \ldots, P_k'(x_k, z)]/P'(x), P_1(x_1), \ldots, P_k(x_k)]A,$$

hence by the same lemma

$$[P_1(x_1) \ldots P_k(x_k)/P_1'(x_1, z) \ldots P_k'(x_k, z)]/[C(x,z)/P'(x,z)]A^m \models [C(x,z), P_1(x_1), \ldots, P_k(x_k)/P'(x), P_1(x_1), \ldots, P_k(x_k)]A,$$

and the latter formula is (congruent to) $[C(x,z)/P(x)]A$. So $[C(x,z)/P(x)]A$ is a strict substitution instance of $A^m$. Since by 2.5.9(2)

$$[C(x,y)/P(x)]A \models [y/z][C(x,z)/P(x)]A,$$

this proves our claim.

Now we can apply induction. As we know, every complex substitution is a composition of simple substitutions. So it is sufficient to show that applying a
simple substitution $S$ to a formula $[y/z]S_0A^m$, where $S_0$ is strict, can also be presented in this form.

Note that we may assume that $r(z) \cap \text{FV}(S) = \emptyset$ — otherwise change the list of extra parameters of $A^m$. So by 2.5.15 we have

$$S[y/z]S_0A^m \triangleq [y/z]SS_0A^m$$

As we have already proved, for $B \triangleq S_0A^m$

$$SB \triangleq [u/t]S_1B^k$$

for some $k$, strict substitution $S_1$ and variable renaming $[t \mapsto u]$. By 2.5.28,

$$B^k \triangleq (S_0A^m)^k \triangleq S_2(A^m)^k \triangleq S_2A^{m+k},$$

where $S_2$ is a strict substitution. Thus

$$[y/z]SB \triangleq [y/z][u/t]S_1S_2A^{m+k},$$

and $S_1S_2$ is strict as required.

Let $\overline{A^m}$ be a universal closure of $A^m$ (for $m \geq 0$); thus $\overline{A^m} = \forall z_1 \ldots \forall z_m A^m$ for a sentence $A$.

For a set of formulas $\Gamma$, let $\text{Sub}(\Gamma)$ be the set of all their substitution instances, $\overline{\text{Sub}(\Gamma)}$ the set of all universal closures of formulas from $\text{Sub}(\Gamma)$. Both these sets are closed under congruence if $\Gamma$ is a set of sentences.

2.6 First-order logics

Definition 2.6.1 An $(N)$-modal predicate logic (m.p.l.) is a set $L \subseteq MF_N$ such that

(m0) $L$ contains classical propositional tautologies;

(m1) $L$ contains the propositional axioms

$$AK_i := \Box_i(p \supset q) \supset (\Box_ip \supset \Box_iq).$$

(m2) $L$ contains the predicate axioms (for some fixed $P, q$ and arbitrary $x, y$):

(Ax12) $\forall xP(x) \supset P(y)$;
(Ax13) $P(y) \supset \exists xP(x)$;
(Ax14) $\forall x(q \supset P(x)) \supset (q \supset \forall xP(x))$;
(Ax15) $\forall x(P(x) \supset q) \supset (\exists xP(x) \supset q)$;

(m3) $L$ is closed under the rules

$$\frac{A, (A \supset B)}{B} \quad (\text{Modus Ponens, or MP});$$
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\[
\begin{align*}
A & \quad (\text{Necessitation, or } \Box\text{-introduction}); \\
\Box_i A & \quad (\text{Generalisation, or } \forall\text{-introduction})
\end{align*}
\]

(for any \(x \in \text{Var}\)).

(m4) \(L\) is closed under \(MF_N\)-substitutions.

Definition 2.6.2 An \((N-)\)modal predicate logic with equality \((m.p.l.\戰)\) is a set \(L \subseteq MF_N\) satisfying (m0)-(m3) from 2.6.1 and also

(m4\(=\)) \(L\) is closed under \(MF_N\)-substitutions;

(m5\(=\)) \(L\) contains the axioms of equality (for arbitrary \(x, y\) and fixed \(P\)):

\[
\begin{align*}
(Ax16) & \quad x = x, \\
(Ax17) & \quad x = y \supset (P(x) \supset P(y)).
\end{align*}
\]

Definition 2.6.3 A superintuitionistic predicate logic \((s.p.l.)\) is a set \(L \subseteq IF\) such that

(s1) \(L\) contains the axioms of Heyting's propositional calculus \(H\) (cf. Section 1.1.2);

(s2) = (m2) \(L\) contains the predicate axioms;

(s3) \(L\) is closed under the rules (MP), \(\forall\text{-introduction}, \text{see} (m3);\)

(s4) \(L\) is closed under IF-substitutions.

Definition 2.6.4 A superintuitionistic predicate logic with equality \((s.p.l.\戰)\) is a set \(L \subseteq IF^=\) satisfying (s1)-(s3) from 2.6.3 and

(s4\(=\)) \(L\) is closed under \(IF^=\)-substitutions;

(s5\(=\)) = (m5\(=\)) \(L\) contains the axioms of equality.

Further, by a 'first-order logic' we mean an arbitrary logic, modal or superintuitionistic, with or without equality.

Elements of a logic are called theorems, and we often write \(L \vdash A\) instead of \(A \in L\).

Definition 2.6.5 A logic \(L\) (modal or superintuitionistic) is called consistent if \(\bot \not\in L\).

\(M_N\) (respectively \(M_N\), \(S, S^=\)) denotes the set of all \(N\)-m.p.l. (respectively, \(N\)-m.p.l.\(=\), s.p.l.; s.p.l.\(=\)). The smallest \(N\)-m.p.l. (respectively, \(N\)-m.p.l.\(=\), s.p.l., s.p.l.\(=\)) is denoted by \(QK_N\) (respectively, by \(QK_N^=, QH, QH^=\)). \(L + \Gamma\) denotes the smallest m.p.l. containing an m.p.l. \(L\) and a set \(\Gamma \subseteq MF\). This notation is obviously extended to other cases (m.p.l.\(=\), s.p.l., s.p.l.\(=\)).

It is well-known that every theorem of \(QH^=\) can be obtained by a formal proof, which is a sequence of formulas that are either substitution instances of axioms or are obtained from earlier formulas by applying inference rules cited in (s3). The same is true for \(QK_N^=\), but with the rules from (m3). The notion of
a formal proof extends to logics of the form $\text{QH}^{(=)} + \Gamma, \text{ QK}_N^{(=)} + \Gamma$, with the only difference that formulas from $\Gamma$ can also be used as axioms. By applying deduction theorems, we can reduce the provability in $L + \Gamma$ to provability in $L$ in a more explicit way, see Section 2.8 below.

**Definition 2.6.6** The quantified version of a modal (respectively, superintuitionistic) propositional logic $\Lambda$ is

$$\text{QA} := \text{QK}_N + \Lambda \quad (\text{respectively, QG} := \text{QH} + \Lambda).$$

**Definition 2.6.7** The propositional part of a predicate logic $L$ is the set of its propositional formulas:

- $L_{\pi} := L \cap L_N$ (for an $N$-modal $L$);
- $L_{\pi} := L \cap L_0$ (for a superintuitionistic $L$).

The following is obvious.

**Lemma 2.6.8**

(1) If $L$ is an $N$-m.p.l. or an s.p.l., then $L_{\pi}$ is a propositional logic of the corresponding kind.

(2) If $L$ is a predicate logic with equality, then $L_{\pi} = (L^0)_{\pi}$.

A well-known example of an s.p.l. is the classical predicate logic $\text{QCL}^{(=)} = \text{Q(CL)}^{(=)} = \text{QH}^{(=)} + EM,$

where $EM = p \lor \neg p$ (see Section 1.1). An s.p.l.(=) $L$ is called intermediate iff $L \subseteq \text{QCL}^{(=)}$. Note that $\text{QCL}^{(=)}$ is included in $\text{QK}_N^{(=)}$ (and thus, in any m.p.l.(=)).

The rule (m4) means that together with a formula $A$, $L$ contains all its $MF_N$-substitution instances (and similarly for (s4)). In particular, $L$ contains every formula congruent to $A$, because it is a substitution instance under the dummy substitution. Hence we easily obtain

**Lemma 2.6.9** If $A \cong B$ then $(A \equiv B) \in L$ (for any m.p.l.(=) or s.p.l.(=) $L$).

**Proof** $A \cong B$ implies $(A \equiv B) \cong (A \equiv A)$, and $(A \equiv A) = [A/p](p \equiv p)$, thus $(A \equiv A) \in L$ by (m0), (m4) (or (s0), (s4)). Hence $(A \equiv B) \in L$.  

**Lemma 2.6.10** Let $A, B$ be formulas in the language of a predicate logic $L$. Then for a variable $x \notin FV(B)$:

(1) $L \vdash \forall x(B \supset A) \supset B \supset \forall xA$,

(2) $L \vdash \forall x(A \supset B) \supset \exists xA \supset B$. 

Proof (1) Consider the substitution $S = [A, B / P(x), q]$; note that $x \notin FV(S)$. By Lemma 2.5.13, up to congruence, $S$ distributes over $\supset$ and $\forall x$ (since $x \notin FV(S)$). Congruence also distributes over $\supset$ and $\forall x$, by 2.3.17. Thus

$$S(Ax14) \equiv \forall x(B \supset A) \supset B \supset \forall x A,$$

and so the latter formula is in $L$.

The proof of (2) is similar. ■

Definition 2.6.11 Let $\Gamma$ be a set of formulas in the language of a predicate logic $L$. An $L$-inference of a formula $B$ from $\Gamma$ a sequence $A_1, \ldots, A_n$, in which $A_n = B$ and every $A_i$ is either a theorem of $L$, or $A_i \in \Gamma$, or $A_i$ is obtained from earlier formulas by applying MP, or $A_i$ is obtained from an earlier formula by $\forall$-introduction over a variable that is not a parameter of any formula from $\Gamma$. If such an inference exists, we say that $B$ is $L$-derivable from $\Gamma$, notation: $\Gamma \vdash_L B$.

Note that we distinguish inferences from proofs; the latter may also use substitution and $\square$-introduction.

From definitions we easily obtain

Lemma 2.6.12 $\vdash_L A$ iff $L \vdash A$.

Proof 'If'. If $L \vdash A$, then $A$ is an $L$-inference (from $\emptyset$).

'Only if'. By induction on the length of an $L$-inference of $A$ from $\emptyset$.

Recall the simplest first-order analogue of the propositional deduction theorem:

Lemma 2.6.13 If $\Gamma \cup \{A\} \vdash_L B$, then $\Gamma \vdash_L A \supset B$.

Proof Standard, by induction on the length of an inference of $B$ from $\Gamma \cup \{A\}$.

(i) If $B \in L \cup \Gamma$, then $A \supset B$ follows by MP from $B$ and $B \supset (A \supset B)$, which is a substitution instance of (Ax1).

(ii) If $B$ is obtained by MP from $C$ and $C \supset B$ and by the induction hypothesis $\Gamma \vdash_L A \supset C$, $A \supset C \supset B$, note that

$$\Gamma \vdash_L (A \supset C \supset B) \supset (A \supset C, \supset A \supset B),$$

from a tautology (or an intuitionistic axiom (Ax2)); hence $\Gamma \vdash_L A \supset B$ by MP.

(iii) Suppose $B = \forall x C$, $\Gamma \vdash_L A \supset C$ by induction hypothesis and $x$ is not a parameter in $\Gamma \cup \{A\}$, then $\Gamma \vdash_L \forall x (A \supset C)$. By Lemma 2.6.10, $L \vdash \forall x (A \supset C) \supset (A \supset B)$, therefore $\Gamma \vdash_L A \supset B$ by MP.

(iv) If $B = A$, then $(A \supset B) = (A \supset A)$, which is $L$-derivable by a standard argument; see any textbook in mathematical logic.

Hence we obtain an equivalent characterisation of $L$-derivability.
Lemma 2.6.14 Let $\Gamma$ be a set of $N$-modal (or intuitionistic) predicate formulas, $L$ an $N$-modal (or superintuitionistic) predicate logic (with or without equality). Then for any $N$-modal (or intuitionistic) formula $B$, $\Gamma \vdash_L B$ iff there exists a finite $X \subseteq \Gamma$ such that

$$L \vdash \bigwedge X \supset B.$$  

As usual, we also include the case $X = \emptyset$, with $\top$ as the empty conjunction.

Of course the notation $\bigwedge X$ makes sense, due to the commutativity and the associativity of conjunction in intuitionistic logic.

Proof Since every inference from $\Gamma$ contains a finite number of formulas from $\Gamma$, it is clear that $\Gamma \vdash_L B$ iff there exists a finite $X \subseteq \Gamma$ such that $X \vdash_L B$.

So we have to show that

(1) $$X \vdash_L B \text{ iff } L \vdash \bigwedge X \supset B.$$  

The proof is by induction on $|X|$.

If $X = \emptyset$, then $\vdash_L B$ iff $L \vdash B$ by 2.6.12.

But $L \vdash B \supset \top \supset B$ (this is an instance of (Ax1)), so by MP, $L \vdash B$ implies $L \vdash \top \supset B$.

The other way round, $L \vdash \top \supset B$ implies $L \vdash B$, since $L \vdash \top$. Therefore $L \vdash B$ iff $L \vdash \top \supset B$.

Suppose (1) holds for $X$ (and any $B$). Then it also holds for $X \cup \{A\}$.

In fact, by 2.6.13 and our assumption

$$X \cup \{A\} \vdash B \text{ iff } X \vdash_L A \supset B \text{ iff } L \vdash \bigwedge X \supset A \supset B.$$  

The latter is equivalent to

$$L \vdash (\bigwedge X) \land A \supset B,$$

due to

(2) $$H \vdash (p \supset q \supset r) \equiv (p \land q \supset r).$$  

(2) follows in a standard way by the deduction theorem from

$$p \supset q \supset r, p \land q \vdash_H r$$

and

$$p \land q \supset r, p, q \vdash_H r.$$  

The next lemmas collects some useful theorems and admissible rules for different types of logics.
Lemma 2.6.15 The following theorems (admissible rules) are in every first-order logic $L$:

(i) Bernays rules:

\[
\begin{align*}
B \supset A & \quad, \quad A \supset B \\
B \supset \forall x A & \quad, \quad \exists x A \supset B
\end{align*}
\]

if $x \not\in \text{FV}(B)$;

(ii) \[
\forall x A \supset [y/x]A, \quad \forall x A \supset A,
\]

\[
[y/x]A \supset \exists x A, \quad A \supset \exists x A;
\]

(iii) variable substitution rule:

\[
A \quad \frac{[y/x]A}{[y/x]A};
\]

(iv) $\forall x (A \supset B) \supset (Qx A \supset Qx B)$;

(v) monotonicity rules for quantifiers

\[
A \supset B \quad \frac{Qx A \supset Qx B}{Qx A \supset Qx B};
\]

(vi) replacement rules for quantifiers

\[
A \equiv B \quad \frac{Qx A \equiv Qx B}{Qx A \equiv Qx B};
\]

(vii) $\forall x (A \land B) \equiv \forall x A \land \forall x B$;

(viii) $\exists x (A \lor B) \equiv \exists x A \lor \exists x B$;

(ix) $\forall x A \equiv A$ if $x \not\in \text{FV}(A)$;

(x) $\exists x A \equiv A$ if $x \not\in \text{FV}(A)$;

(xi) $\forall x (C \supset A) \equiv (C \supset \forall x A)$ if $x \not\in \text{FV}(C)$;

(xii) $\forall x (A \supset C) \equiv (\exists x A \supset C)$ if $x \not\in \text{FV}(C)$;

(xiii) $\forall x \neg A \equiv \neg \exists x A$;

(xiv) $\exists x (C \supset A) \supset (C \supset \exists x A)$ if $x \not\in \text{FV}(C)$;

(xv) $\exists x (A \supset C) \supset (\forall x A \supset C)$ if $x \not\in \text{FV}(C)$;

(xvi) $\exists x \neg A \supset \neg \forall x A$;

(xvii) $\exists x (A \lor C) \equiv \exists x A \lor C$ if $x \not\in \text{FV}(C)$;
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(xviii) \( Qx(A \land C) \equiv Qx A \land C \), if \( x \notin FV(C) \), \( Q \in \{\forall, \exists\}; \)

(xix) \( \exists x(A \land B) \supset \exists x A \land \exists x B; \)

(xx) \( \forall x A \lor \forall x B \supset \forall x(A \lor B); \)

(xxi) \( \forall x A \lor C \supset \forall x(A \lor C) \) if \( x \notin FV(C); \)

(xxii) \( QxQyA \equiv QyQxA \) for \( Q \in \{\forall, \exists\}; \)

(xxiii) \( QxA \equiv Q(x \cdot \sigma)A \) for a quantifier \( Q \), a distinct list \( x \) and a permutation \( \sigma \) of \( I_n \), where \( n = |x|; \)

(xxiv) \( \exists x\forall yA \supset \forall y\exists xA; \)

(xxv) \( \forall xA \supset [y/x]A \) for a variable substitution \([y/x]\); \)

(xxvi) \( \forall x(A \equiv B) \supset (QxA \equiv QxB); \)

(xxvii) \( \forall(A \equiv A') \supset \forall([A/P(x)]B \equiv [A'/P(x)]B), \)
if \( B \) is non-modal (moreover, if \( P(x) \) is not within the scope of modal operators in \( B); \)

(xxviii) \( \frac{A \equiv A'}{[A/P(x)]B \equiv [A'/P(x)]B} \) \hspace{1cm} (replacement rule)

with the same restriction as in (xxvii).

So (xxviii) shows that up to equivalence, the universal closure \( \forall A \) does not depend on the order of quantifiers.

Similarly to the propositional case (Section 1.1), the replacement rule (xxviii) can be written as follows:

\[
\frac{A \equiv A'}{B(\ldots A \ldots) \equiv B(\ldots A' \ldots)}.
\]

Proof

(i) Readily follows from 2.6.10.

(ii) By Lemma 2.5.13 we obtain

\[
[A/P(x)](\forall xP(x) \supset P(y)) \equiv \forall xA \supset [y/x]A
\]

(note that \( x \notin FV[A/P(x)] \)). So since \( L \) contains (Ax12), it also contains (ii).

The particular case of this is \( \forall xA \supset A \). Hence \( \forall xA \supset A \) easily follows by induction on \( |x| \) and the transitivity of \( \supset \).

The dual claims for \( \exists \) are proved in a similar way.
(iii) If $L \vdash A$, then $L \vdash \forall x A$. Since $L \vdash \forall x A \supset [y/x] A$ by (ii), we obtain $L \vdash [y/x] A$ by MP. Therefore $L$ is closed under variable substitution, since every variable substitution is a composition of simple substitutions.

(iv) By the deduction theorem, it is sufficient to show

$$\forall x (A \supset B) \vdash_L Qx A \supset QxB.$$ 

First consider the case $Q = \forall$. We have the following 'abridged' $L$-inference from $\forall x (A \supset B)$:

1. $\forall x (A \supset B) \supset A \supset B$ by (ii)
2. $\forall x (A \supset B)$ by assumption
3. $A \supset B$ by 1, 2, MP
4. $\forall x A \supset A$ by (ii)
5. $\forall x A \supset B$ by 3, 4, transitivity
6. $\forall x A \supset \forall x B$ by 5, (i).

Here we apply the transitivity rule and the Bernays rule to $L$-derivability from $\Gamma$; the reader can easily see that they are really admissible in this situation.

For the case $Q = \exists$ the argument slightly changes in items 4–6.

4. $B \supset \exists x B$ by (ii)
5. $A \supset \exists x B$ by 3, 4, transitivity
6. $\exists x A \supset \exists x B$ by 5, (i).

(v) If $L \vdash A \supset B$, then $L \vdash \forall x (A \supset B)$ by generalisation. Since $L \vdash \forall x (A \supset B) \supset Qx A \supset QxB$ by (iv), we obtain $L \vdash Qx A \supset QxB$ by MP.

(vi) If $L \vdash A \equiv B$, then $L \vdash A \supset B$, $B \supset A$ by (Ax3), (Ax4)

17 and MP. Hence $L \vdash Qx A \supset QxB$, $QxB \supset QxA$ by (v), and thus $L \vdash QxA \equiv QxB$ by ∧-introduction $\frac{C, D}{C \land D}$, which is admissible in $L$.

(vii) Since $L \vdash A \land B \supset A$ by (Ax3) and substitution, it follows that $L \vdash \forall x (A \land B) \supset \forall x A$, by (v), and thus $\forall x (A \land B) \vdash_L \forall x A$.

Similarly from (Ax4) we obtain

$$\forall x (A \land B) \vdash_L \forall x B;$$

hence

$$\forall x (A \land B) \vdash_L \forall x A \land \forall x B;$$

\footnote{In the modal case we may use (Ax3), (Ax4) as classical tautologies.}
by \wedge\text{-}introduction, and therefore

\[ L \vdash \forall x (A \land B) \supset \forall x A \land \forall x B, \]

by the deduction theorem.

To show the converse we may also use the deduction theorem. In fact, we have the following abridged inference from \( \forall x A \land \forall x B \):

1. \( \forall x A \land \forall x B \) by assumption
2. \( \forall x A \land \forall x B \supset \forall x A \) by (Ax3), substitution
3. \( \forall x A \) by 1,2, MP
4. \( \forall x A \supset A \) by (ii)
5. \( A \) by 3,4, MP.

A similar argument shows \( \forall x A \land \forall x B \vdash L B \).

Hence \( \forall x A \land \forall x B \vdash L A \land B \), by \wedge\text{-}introduction and therefore \( \forall x A \land \forall x B \vdash L \forall x (A \land B) \).

(viii) It is sufficient to show

\[ L \vdash \exists x A \lor \exists x B \supset \exists x (A \lor B) \]

and

\[ L \vdash \exists x (A \lor B) \supset \exists x A \lor \exists x B. \]

For the first, we can use the \lor\text{-}introduction rule:

\[
\frac{X \supset Z, \; Y \supset Z}{X \lor Y \supset Z},
\]

which is admissible in \( L \), due to (Ax5).

So it remains to show

\[ L \vdash \exists x A \supset \exists x (A \lor B), \; \exists x B \supset \exists x (A \lor B). \]

But these follow by (v) from \( A \supset A \lor B \), \( B \supset A \lor B \), which are substitution instances of (Ax6), (Ax7).

The converse \( L \vdash \exists x (A \lor B) \supset \exists x A \lor \exists x B \) follows by the Bernays rule from

\[ L \vdash A \lor B \supset \exists x A \lor \exists x B. \]

For the latter we can also use \lor\text{-}introduction after we show

\[ L \vdash A \supset \exists x A \lor \exists x B, \; B \supset \exists x A \lor \exists x B. \]

But \( A \supset \exists x A \lor \exists x B \) follows by transitivity from \( A \supset \exists x A \) (ii) and \( \exists x A \supset \exists x A \lor \exists x B \) (Ax6). The argument for \( B \supset \exists x A \lor \exists x B \) is similar.
(ix) \( L \vdash \forall x A \supset A \) by (ii). \( L \vdash A \supset \forall x A \) follows by the Bernays rule from \( L \vdash A \supset A \).

(x) The proof is similar to (ix).

(xi) We have \( \forall x (C \supset A) \vdash_L C \supset \forall x A \) by Bernays' rule, and thus \( L \vdash \forall x (C \supset A) \supset \forall x A \) by Deduction theorem.

For the converse, first note that

\[
C \supset \forall x A, \ C \vdash_L A
\]

by the abridged inference

\[
C, \ C \supset \forall x A, \ \forall x A, \ \forall x A \supset A, \ A,
\]

hence

\[
C \supset \forall x A \vdash_L C \supset A
\]

by Deduction theorem, and thus

\[
C \supset \forall x A \vdash_L \forall x (C \supset A),
\]

since \( x \notin FV(C) \). Therefore

\[
L \vdash C \supset \forall x A \supset \forall x (C \supset A).
\]

(xii) Along the same lines as in (xi), using the second Bernays' rule and the theorem \( A \supset \exists x A \). We leave the details to the reader.

(xiii) Readily follows from (xii), with \( C = \bot \).

(xiv) By Deduction theorem, this reduces to \( \exists x C \supset A, C \vdash_L \exists x A \). The latter follows by the abridged inference

\[
C, \ C \supset \exists x C, \ \exists x C, \ \exists x C \supset A, \ A, \ A \supset \exists x A, \ \exists x A.
\]

(xv) By the Bernays rule and deduction theorem from

\[
A \supset C \vdash_L \forall x A \supset C.
\]

By Deduction theorem, the latter reduces to

\[
A \supset C, \forall x A \vdash_L C,
\]

which we leave as an easy exercise for the reader.

(xvi) = (xiv) for \( C = \bot \).
(xvii) By (x), $L \vdash \exists x C \equiv C$, so the admissible replacement rule

\[
\begin{array}{c}
B_1 \equiv B_2 \\
\hline
A \lor B_1 \equiv A \lor B_2
\end{array}
\]

yields

$L \vdash \exists x A \lor \exists x C \equiv \exists x A \lor C$.

Since also

$L \vdash \exists x (A \lor C) \equiv \exists x A \lor \exists x C$

by (viii), and we obtain (xvi) by transitivity for $\equiv$.

(xviii) If $Q = \forall$, the argument is similar to (xvi), using (ix), (vii), and the replacement rule

\[
\begin{array}{c}
B_1 \equiv B_2 \\
\hline
A \land B_1 \equiv A \land B_2
\end{array}
\]

Let $Q = \exists$. Then

$L \vdash \exists x (A \land C) \supset \exists x A \land C$

follows from (xvii), (x), and the replacement rule

\[
\begin{array}{c}
A_1 \supset A_2 \land B_1, B_1 \equiv B_2 \\
\hline
A_1 \supset A_2 \land B_2
\end{array}
\]

for $A_1 = \exists x (A \land C)$, $A_2 = \exists x A$, $B_1 = \exists x C$, $B_2 = C$.

Finally, to show

$L \vdash \exists x A \land C \supset \exists x (A \land C)$

we argue as follows. First we obtain

$C \vdash_L A \supset A \land C$

by the deduction theorem and $\land$-introduction. Hence

$C \vdash_L \exists x A \supset \exists x (A \land C)$

by (v); this rule is still admissible in $L$-inferences from $C$, since $\forall x$-introduction is admissible.

So by the deduction theorem,

$L \vdash C \supset \exists x A \supset \exists x (A \land C)$.

The latter formula is equivalent to

$\exists x A \land C \supset \exists x (A \land C)$.

In fact,

$\exists x A \land C \vdash_L \exists x (A \land C)$,

since $\exists x A \land C \vdash_L C$ and $\exists x A \land C \vdash \exists x A$, and we may use $C \supset \exists x A \supset \exists x (A \land C)$ and MP to obtain $\exists x (A \land C)$.

Therefore

$L \vdash \exists x A \land C \supset \exists x (A \land C)$. 
(xix) The proof is similar to (vii). From (Ax3), (Ax4) by monotonicity we obtain
\[ L \vdash \exists x(A \land B) \supset \exists xA, \exists x(A \land B) \supset \exists xB. \]
Hence \( L \vdash \exists x(A \land B) \supset \exists xA \land \exists xB \) by \( \land \)-introduction and the deduction theorem.

(xx) The proof is similar to (viii).
First we note that
\[ L \vdash \forall xA \supset A \lor B \]
by transitivity from \( \forall xA \supset A, A \supset A \lor B. \)
Similarly
\[ L \vdash \forall xB \supset A \lor B. \]
Hence by \( \lor \)-introduction,
\[ L \vdash \forall xA \lor \forall xB \supset A \lor B, \]
and (xviii) follows by the Bernays rule.

(xxii) By (ii), \( L \vdash \forall yA \supset A \); hence
\[ L \vdash \forall x\forall yA \supset \forall xA \]
by monotonicity and
\[ L \vdash \forall x\forall yA \supset \forall y\forall xA, \]
by the Bernays rule.
The converse is obtained in the same way.
The case of \( \exists \) is similar.

(xxiii) Since \( \sigma \) is a composition of elementary transpositions, it is sufficient to consider \( \sigma = \sigma_{i,i+1}^n. \) So let \( x = yx_ix_{i+1}z, \) then \( x \cdot \sigma = yx_{i+1}x_iz. \) We have (in \( L \))
\[ \vdash Qx_ix_{i+1}QzA \equiv Qx_{i+1}Qx_iQzA \]
by (xxii), hence
\[ QyQx_iQx_{i+1}QzA \equiv QyQx_{i+1}Qx_iQzA \]
by (vi), i.e. we obtain
\[ QxA \equiv Q(x \cdot \sigma)A. \]

(xxiv) Since \( L \vdash \forall yA \supset A, \) we obtain \( L \vdash \exists x\forall yA \supset \exists xA \) by monotonicity; hence
\[ L \vdash \exists x\forall yA \supset \forall y\exists xA \]
by the Bernays rule.
First consider the case when $x \cap y = \emptyset$. We argue by induction on $n = |x|$. The base $n = 1$ was proved in (ii).

Next, if $x = x_1x'$, $y = y_1y'$, and we know that $L \vdash \forall x' A \supset [y'/x']A$, then by (v),

$$L \vdash \forall x' A \supset \forall x_1[y'/x']A.$$

By (v) again,

$$L \vdash \forall x_1[y'/x']A \supset [y_1/x_1][y'/x']A,$$

hence

$$L \vdash \forall xA \supset [y_1/x_1][y'/x']A,$$

by transitivity. Since $x_1 \notin y'$, the conclusion is $[y/x]A$ as we need.

Now in the general case, let $z$ be a distinct list of new variables, $|z| = n$. Then as we have proved, $L \vdash \forall xA \supset [z/x]A$, and thus

$$L \vdash \forall xA \supset \forall z[z/x]A$$

by the Bernays rule. We also have

$$L \vdash \forall z[z/x]A \supset [y/z][z/x]A$$

from the above, so by transitivity

$$L \vdash \forall xA \supset [y/z][z/x]A.$$

Since $[y/z][z/x]A \Rightarrow [y/x]A$, this completes the argument.

We have the following theorems in $L$:

1. $A \equiv B \supset A \supset B$ (Ax3)
2. $\forall x(A \equiv B) \supset \forall x(A \supset B)$ 1, monotonicity (v)
3. $\forall x(A \supset B) \supset QxA \supset QxB$ (iv)
4. $\forall x(A \equiv B) \supset QxA \supset QxB$ 2, 3, transitivity.

Hence $\forall x(A \equiv B) \vdash_L QxA \supset QxB$. In the same way (using Ax4) we obtain

$$\forall x(A \equiv B) \vdash_L QxB \supset QxA.$$

Hence by propositional logic

$$\forall x(A \equiv B) \vdash_L QxA \equiv QxB,$$

which implies (xxvi) by the deduction theorem.

To simplify the notation, we write $B(A)$ instead of $[A/P(x)]B$. So we show

$$\bar{\forall}(A \equiv A') \vdash_L \bar{\forall}(B(A) \equiv B(A'))$$

by induction on the length of $B$ and then apply the deduction theorem.
If \( B = B_1 \ast B_2 \) for a propositional connective \( \ast \), and by the induction hypothesis we have
\[
\forall (A \equiv A') \vdash L \forall (B_1(A) \equiv B_1(A')) , \forall (B_2(A) \equiv B_2(A')) ,
\]
hence we deduce (by (xxv))
\[
B_1(A) \equiv B_1(A') , B_2(A) \equiv B_2(A') .
\]
Now we can apply the admissible propositional rule
\[
\begin{array}{c}
A_1 \equiv A'_1 , A_2 \equiv A'_2 \\
\hline
A_1 \ast A_2 \equiv A'_1 \ast A'_2
\end{array}
\]
and obtain \( B(A) \equiv B(A') \). Since \( \forall (A \equiv A') \) is closed, \( \forall \)-introduction is also applicable.

If \( B = Q_y B_1 \), we change it to a congruent formula so that \( y \notin FV[A/P(x)] \).
Then \( B(A) = Q_y B_1(A) \), \( B(A') = Q_y B_1(A') \). If by the induction hypothesis
\[
\forall (A \equiv A') \vdash L \forall (B_1(A) \equiv B_1(A')) ,
\]
then
\[
\forall (A \equiv A') \vdash L \forall (B_1(A) \equiv B_1(A')).
\]
Hence we deduce
\[
Q_y B_1(A) \equiv Q_y B_1(A')
\]
by (xxvi) and MP and then apply generalisation.

In the case when \( B = \Box_i B_1 \) the propositional replacement rule (1.1.1) can be used, the details are left to the reader.

(xxviii) If \( L \vdash A \equiv A' \), then by generalisation \( L \vdash \forall (A \equiv A') \). Hence by (xxvii) and MP, \( L \vdash \forall (B(A) \equiv B(A')) \). Now we can eliminate \( \forall \) by (xxv) and MP.

Lemma 2.6.16 Theorems in logics with equality:

(1) \( x = y \supset y = x \);

(2) \( x = y \land y = z \supset x = z \);

(3) \( x = y \supset [x/z]A \equiv [y/z]A \);

(4) \( x_1 = y_1 \land \ldots \land x_n = y_n \supset P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n) \).

Proof
(1) From (Ax17) by substitution \([x = z/P(x)]\) we obtain
\[ L \vdash x = y \supset x = z \supset y = z. \]
Hence by substitution \([x/z]\) (2.6.15 (iii))
\[ L \vdash x = y \supset x = x \supset y = x. \]
This implies \(x = y \vdash y = x\) (due to (Ax16)), whence \(L \vdash x = y \supset y = x\)
by the deduction theorem.

(2) From (Ax17)
\[ y = z \supset P(y) \supset P(z) \]
by substitution \([x = y/P(y)]\) we have
\[ y = z \supset x = y \supset x = z. \]
This is equivalent to (2) by H.

(3) By (Ax17)
\[ x = y \vdash L P(x) \supset P(y) \]
and
\[ y = x \vdash L P(y) \supset P(x). \]
Hence by (1), \(x = y \vdash L P(y) \supset P(x)\), so by \&-introduction \(x = y \vdash L P(x) \equiv P(y)\), and thus
\[ L \vdash x = y \supset P(x) \equiv P(y) \] (#1)
by the deduction theorem.
Now we apply the substitution
\[ S := [B/P(x)], \]
where
\[ B := [xx_1/xx]A \]
and \(x_1 \notin V(A)\). Then
\[ SP(y) \equiv [y/x]B \equiv [y/x][xx_1/xx]A. \]
Note that
\[ [y/x] \cdot [xx_1/xx] = [x_1 y/xx], \]
since this substitution sends \(x\) to \(x_1\), \(z\) to \(y\) and does not change other variables. So by 2.3.25(4),
\[ SP(y) \equiv [x_1 y/xx]A, \]
and thus (§1) implies
\[ L \vdash x = y \supset [x_1x/xz]A \equiv [x_1y/sz]A. \]

Hence
\[ L \vdash [x/x_1](x = y \supset [x_1x/xz]A \equiv [x_1y/xz]A) \]
by 2.6.15(iii). The latter formula is congruent to
\[ x = y \supset [x/x_1][x_1/xz]A \equiv [x/x_1][x_1y/xz]A. \]

Now we can again apply 2.3.25(4):
\[
\begin{align*}
[x/x_1][x_1x/xz]A & \equiv [xx/zz_x]A, \\
[x/x_1][x_1y/xz]A & \equiv [yx/zz_x]A.
\end{align*}
\]

It remains to note that
\[
\begin{align*}
[xx/zz_x]A & \equiv [x/z]A, \\
[yx/zz_x][x_1y/xz]A & \equiv [y/z]A,
\end{align*}
\]
since \( x_1 \) does not occur in \( A \).

(4) By the deduction theorem, it suffices to show
\[ x_1 = y_1 \land \ldots \land x_n = y_n \vdash_L P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n). \quad (\#2) \]

For this we show by induction that
\[
\begin{align*}
x_1 = y_1 \land \ldots \land x_m = y_m & \vdash_L [x_1, \ldots, x_m/z_1, \ldots, z_m]P(z) \equiv [y_1, \ldots, y_m/z_1, \ldots, z_m]P(z). \quad (\#3_m)
\end{align*}
\]
for a list of new variables \( z = (z_1, \ldots, z_n) \).

The case \( m = 0 \) is trivial.

Suppose \( (\#3_m) \) holds; to check \( (\#3_{m+1}) \), assume
\[ x_1 = y_1 \land \ldots \land x_m = y_m \land x_{m+1} = y_{m+1}. \quad (\#4) \]
Then by \((\forall z_{m+1})\)-introduction (since \( z_{m+1} \) is new)
\[ \forall z_{m+1}(A_m \equiv B_m), \]
where
\[
\begin{align*}
A_m := [x_1, \ldots, x_m/z_1, \ldots, z_m]P(z), \\
B_m := [y_1, \ldots, y_m/z_1, \ldots, z_m]P(z).
\end{align*}
\]

Hence by 2.6.15 (ii) and MP
\[ [x_{m+1}/z_{m+1}]A_m \equiv [x_{m+1}/z_{m+1}]B_m. \quad (\#5) \]
The assumption \( (\#4) \) implies \( x_{m+1} = y_{m+1} \), so by (iii) we have
\[ [x_{m+1}/z_{m+1}]B_m \equiv [y_{m+1}/z_{m+1}]B_m. \quad (\#6) \]
From \( (\#5), (\#6) \), by transitivity we obtain \([x_{m+1}/z_{m+1}]A_m \equiv [y_{m+1}/z_{m+1}]B_m \), i.e. \( A_{m+1} \equiv B_{m+1} \). Now since \( (\#2) \) is \( (\#3_n) \), the claim is proved.
Lemma 2.6.17 Theorems in QCL (and thus, in any m.p.l.):

(1) \( \exists x (A \supset C) \equiv (\forall x A \supset C) \) if \( x \notin \text{FV}(C) \);

(2) \( \exists x \neg A \equiv \neg \forall x A \);

(3) \( \exists x (C \supset A) \equiv (C \supset \exists x A) \) if \( x \notin \text{FV}(C) \);

(4) \( \forall x (A \lor C) \equiv \forall x A \lor C \) if \( x \notin \text{FV}(C) \).

Proof

1. We have in QCL:
   1. \( \neg \exists x (A \supset C) \equiv \forall x \neg (A \supset C) \) (Lemma 2.6.15(xiii))
   2. \( \neg (A \supset C) \equiv A \land \neg C \) (by a propositional tautology)
   3. \( \forall x \neg (A \supset C) \equiv \forall x (A \land \neg C) \) (2, replacement)
   4. \( \forall x (A \land \neg C) \equiv \forall x A \land \neg C \) (2.6.15(xix))
   5. \( \forall x A \land \neg C \equiv \neg (\forall x A \supset C) \) (by a propositional tautology).

   Hence we obtain

   (a) \( \neg \exists x (A \supset C) \equiv \neg (\forall x A \supset C) \) (by transitivity from 1, 3, 4, 5).

   This implies (1), due to the admissible rule

   \[
   \frac{\neg A \equiv \neg B}{A \equiv B}.
   \]

(ii) Take \( C = \perp \) in (1)

(iii) We have in QCL:

   (a) \( (C \supset A) \equiv (\neg C \lor A) \) (from a propositional tautology)
   (b) \( \exists x (C \supset A) \equiv \exists x (\neg C \lor A) \) (1, replacement)
   (c) \( \exists x (\neg C \lor A) \equiv \neg C \lor \exists x A \) (2.6.15(xvii))
   (d) \( \neg C \lor \exists x A \equiv C \supset \exists x A \) (from a propositional tautology)
   (e) \( \exists x (C \supset A) \equiv C \supset \exists x A \) (by transitivity from 2, 3, 4).

(iv) We have in QCL:

   (a) \( A \lor C \equiv \neg C \supset A \) (from a propositional tautology)
   (b) \( \forall x (A \lor C) \equiv \forall x (\neg C \supset A) \) (1, replacement)
   (c) \( \forall x (\neg C \supset A) \equiv \neg C \supset \forall x A \) (2.6.15(xi))
   (d) \( \neg C \supset \forall x A. \equiv \forall x A \lor C \) (from a propositional tautology)
(e) \( \forall x (A \lor C) \equiv \forall x A \lor C \) (by transitivity from 2, 3, 4).

Lemma 2.6.18 Theorems in modal logics (where \( \Box \in \{ \square_i, \Diamond_i \} \), \( x \) is a list of variables):

1. \( \Box \forall x A \supset \forall x \Box A \);

2. \( \exists x \Box A \supset \Box \exists x A \);

3. \( x = y \supset \Box_\alpha (x = y) \) for \( N \)-modal logics with equality, \( \alpha \in \Gamma_N^\infty \).

Proof

1. \( L \vdash \forall x A \supset A \) (2.6.15 (ii)), hence \( L \vdash \Box \forall x A \supset \Box A \) by monotonicity (1.1.1), which is also admissible in the predicate case. Therefore \( L \vdash \Box \forall x A \supset \forall x \Box A \), by the Bernays rule.

2. Similar to (1), using \( A \supset \exists x A \).

3. Let us first prove \( x = y \supset \Box_i (x = y) \). So assuming \( x = y \), we prove \( \Box_i (x = y) \).

   1. \( x = y \supset \Box_i (z = x) \supset \Box_i (z = y) \) Ax17, substitution \([\Box_i (z = x) / P(x)]\).

   2. \( \Box_i (z = x) \supset \Box_i (z = y) \) 1, \( x = y \), MP.

   3. \( \forall z (\Box_i (z = x) \supset \Box_i (z = y)) \) 2, \( \forall z \)-introduction (if \( z \) is new).

   4. \( \Box_i (x = x) \supset \Box_i (x = y) \) 3, Axi12, MP.

   5. \( \Box_i (x = x) \) Ax16, \( \Box \)-introduction.

   6. \( \Box_i (x = y) \) 4, 5, MP.

   Hence \( L \vdash x = y \supset \Box_i (x = y) \).

   For arbitrary \( \alpha \) apply induction and monotonicity rules, cf. Lemma 1.1.1.

We use special notation for some formulas.

Intuitionistic formulas:

\[ \begin{align*}
CD & := \forall x (P(x) \lor q) \supset \forall x P(x) \lor q \text{ (the constant domain principle)}; \\
CD^- & := \forall x (\neg P(x) \lor q) \supset \forall x \neg P(x) \lor q; \\
Ma^- & := \neg \exists x P(x) \supset \exists x \neg P(x) \text{ (strong Markov principle);} \\
Ma^+ & := \exists x P(x) \lor \exists x \neg P(x); \\
UP & := (\neg p \supset \exists x Q(x)) \supset \exists x (\neg p \supset Q(x)); \\
KF & := \neg \forall x (P(x) \lor \neg P(x)); \\
AP^+_1 & := \forall x_1 (Q_1(x_1) \lor \neg Q_1(x_1)); \\
AP^+_n & := \forall x_n (Q_n(x_n) \lor (Q_n(x_n) \supset AP^+_{n-1})) \quad (n > 1); \\
DE & := \forall x \forall y (x = y \lor \neg x = y) \text{ (the decidable equality principle);} \\
SE & := \forall x \forall y (\neg \neg x = y \supset x = y) \text{ (the stable equality principle);} \\
AU' & := \exists x P(x) \supset \forall x P(x); \\
AU_n & := \forall \left( \bigwedge_{0 \leq i \leq n} P_i(x_i) \supset \bigvee_{0 \leq i < j \leq n} P_i(x_j) \right) \quad (n > 0); \\
AU'_n & := \forall \left( \bigvee_{0 \leq i < j \leq n} (x_i = x_j) \right) \quad (n > 0). 
\end{align*} \]
Modal formulas:

\[ Ba_i := \forall x \Box_i P(x) \supset \Box_i \forall x P(x) \] (Barcan formula for \( \Box_i \));

\[ CE_i := \forall x \forall y (x \neq y \supset \Box_i (x \neq y)) \] (the closed equality principle for \( \Box_i \)).

In particular,

\[ AU_1 \equiv \forall x \forall y (P(x) \supset P(y)), \quad AU_1^- \equiv \forall x \forall y (x = y). \]

All the above intuitionistic formulas except \( AU_1', AU_n, \) and \( AU_n^- \) are classical theorems.

Classically both formulas \( AU_n \) and \( AU_n^- \) state that the individual domain contains at most \( n \) elements, so they are logically equivalent. This also holds in intuitionistic logic:

**Lemma 2.6.19**  

1. \( \mathbf{QH}^- \vdash AU_n^- \supset AU_n \) (and so \( \mathbf{QK}_N^- \vdash AU_n^- \supset AU_n \)).

2. \( \mathbf{QH}^- + AU_n = \mathbf{QH}^- + AU_n^- \) (and \( \mathbf{QK}_N^- + AU_n = \mathbf{QK}_N^- + AU_n^- \)).

3. \( \mathbf{QH} + AU_1' = \mathbf{QH} + AU_1 \).

**Proof**

1. Since \( P_i(x_i) \land (x_i = x_j) \) implies \( P_i(x_j) \).

2. Consider the formula

\[ AU_n^- := \bigwedge_i P_i(x_i) \supset \bigvee_{i < j} P_i(x_j) \]

(the 'quantifier-free matrix' of \( AU_n \)) and the substitution

\[ S := [x_1 = z, \ldots, x_n = z / P_1(z), \ldots, P_n(z)]. \]

Then

\[ S(AU_n^-) = \bigwedge_{i < n} x_i = x_i \supset \bigvee_{i < j} x_i = x_j \]

implies \( \bigvee_{i < j} x_i = x_j \); therefore \( \mathbf{QH}^- + AU_n \vdash AU_n^- \).

3. \( [T/P_1(x)]AU_1 \) is equivalent to \( AU_1' \). So \( \mathbf{QH} + AU_1 \vdash AU_1' \). The converse is trivial. \( \blacksquare \)

Note that the argument is not valid for \( S(AU_n) \), because \( S \) renames bound variables \( x_i \) in \( AU_n \).

**Lemma 2.6.20**

1. For any modal formula \( A \) and a list of variables \( x \)

\[ \mathbf{QS4} \vdash \Box \forall x \Box A \equiv \Box \forall x A, \quad \Diamond \exists x \Diamond A \equiv \Diamond \exists x A. \]

\( ^{18} \)First introduced by A. Grzegorczyk.
(2) For any superintuitionistic formula \( A \)

\[
QH + KF \vdash \forall x \neg \neg A \supset \neg \forall x A;
\]

moreover,

\[
QH + KF = QH + \forall x \neg P(x) \supset \neg \forall x P(x).
\]

**Proof**

(1) \( \Box \forall x A \supset \forall x \Box A \) follows from \( \Box p \supset p \).

The other way round, we obtain \( \Box \forall x A \supset \forall x \Box A \) by 2.6.16(1), hence \( \Box^2 \forall x A \supset \Box \forall x \Box A \) by monotonicity, and thus \( \Box \forall x A \supset \Box^2 \forall x A \) (since \( S4 \vdash \Box p \supset \Box^2 p \)).

For the second formula the proof is similar, using \( A \supset \exists x A \).

(2) For \( L := QH + \forall x \neg P(x) \supset \neg \forall x P(x) \) let us show that \( L \vdash KF \).

In fact,

\[
L \vdash \forall x \neg ((P(x) \lor \neg P(x)) \supset \neg \forall x(P(x) \lor \neg P(x)))
\]

by substitution.

On the other hand,

\[
H \vdash \neg (p \lor \neg p)
\]

by the Glivenko theorem, hence

\[
QH \vdash \neg \neg (P(x) \lor \neg P(x)),
\]

and thus

\[
QH \vdash \forall x \neg (P(x) \lor \neg P(x)).
\]

So by MP it follows that \( L \vdash \neg \forall x(P(x) \lor \neg P(x))(= KF) \).

The other way round, let us show that

\[
QHK := QH + KF \vdash \forall x \neg P(x) \supset \neg \forall x P(x).
\]

By the deduction theorem, this reduces to

\[
\forall x \neg P(x) \vdash_{QHK} \neg \forall x P(x)
\]

and next to

\[
\forall x \neg P(x), \neg \forall x P(x) \vdash_{QHK} \bot.
\]

It suffices to show that

\[
\forall x \neg P(x), \neg P(x) \vdash_{QH} \neg \forall x(P(x) \lor \neg P(x)),
\]

which follows from

(I) \[
\forall x \neg P(x), \forall x(P(x) \lor \neg P(x)) \vdash_{QH} P(x).
\]

To show (I), note that

(II) \[
\forall x \neg P(x), \forall x(P(x) \lor \neg P(x)) \vdash_{QH} \neg \neg P(x), P(x) \lor \neg P(x)
\]

and

(III) \[
\neg \neg P(x), P(x) \lor \neg P(x) \vdash_{QH} P(x),
\]

since \( P(x) \vdash P(x) \) and \( \{ \neg \neg P(x), \neg P(x) \} \) is inconsistent.
2.7 First-order theories

In this section we again consider formulas with extra individual constants.

**Definition 2.7.1** Let $L$ be an $N$-modal ($\equiv$) (respectively, an s.p. ($\equiv$)). An $N$-modal (respectively, an intuitionistic) formula with extra constants $A$ is called $L$-provable, or an $L$-theorem (notation: $\vdash_L A$) if it has a maximal generator in $L$, i.e. if it can be presented in a form $A = [c/x] B$, with $B \in L$, a distinct list of constants $c$, and a distinct list of variables $x$ (cf. Lemma 2.4.4).

For $A$ without constants, we also assume that $\vdash_L A$ iff $A \in L$.

**Lemma 2.7.2** $\vdash_L A$ iff $A$ has a generator in $L$ (for $A$ containing constants).

**Proof** ‘Only if’ follows from 2.7.1. The other way round, if $A$ has a generator $B \in L$, then its maximal generator $A_1$ can be presented as $[y/x] B$ (Lemma 2.4.5); so $A_1 \in L$ by 2.6.15 (iii).

$L$-provability does not depend on the presentation of a formula:

**Lemma 2.7.3** If $B, B'$ are maximal generators of $A$ and $B \in L$, then $B' \in L$.

**Proof** By Lemma 2.4.5, $B' \equiv [z/y] B$ for some variable substitution $[z/y]$. So $B' \in L$, by Lemma 2.6.15.

$L$-provability respects $\land$-introduction:

**Lemma 2.7.5** If $\vdash_L A$, then $\vdash_L \Box_i A$.

**Proof** Suppose $A = [c/x] A_1$ for distinct $c, x$, and $A_1 \in L$. Then $\Box_i A_1 \in L$ and $\Box_i A = [c/x] \Box_i A_1$.

$L$-provability also respects substitution into propositional formulas:

**Lemma 2.7.6** If $L$ is a predicate logic (of any kind), $S$ is a propositional formula, $A \in L$, $S = [B_1, \ldots, B_n/q_1, \ldots, q_n]$ is a substitution of formulas with constants for propositional letters, then $\vdash_L S A$.

**Proof** Let $G_i$ be a maximal generator of $B_i$, so $B_i = [c_i/x_i] G_i$ for an injective $[c_i/x_i]$. Take distinct lists $y_1, \ldots, y_n$ of variables non-occurring in any $G_i$ and put $G'_i := [y_i/x_i] G_i$, $G' := G'_1, \ldots, G'_n$, $q := q_1, \ldots, q_n$, $c := c_1 \ldots c_n$, $y := y_1 \ldots y_n$. Then by Lemma 2.4.2(3), $B_i \equiv [c_i/y_i] G'_i$, and thus

$$S A \equiv [[c/y] G'/q] A \equiv [c/y] [G'/q] A$$

(Lemma 2.5.18). Since $[G'/q] A \in L$, it follows that $\vdash_L S A$. 


Definition 2.7.7 An $N$-modal theory (respectively, an intuitionistic simple theory, with or without equality) is a set of $N$-modal (respectively, intuitionistic) sentences with individual constants.

$D_{\Gamma}$ denotes the set of individual constants occurring in a theory $\Gamma$. So according to the terminology from Section 2.3, $\Gamma$ is a set of $N$-modal or intuitionistic $D_{\Gamma}$-sentences: $\Gamma \subseteq MS_{N}^{(=)}(D_{\Gamma})$ or $\Gamma \subseteq IS^{(=)}(D_{\Gamma})$. The set of all $D_{\Gamma}$-sentences (of the corresponding kind) $L^{(=)}(\Gamma) := MS_{N}^{(=)}(D_{\Gamma})$ (or $IS^{(=)}(D_{\Gamma})$) is called the language of $\Gamma$.

The next definition is an analogue of 2.6.11.

Definition 2.7.8 An $L$-inference of a formula with constants $B$ from a theory $\Gamma$ is a sequence $A_{1}, \ldots, A_{n}$ in which $A_{1} = B$ and every $A_{i}$ is either $L$-provable, or $A_{i} \in \Gamma$, or $A_{i}$ is obtained from earlier formulas by applying MP or $\forall$-introduction.

If there exists an $L$-inference of $B$ from $\Gamma$, we say that $B$ is $L$-derivable from $\Gamma$ (or an $L$-theorem of $\Gamma$); notation: $\Gamma \vdash_{L} B$.

Then we easily obtain an analogue of 2.6.14:

Lemma 2.7.9 Let $\Gamma$ be an $N$-modal or intuitionistic first-order theory (with or without equality), $L$ a predicate logic of the corresponding kind. Then for any $N$-modal or intuitionistic formula $B$ (perhaps, with extra constants)

$$\Gamma \vdash_{L} B \text{ iff there exists a finite } X \subseteq \Gamma \text{ such that } \vdash_{L} \bigwedge X \supseteq B.$$ 

Proof The argument is essentially the same as for 2.6.14; we check that

$$X \vdash_{L} B \iff \vdash_{L} \bigwedge X \supseteq B$$

for a finite theory $X$ and a formula with constants $B$, by induction on $|X|$.

If $X = \emptyset$, (*) means

$$\vdash_{L} B \iff \vdash_{L} \top \supseteq B,$$

which follows in the same way as in 2.6.14.

For the induction step we need the equivalence

$$\vdash_{L} \bigwedge X \supseteq A \supseteq B \iff \vdash_{L} (\bigwedge X) \supseteq A \supseteq B,$$

which follows by Lemma 2.7.6 from (2) in the proof of 2.6.13.

Lemma 2.7.10 Let $\Gamma$ be an $N$-modal theory, $\Box_{i}\Gamma := \{\Box_{i}A \mid A \in \Gamma\}$.

(1) If $\Gamma \vdash_{L} A$, then $\Box_{i}\Gamma \vdash_{L} \Box_{i}A$.

(2) If $\Gamma \vdash_{L} A \supseteq B$, then $\Box_{i}\Gamma \vdash_{L} \Box_{i}A \supseteq \Box_{i}B$.

Proof (1) We apply 2.7.9. If $L \vdash \bigwedge X \supseteq A$ for a finite $X \subseteq \Gamma$, then by 2.7.5, $L \vdash \Box_{i}(\bigwedge X) \supseteq \Box_{i}A$, and thus $L \vdash \bigwedge \Box_{i}X \supseteq \Box_{i}A$. Hence $\Box_{i}\Gamma \vdash_{L} \Box_{i}A$.

(2) By (1), $\Gamma \vdash_{L} A \supseteq B$ implies $\Box_{i}\Gamma \vdash_{L} \Box_{i}(A \supseteq B)$. Then we can apply the axiom $AK_{i}$ and MP.
Another useful fact is the following lemma on new constants.

**Lemma 2.7.11** Let $L$ be an $N$-modal or superintuitionistic logic, $\Gamma$ a modal (respectively, intuitionistic) theory, $A(x)$ a formula with constants (resp., modal or intuitionistic), $x$ a variable not bound in $A(x)$, and assume that a constant $c$ does not occur in $\Gamma \cup \{A(x)\}$. Then the following conditions are equivalent:

1. $\Gamma \vdash_L A(c)$;
2. $\Gamma \vdash_L A(x)$;
3. $\Gamma \vdash_L \forall x A(x)$.

**Proof** (1) $\Rightarrow$ (2). Assume $\Gamma \vdash_L A(c)$. Then by 2.7.9, $\vdash_L \wedge \Gamma_1 \supset A(c)$ for some finite $\Gamma_1 \subseteq \Gamma$. Let $B := \wedge \Gamma_1$. Then for some injective $[yx \mapsto dc]$, 

$$B \supset A(c) = [dc/\gamma x](B_0 \supset A_0(x)),$$

$$(B_0 \supset A_0(x)) \in L.$$

Since by assumption $c$ does not occur in $\Gamma$, it does not occur in $B$, so we have 

$$B = [dc/\gamma x]B_0 = [d/\gamma]B_0,$$

$$A(c) = [dc/\gamma x]A_0(x) = [d/\gamma]A_0(c).$$

Hence 

$$A(x) = A(c)[c \mapsto x] = [d/\gamma]A_0(x),$$

and thus 

$$B \supset A(x) = [d/\gamma](B_0 \supset A_0(x)).$$

So $\vdash_L B \supset A(x)$, therefore $\Gamma \vdash_L A(x)$.

(2) $\Rightarrow$ (3). Assume $\Gamma \vdash_L A(x)$, then by 2.7.9, for some finite $\Gamma_1 \subseteq \Gamma$ we have $\vdash_L B \supset A(x)$, where $B = \wedge \Gamma_1$. So for some injective $[y \mapsto d]$, $B \supset A(x) = [d/y](B_0 \supset A_0(x))$ and $(B_0 \supset A_0(x)) \in L$.

Since $B$ is closed, $x \notin FV(B_0)$, thus by the Bernays rule, $(B_0 \supset \forall x A_0(x)) \in L$. We also have 

$$[d/y](B_0 \supset \forall x A_0(x)) = B \supset \forall x A(x),$$

and so the latter formula is $L$-provable. Therefore $\Gamma \vdash_L \forall x A(x)$.

(3) $\Rightarrow$ (1). Let $A_0(x)$ be a maximal generator of $A(x)$, then $A(x) = [d/\gamma]A_0(x)$, and also 

$$\forall x A(x) \supset A(c) = [dc/\gamma x](\forall x A_0(x) \supset A_0(x)).$$

But $(\forall x A_0(x) \supset A_0(x)) \in L$ by 2.6.13 (ii), so 

$$\vdash_L \forall x A(x) \supset A(c).$$

Then $\vdash_L \forall x A(x)$ implies $\Gamma \vdash_L A(c)$ by MP.

In the intuitionistic case it is convenient to use theories of another kind.
Definition 2.7.12 A intuitionistic double theory (with or without equality) is a pair \((\Gamma, \Delta)\), in which \(\Gamma, \Delta\) are intuitionistic sentences (respectively, with or without equality). \(D(\Gamma, \Delta)(= D_{\Gamma \cup \Delta})\) denotes the set of constants occurring in \(\Gamma \cup \Delta\); the language of \((\Gamma, \Delta)\) is \(L(\Gamma, \Delta) := IF(=)(D_{(\Gamma, \Delta)})\).

Definition 2.7.13 Let \(L\) be an s.p.l., \((\Gamma, \Delta)\) an intuitionistic theory. An intuitionistic formula \(A\) (with constants) is \(L\)-provable in \((\Gamma, \Delta)\) if \(\Gamma \vdash_{L} A \lor \lor \Delta_{1}\) for some finite \(\Delta_{1} \subseteq \Delta\).

So, as we assume \(\lor \emptyset := \bot, \Gamma \vdash_{L} A\) implies \((\Gamma, \Delta) \vdash_{L} A\).

This provability respects MP as well:

Lemma 2.7.14 If \((\Gamma, \Delta) \vdash_{L} C\) and \((\Gamma, \Delta) \vdash_{L} C \supset B\), then \((\Gamma, \Delta) \vdash_{L} B\).

Proof First note that \(\Gamma \vdash_{L} A \lor \lor \Delta_{1}\) implies \(\Gamma \vdash_{L} A \lor \lor \Delta_{2}\) for any \(\Delta_{2} \supseteq \Delta_{1}\), since \(\vdash_{QH} A \lor \lor \Delta_{1} \supset A \lor \lor \Delta_{2}\). The latter follows from the intuitionistic tautology
\[ p \lor q \supset p \lor (q \lor r). \]
So if \((\Gamma, \Delta) \vdash_{L} C\) and \((\Gamma, \Delta) \vdash_{L} C \supset B\), then \(\Gamma \vdash_{L} C \lor \lor \Delta_{1}\) and
\[ \Gamma \vdash_{L} (C \supset B) \lor \lor \Delta_{1} \]
for some finite \(\Delta_{1} \subseteq \Delta\). But by 2.7.6
\[ \vdash_{QH} (C \lor \lor \Delta_{1}) \land ((C \supset B) \lor \lor \Delta_{1}) \supset B \lor \lor \Delta_{1}, \]
since
\[ \vdash_{H} (p \lor r) \land ((p \supset q) \lor r) \supset q \lor r \]
(the latter follows from \(p \lor r, (p \supset q) \lor r \vdash_{H} q \lor r\), which we leave to the reader). Hence \(\Gamma \vdash_{L} B \lor \lor \Delta_{1}\), and thus \((\Gamma, \Delta) \vdash_{L} B\). \(\blacksquare\)

2.8 Deduction theorems

We begin with an analogue of Lemma 2.6.13.

Lemma 2.8.1 For a predicate logic \(L\), a first-order theory \(\Gamma\) and formulas with constants \(A, B\) of the corresponding kind
\[ \Gamma \cup \{A\} \vdash_{L} B \Rightarrow \Gamma \vdash_{L} A \supset B. \]

Proof By an easy modification of the proof of 2.6.13, using Lemma 2.7.6. The details are left to the reader. \(\blacksquare\)

Lemma 2.8.2 Let \(A\) be a predicate (intuitionistic) formula and let \(S\) be an \(IF(=)\)-substitution such that \(FV(A) \cap FV(S) = \emptyset\). Then
\[ QH(=) \vdash \forall S A \supset S \forall A. \]
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**Proof** Let \( FV(A) = r(x) \), so \( \forall A = \forall x A \). Let \( A^o \) be a clean version of \( A \), such that \( BV(A^o) \cap FV(S) = \emptyset \). Then \( SA = SA^o \), and thus

1. \( \forall SA \Rightarrow \forall SA^o \),
2. \( S(\forall A) = \forall x SA^o \),

since \( r(x) \cap FV(S) = \emptyset \). But by Lemma 2.6.15 (xx),

\[
QH^{(=)} \vdash \forall SA^o = \forall y \forall x SA^o,
\]

where \( FV(SA^o) = r(xy) \). By 2.6.15 (ii),

\[
QH^{(=)} \vdash \forall y \forall x SA^o \supset \forall x SA^o.
\]

Hence

\[
QH^{(=)} \vdash \forall SA^o \supset \forall x SA^o,
\]

and thus by (1), (2) we obtain \( QH^{(=)} \vdash \forall SA \supset S(\forall A) \).

Here is an example showing that the requirement

\( FV(A) \cap FV(S) = \emptyset \)

is necessary. Let

\[
A = P(x), \quad S = [Q(x, y)/P(y)],
\]

so we have \( FV(A) \cap FV(S) = \{x\} \). Then

\[
\forall SA \Rightarrow \forall xQ(x, x), \quad S(\forall A) \Rightarrow \forall y Q(x, y),
\]

but

\[
QH \not\vdash \forall xQ(x, x) \supset \forall yQ(x, y).
\]

**Theorem 2.8.3** *(Deduction theorem for superintuitionistic logics)* Let

\( L \) be an s.p.l.(\( = \)), \( \Gamma \) an intuitionistic theory. Then for any \( A \in IF^{(=)} \)

\[
L + \Gamma \vdash A \iff \overline{Sub}(\Gamma) \vdash_L A.
\]

**Proof** *(If.)* \( Sub(\Gamma) \subseteq L + \Gamma \), hence \( \overline{Sub}(\Gamma) \subseteq L + \Gamma \). So \( \overline{Sub}(\Gamma) \vdash_L A \) implies \( L + \Gamma \vdash L A \), and thus \( L + \Gamma \vdash A \), since \( L + \Gamma \) is closed under \( L \)-provability.

*(Only if.)* It is sufficient to show that the set \( \{A \mid \overline{Sub}(\Gamma) \vdash_L A\} \) is a superintuitionistic logic. The conditions (s1)–(s3) from Definition 2.6.3 are obvious (e.g. for generalisation we apply the Bernays rule, since \( \overline{Sub}(\Gamma) \) consists of sentences).

To check (s4), assume that \( \forall A_1, \ldots, \forall A_k \vdash L A \) for \( A_1, \ldots, A_k \in Sub(\Gamma) \). Consider a substitution \( S = [C(xy)/P(x)] \), and let \( z \) be a distinct list such that

\[
r(z) \cap r(y) \cup \bigcup_{i=1}^k FV(A_i) \cup FV(A) = \emptyset.
\]

Let us show that \( \overline{Sub}(\Gamma) \vdash_L SA \).

We may also assume that \( BV(C) \cap r(xyz) = \emptyset \). Consider another substitution \( S' = [C'/P(x)] \), where \( C' := [z/y]C \).
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Since \( \forall A_1, \ldots, \forall A_k \vdash_L A \), we have \( \left( \bigwedge_{i=1}^{k} \forall A_i \supset A \right) \in L \), and thus

\[
S' \left( \bigwedge_{i=1}^{k} \forall A_i \supset A \right) \in L.
\]

But

\[
S' \left( \bigwedge_{i=1}^{k} \forall A_i \supset A \right) = \left( \bigwedge_{i=1}^{k} S' \forall A_i \supset S'A \right),
\]

and so

(1) \( L \vdash \bigwedge_{i=1}^{k} \forall S'(\forall A_i) \supset S'A. \)

On the other hand, since \( FV(S') \cap FV(A_i) = \emptyset \), by Lemma 2.8.2, we have

\[
QH \vdash \forall S'A_i \supset S'(\forall A_i),
\]

and thus

\[
QH \vdash \bigwedge_{i=1}^{k} \forall S'A_i \supset \bigwedge_{i=1}^{k} S'(\forall A_i),
\]

Hence \( \overline{\text{Sub}(\Gamma)} \vdash_L \bigwedge_{i=1}^{k} S'(\forall A_i) \), and by (1) we obtain

\[
\overline{\text{Sub}(\Gamma)} \vdash_L S'A,
\]

which implies

\[
\overline{\text{Sub}(\Gamma)} \vdash_L \forall z S'A,
\]

by \( \forall \)-introduction. But by 2.5.27(ii),

\[
[y/z]S'A \triangleq SA,
\]

and thus by 2.6.15(ii),

\[
QH \vdash \forall z S'A \supset SA.
\]

Therefore \( \overline{\text{Sub}(\Gamma)} \vdash_L SA \), as stated.

Note that the previous proof can be simplified if \( \Gamma \) is a set of sentences: then we choose substitution instances \( A_\alpha \) in such a way that \( FV(A_\alpha) \cap FV[C/P] = \emptyset \), and we can readily apply Lemma 2.8.2.

For an \( N \)-modal theory \( \Delta \) put

\[
\Box^\infty \Delta := \{ \Box_{\alpha} B \mid B \in \Delta, \alpha \in I_N^\infty \}.
\]

Theorem 2.8.4 (Deduction theorem for modal logics) Let \( L \) be a \( N \)-m.p.l. (=), \( \Gamma \) an \( N \)-modal theory. Then for any \( N \)-modal formula \( A \)

\[
L + \Gamma \vdash A \iff \Box^\infty \overline{\text{Sub}(\Gamma)} \vdash_L A.
\]
2.8. DEDUCTION THEOREMS

Proof \( \Box^\infty \text{Sub}(\Gamma) \vdash_L A \) clearly implies \( L + \Gamma \vdash A \). For the converse, note that the set \( \{ A \mid \Box^\infty \text{Sub}(\Gamma) \vdash_L A \} \) is substitution closed; this is proved as in the previous theorem. It is also closed under \( \Box \)-introduction. In fact,
\[
\Box_{\alpha_1} B_1, \ldots, \Box_{\alpha_n} B_n \vdash_L A \text{ iff } \vdash_L \Box_{\alpha_1} B_1 \land \ldots \land \Box_{\alpha_n} B_n \supset A.
\]
By monotonicity and the distribution of \( \Box_i \) over \( \land \) in \( K_N \), this implies
\[
\vdash_L \Box_i \Box_{\alpha_1} B_1 \land \ldots \land \Box_i \Box_{\alpha_n} B_n \supset \Box_i A.
\]
Hence \( \Box^\infty \text{Sub}(\Gamma) \vdash_L \Box_i A \).

Definition 2.8.5 An m.p.l.(\( = \)) is called conically expressive if its propositional part is conically expressive.

Lemma 2.8.6 For any conically expressive N-m.p.l.(\( = \)) \( L \), theory \( \Delta \) and formula \( A \),
\[
\Box^* \Delta \vdash_L A \text{ iff } \Box^\infty \Delta \vdash_L A,
\]
where
\[
\Box^* \Delta := \{ \Box^* B \mid B \in \Delta \}.
\]
Proof 'If' readily follows from \( L \vdash \Box^* p \supset \Box \alpha p \). For the general proof of the converse we need the strong completeness of \( QK_N \), see below. But in particular cases (like \( QS4 \) or \( QK4 \)), \( \Box^* B \) is equivalent to a finite conjunction of formulas from \( \Box^\infty \{ B \} \), so \( \Box^* \Delta \vdash_L A \) obviously implies \( \Box^\infty \Delta \vdash_L A \).

Hence we obtain a simplified version of the deduction theorem for conically expressive modal logics:

Theorem 2.8.7 Let \( L \) be a conically expressive N-m.p.l.(\( = \)), \( \Gamma \) an N-modal theory. Then for any N-modal formula \( A \)
\[
L + \Gamma \vdash A \text{ iff } \Box^* \text{Sub}(\Gamma) \vdash_L A,
\]
Proof Follows from 2.8.4 and 2.8.6.

Here is a simple application of the deduction theorem.

Lemma 2.8.8 \( \text{QH} + W^* \vdash KF \), where
\[
W^* = \forall x((P(x) \supset \forall y P(y)) \supset \forall y P(y)) \supset \forall x P(x),
\]
\[
K = \forall x \neg \neg P(x) \supset \neg \forall x P(x).
\]
Proof It is sufficient to show that
\[
W^*, \forall x \neg \neg P(x), \neg \forall x P(x) \vdash_{\text{QH}} \bot,
\]
or, equivalently,
\[
W^*, \forall x \neg \neg P(x), \neg \forall x P(x) \vdash_{\text{QH}} \forall x P(x).
\]
But this is obvious, since the premise \( \forall x((P(x) \supset \forall y P(y)) \supset \forall y P(y)) \) of \( W^* \) is equivalent to \( \forall x((P(x) \supset \bot) \supset \bot) \), i.e. to \( \forall x \neg \neg P(x) \), in the presence of \( \neg \forall x P(x) \).
2.9 Perfection

Let $A_1 \ldots A_m$ be a list of formulas (not necessarily distinct); we define their disjoint conjunction

$$A_1 \land \ldots \land A_m := S_1 A_1 \land \ldots \land S_m A_m,$$

where $S_i$ is a formula substitution transforming $A_i$ in such a way that predicate letters in all conjuncts become disjoint. E.g. we can put

$$S_i P^n_k(x) := P^n_{m_k+i}(x);$$

then $P^m_l$ occurs in $S_i A_i$ only if $l \equiv i \pmod{m}$. The formula

$$\overline{A_1^k \land \ldots \land A_m^k}$$

is equivalent to

$$\overline{\forall (A_1^k \land \ldots \land A_m^k)},$$

in all our logics, see below.

Similarly we define a disjoint disjunction:

$$A_1 \lor \ldots \lor A_m := S_1 A_1 \lor \ldots \lor S_m A_m.$$

For a theory $\Theta$ put

$$\Theta^\lor := \{\overline{A^k} | A \in \Theta, k \geq 0\},$$

$$\Theta^\land = \{A_1 \land \ldots \land A_m | m > 0, A_1, \ldots, A_m \in \Theta\}.$$

Thus $\Theta^\land$ contains conjunctions of variants of formulas from $\Theta$ in disjoint predicate symbols, and

$$\Theta^{\land \lor} = \{(A_1 \land \ldots \land A_k)^k | k > 0, A_1, \ldots, A_k \in \Theta\}.$$

Obviously, $H + \Theta = H + \Theta^\lor = H + \Theta^\land = H + \Theta^{\land \lor}$.

For theories $\Theta_1, \ldots, \Theta_m$ we also define the disjoint disjunction

$$\Theta_1 \lor \ldots \lor \Theta_m := \{A_1 \lor \ldots \lor A_m | A_1 \in \Theta_1, \ldots, A_m \in \Theta_m\}$$

(the set of disjunctions of variants of formulas from $\Theta_1, \ldots, \Theta_m$) and the extended disjoint disjunction:

$$(\Theta_1 \lor \ldots \lor \Theta_m)^* = \{(A_1 \lor \ldots \lor A_m)^k | k > 0, A_1 \in \Theta_1, \ldots, A_m \in \Theta_m\}.$$
2.9. PERFECTION

• $\Theta_1$ $L$-implies $\Theta_2$ (notation: $\Theta_1 \leq_L \Theta_2$) if
  \[ \forall B \in \Theta_2 \exists A \in \Theta_1 L \vdash A \supset B. \]

• $\Theta_1$ and $\Theta_2$ are $L$-equivalent (notation: $\Theta_1 \sim_L \Theta_2$) if $\Theta_1 \leq_L \Theta_2$ and $\Theta_2 \leq_L \Theta_1$.

• $\Theta_1$ sub-$L$-implies $\Theta_2$ (notation: $\Theta_1 \leq_{L\text{ub}} \Theta_2$) if $\text{Sub}(\Theta_1) \leq_L \text{Sub}(\Theta_2)$;

• $\Theta_1$ is sub-$L$-equivalent to $\Theta_2$ (notation: $\Theta_1 \sim_{L\text{ub}} \Theta_2$) if $\text{Sub}(\Theta_1) \sim_L \text{Sub}(\Theta_2)$.

Here are some simple properties of these relations.

Lemma 2.9.2

(1) $\Theta_1 \leq_L \Theta_2 \implies \Theta_1 \leq_{L\text{ub}} \Theta_2$.

(2) $\Theta_1 \sim_L \Theta_2 \implies \Theta_1 \sim_{L\text{ub}} \Theta_2$.

(3) $\Theta_1 \leq_{L\text{ub}} \Theta_2$ iff $\text{Sub}(\Theta_1) \leq_L \Theta_2$.

(4) $\Theta_1 \leq_{L\text{ub}} \Theta_2 \implies L + \Theta_2 \subseteq L + \Theta_1$.

(5) $\Theta_1 \sim_{L\text{ub}} \Theta_2 \implies L + \Theta_1 = L + \Theta_2$.

Proof

(1) Note that $L \vdash A \supset B$ implies $L \vdash SA \supset SB$ for any substitution $S$.

(2) Follows from (1).

(3) By (1), $\text{Sub}(\Theta_1) \leq_L \Theta_2 \implies \text{Sub}(\Theta_1) = \text{Sub}(\text{Sub}(\Theta_1)) \leq_L \text{Sub}(\Theta_2)$. The converse is trivial.

(4) $\Theta_1 \leq_{L\text{ub}} \Theta_2$ clearly implies $\Theta_2 \subseteq L + \Theta_1$.

(5) Follows from (4).

Lemma 2.9.2(2), (5) allow us sometimes to identify $L$-(sub)-equivalent sets of formulas.

Definition 2.9.3 A set of sentences $\Theta$ is called $\forall$-perfect in a logic $L$ if the following holds

($\forall$-p) for any $A \in \text{Sub}(\Theta)$ there exists $B \in \text{Sub}(\Theta)$ such that $L \vdash B \supset \bigvee A$,

i.e. $\text{Sub}(\Theta) \leq_L \overline{\text{Sub}(\Theta)}$.

Lemma 2.9.4 ($\forall$-p) follows from its weaker version
(∀-p−) for any \( A \in \Theta \), \( k \geq 0 \) there exists \( B \in \text{Sub}(\Theta) \) such that \( L \vdash B \supset \overline{A^k} \).

**Proof** In fact, assume (∀-p−). If \( A' \in \text{Sub}(\Theta) \), then \( \overline{A'} = S\overline{A^k} \) for some substitution \( S \) and \( A \in \Theta \), \( k \geq 0 \). By (∀-p−), \( L \vdash B \supset \overline{A^k} \) for some \( B \in \text{Sub}(\Theta) \); so

\[
L \vdash S(B \supset \overline{A^k}) = (SB \supset \overline{A'})
\]

Since \( SB \in \text{Sub}(\Theta) \), (∀-p) holds for \( A' \).

**Definition 2.9.5** A set of sentences \( \Theta \) is called \( \wedge \)-perfect in a logic \( L \) if

(∀-p−) for any \( A_1, \ldots, A_m \in \text{Sub}(\Theta) \) there exists \( B \in \text{Sub}(\Theta) \) such that \( L \vdash B \supset A_1 \wedge \ldots \wedge A_m \).

Note that the condition (∀-p) for \( m = 2 \) implies (∀-p) for arbitrary \( m \), by induction.

**Lemma 2.9.6** \( \Theta \) is \( \wedge \)-perfect in \( L \) iff \( \text{Sub}(\Theta) \leq_L \Theta^\wedge \).

**Proof** Note that if every \( A_i \) is a substitution instance of \( A_i' \in \Theta \), then

\[
A_1 \wedge \ldots \wedge A_m
\]

is a substitution instance of \( A_1' \wedge \ldots \wedge A_m' \).

**Definition 2.9.7** A set \( \Theta \) is called \( \forall \wedge \)-perfect (in \( L \)) if it is both \( \forall \)-perfect and \( \wedge \)-perfect.

Obviously, a perfect set (of any kind) in \( L \) is also perfect in all every \( L' \supset L \). The next proposition suggests equivalents to 2.9.7.

**Proposition 2.9.8** Let \( \Theta \) be a set of sentences. Then the following conditions are equivalent:

1. \( \Theta \) is \( \forall \wedge \)-perfect in \( L \);
2. \( \text{Sub}(\Theta) \leq_L \Theta^\forall \wedge \);
3. for any \( A_1, \ldots, A_m \in \text{Sub}(\Theta) \) there exists \( B \in \text{Sub}(\Theta) \) such that

\[
L \vdash B \supset \overline{A_1} \wedge \ldots \wedge \overline{A_m}
\]

(or equivalently, \( L \vdash B \supset \overline{(A_1 \wedge \ldots \wedge A_m)} \));
4. \( L + \Theta \vdash A \) iff \( L \vdash B \supset A \) for some \( B \in \text{Sub}(\Theta) \).

Informally speaking, (4) means that a \( \forall \wedge \)-perfect set of axioms allows for the simplest natural form of the deduction theorem.

**Proof** (1) \( \Rightarrow \) (2). If \( L \vdash B_i \supset \overline{A_i^m} \) for \( i = 1, \ldots, m \) and \( L \vdash B \supset B_1 \wedge \ldots \wedge B_m \), then \( L \vdash B \supset \overline{A_1^m} \wedge \ldots \wedge \overline{A_m^m} \).

(2) \( \Rightarrow \) (3). If \( \overline{A_i'} \in \text{Sub}(\overline{A_i^{k_i}}) \), for \( i = 1, \ldots, m \), then for \( k \geq \max(m, k_1, \ldots, k_m) \), a substitution instance of \( (A_1 \wedge \ldots \wedge A_k)^k \) implies \( \overline{A_1'} \wedge \ldots \wedge \overline{A_m'} \) in \( \text{QH} \) (where e.g. \( A_i = A_m \) for \( m < i \leq k \)).

(3) \( \Rightarrow \) (4). If \( L + \Theta \vdash A \), then by the deduction theorem 2.8.3, \( L \vdash \overline{A_1} \wedge \ldots \wedge \overline{A_m} \supset A \) for some \( A_1, \ldots, A_m \in \text{Sub}(\Theta) \).

The converse implications (and (4) \( \Rightarrow \) (1)) are obvious.
2.9. PERFECTION

Similarly in the case of $\forall$- or $\wedge$-perfection we can somewhat simplify the deduction theorem:

**Lemma 2.9.9**  
(1) If $\Theta$ is $\forall$-perfect in $L$, then $L + \Theta \vdash A$ iff $\text{Sub}(\Theta) \vdash_L A$.

(2) If $\Theta$ is $\wedge$-perfect in $L$, then $L + \Theta \vdash A$ iff $L \vdash \forall B \supset A$ for some $B \in \text{Sub}(\Theta)$.

**Proof**  
(1) By 2.8.3, $L + \Theta \vdash A$ iff $\text{Sub}(\Theta) \vdash_L A$, so $\text{Sub}(\Theta) \vdash_L A$ implies $L + \Theta \vdash A$.

The other way round, if $\text{Sub}(\Theta) \vdash_L A$, then

$$\forall A_1 \wedge \ldots \wedge \forall A_m \vdash_L A$$

for some $A_1, \ldots, A_m \in \text{Sub}(\Theta)$. By $(\forall\text{-p})$ there exist $B_i \in \text{Sub}(\Theta)$ such that $L \vdash B_i \supset \forall A_i$; thus $\text{Sub}(\Theta) \vdash_L A$.

(2) ‘If’ is again obvious. The other way round, suppose $\text{Sub}(\Theta) \vdash_L A$, i.e.

$$\forall A_1 \wedge \ldots \wedge \forall A_m \vdash L A$$

for $A_1, \ldots, A_m \in \text{Sub}(\Theta)$. Since

$$\textbf{QH} \vdash \forall A_1 \wedge \ldots \wedge \forall A_m \equiv \forall(A_1 \wedge \ldots \wedge A_m),$$

by the deduction theorem we obtain

$$L \vdash \forall(A_1 \wedge \ldots \wedge A_m) \supset A.$$ 

By $(\wedge\text{-p})$ then

$$L \vdash B \supset A_1 \wedge \ldots \wedge A_m$$

for some $B \in \Theta$; hence by monotonicity

$$L \vdash \forall B \supset \forall(A_1 \wedge \ldots \wedge A_m)$$

and eventually

$$L \vdash \forall B \supset A.$$  

**Lemma 2.9.10**  
(1) For any theory $\Theta$, the theories $\Theta^\forall, \Theta^\wedge, \Theta^{\forall\wedge}$ are respectively $\forall$-perfect, $\wedge$-perfect, and $\forall\wedge$-perfect.

(2) For any theories $\Theta_1, \ldots, \Theta_m$, the theory $(\Theta_1 \hat{\lor} \ldots \hat{\lor} \Theta_m)^*$ is $\forall$-perfect

**Proof**  
(1) Straightforward.

(2) By 2.6.15(xx)

$$\textbf{QH} \vdash \forall C_1 \lor \ldots \lor \forall C_m \supset \forall(C_1 \lor \ldots \lor C_m),$$

hence

$$\textbf{QH} \vdash (A_1 \hat{\lor} \ldots \hat{\lor} A_m)^{k+1} \supset ((A_1 \hat{\lor} \ldots \hat{\lor} A_m)^k)^1.$$ 

Thus $(\forall\text{-p}^-)$ holds.
So every recursively axiomatisable s.p.1. has a recursive \( \land \lor \)-perfect (in \( \mathcal{QH} \)) axiomatisation.\(^{19}\)

Sometimes one can construct simpler perfect axiomatisations for \( \mathcal{QH} + \Theta \) (or for \( L + \Theta \), with a superintuitionistic \( L \)). E.g. if a set \( \Theta_0 \) is \( \lor \)-perfect, then \( \Theta_0^* \) (or \( (\Theta_0 \cup \Theta^V)^{\lor} \) for any \( \Theta \)) is \( \land \lor \)-perfect, etc.

To construct the sets \( \Theta^V, \Theta^\land, \Theta^{V\land} \) from \( \Theta \) we usually need infinitely many additional universal quantifiers or conjuncts, so these sets are infinite for any finite set \( \Theta \) (unless \( \Theta \subseteq \{ \bot, \top \} \)).\(^{20}\) But sometimes finite perfect extensions also exist.

**Lemma 2.9.11** If \( \Theta_1, \ldots, \Theta_m \) are \( \land \lor \)-perfect, then

1. \( (\Theta_1 \lor \ldots \lor \Theta_m)^* \) is \( \land \lor \)-perfect;
2. \( \mathcal{QH} + (\Theta_1 \lor \ldots \lor \Theta_m)^* = \mathcal{QH} + \Theta_1 \lor \ldots \lor \Theta_m \).

**Proof** \( \lor \)-perfection follows from Lemma 2.9.10. To show \( \land \)-perfection, consider \( C_1, \ldots, C_k \in (\Theta_1 \lor \ldots \lor \Theta_m)^* \), with

\[
C_i = A_{i1} \lor \ldots \lor A_{im},
\]

\( A_{ij} \in \Theta_j \). As noted above, we may assume that different \( C_i \) have no common predicate letters. Then every \( C_i \) can be presented as

\[
C_i = \overline{\lor}(A'_{i1} \lor \ldots \lor A'_{im}),
\]

where \( A'_{ij} \in \text{Sub}(\Theta_j) \). By \( \forall \land \)-perfection (2.9.8(3)), there exist \( B_j \in \text{Sub}(\Theta_j) \) such that

\[
L \vdash B_j \supset \overline{\lor} \bigwedge_{i=1}^k A'_{ij}.
\]

Then

\[
L \vdash \left( \bigvee_{j=1}^m B_j \right) \supset \bigwedge_{i=1}^k C_i,
\]

with \( \left( \bigvee_{j=1}^m B_j \right) \in \text{Sub}(\Theta_1 \lor \ldots \lor \Theta_m) \).

**Remark 2.9.12** Another kind of perfection was introduced in [Yokota, 1989], cf. also [Skvortsov, 2004]. A theory \( \Theta \) is called *arity-perfect* in a logic \( L \) if for any \( A \in \text{Sub}(\Theta) \) there exists a closed substitution instance \( B' \) of some \( B \in \Theta \) such that \( L \vdash B' \supset A \) (and thus \( L \vdash B' \supset \forall A \)).

\(^{19}\)Of course every s.p.1. itself is \( \land \lor \)-perfect, but such a trivial axiomatisation is not recursive.

\(^{20}\)Formally speaking, \( \Theta^V \) and \( \Theta^\land \) are infinite even for \( \Theta = \{ \bot \} \), but repetitions and dummy quantifiers can be eliminated, so we can say that \( \{ \bot \}^V = \{ \bot \}^{V\land} = \{ \bot \} \).
2.10. INTERSECTIONS

Our notion of \( \forall \)-perfection is weaker, because \( B \) in 2.9.3(\( \forall \)-p) is not necessarily closed\(^{21} \).

Note that the sets \( \Theta^\wedge \), \( \Theta^\forall \) are arity-perfect.

**Lemma 2.9.13** \( L \)-sub-equivalence preserves all forms of perfection.

### 2.10 Intersections

**Proposition 2.10.1** Let \( L \) be a superintuitionistic predicate logic, and let \( \Gamma, \Gamma' \) be sets of sentences such that formulas from \( \Gamma \) and \( \Gamma' \) have no common predicate letters. Then:

1. \( (L + \Gamma) \cap (L + \Gamma') = L + \{ A^m \lor B^n \mid A \in \Gamma, B \in \Gamma', m, n \in \omega \} = L + \{ A^m \lor B^m \mid A \in \Gamma, B \in \Gamma', m \in \omega \} \)

2. (cf. [Ono, 1973, Theorem 5.5]) If \( L \vdash CD \) then

\[
(L + \Gamma) \cap (L + \Gamma') = L + \{ A \lor B \mid A \in \Gamma, B \in \Gamma' \}
\]

3. If \( L + \Gamma \vdash CD \) and \( L + \Gamma' \vdash CD \), then

\[
(L + \Gamma) \cap (L + \Gamma') = L + CD + \{ A \lor B \mid A \in \Gamma, B \in \Gamma' \}
\]

**Proof**

1. Let \( (L + \Gamma) \cap (L + \Gamma') \vdash A \). Then by Theorem 2.8.3,

\[
L \vdash \bigwedge_{i=1}^{k} \forall A_i \supset A \quad \text{and} \quad L \vdash \bigwedge_{j=1}^{l} \forall B_j \supset A
\]

for some \( A_1, \ldots, A_k \in \text{Sub}(\Gamma) \), \( B_1, \ldots, B_l \in \text{Sub}(\Gamma') \). Thus

\[
L \vdash \bigwedge_{i,j} (\forall A_i \lor \forall B_j \supset A).
\]

Assuming that \( |FV(A_i)| = m, |FV(B_j)| = n \), we obtain that \( \forall A_i \lor \forall B_j \) is a substitution instance of \( A^m \lor B^n \). So we obtain the first equality in (1).

Now note that by substitution and omitting dummy quantifiers,

\[
L + A^m \lor B^n \vdash A^m \lor B^n,
\]

if \( m \leq m_1, n \leq n_1 \). Hence

\[
L + A^s \lor B^s \vdash A^m \lor B^n,
\]

where \( s = \max(m, n) \), and the second equality follows.

\(^{21}\)Recent counterexamples by D. Skvortsov show that \( \forall \)-perfection is strictly weaker than arity-perfection.
(2) We use (1). Assume that $A \in \Gamma$, $B \in \Gamma'$. Obviously, $L + \overline{A^0} \lor \overline{B^0} \vdash A \lor B$, thus $A \lor B \in (L + \Gamma) \cap (L + \Gamma')$. So it remains to show that for any $m \in \omega$

\begin{equation}
L + A \lor B \vdash \overline{A^m} \lor \overline{B^m}.
\end{equation}

Let $\overline{A^m} = \forall u_1 \ldots \forall u_m A^m$, $\overline{B^m} = \forall v_1 \ldots \forall v_m B^m$, and

\[ C := \forall u_1 \forall v_1 \ldots \forall u_m \forall v_m (A^m \lor B^m). \]

Then $L + A \lor B \vdash C$ and $L \vdash C \supset \overline{A^m} \lor \overline{B^m}$, by CD. Hence (\ref{eq:1}) follows.

(3) follows from (2): take $L + CD$ as $L$.

\[ \square \]

Remark 2.10.2 The assumption $L \vdash CD$ in Proposition 2.10.1(2) is necessary. [Ono, 1972/73] gives an example of predicate logics for which (\ref{eq:2}) does not hold.

Similarly one can describe intersections of modal predicate logics.

Proposition 2.10.3 Assume that $L$ is a $N$-modal predicate logic, and $\Gamma, \Gamma'$ have no common predicate letters. Then

(1) $(L + \Gamma) \cap (L + \Gamma') = L + \{ \Box_{\alpha} \overline{A^m} \lor \Box_{\beta} \overline{B^m} \mid A \in \Gamma, B \in \Gamma'; m, n \in \omega; \alpha, \beta \in I_N^\infty \}.$

For particular classes of 1-modal logics this presentation can be simplified, cf. Section 1.1:

(a) For logics above $\mathbf{QT}$:

\[ (\Lambda + \Gamma) \cap (\Lambda + \Gamma') = L + \{ \Box^r \overline{A^m} \lor \Box^s \overline{B^m} \mid A \in \Gamma, B \in \Gamma'; m, n, r, s \in \omega \}. \]

(b) For logics above $\mathbf{QK4}$:

\[ (\Lambda + \Gamma) \cap (\Lambda + \Gamma') = L + \{ \Box^r \overline{A^m} \lor \Box^s \overline{B^m} \mid A \in \Gamma, B \in \Gamma'; m, n, r \in \omega; s \in \{0, 1\} \}. \]

(c) For logics above $\mathbf{QS4}$:

\[ (\Lambda + \Gamma) \cap (\Lambda + \Gamma') = L + \{ \Box \overline{A^m} \lor \Box \overline{B^m} \mid A \in \Gamma, B \in \Gamma'; m, n \in \omega \}. \]

(2) If $L \vdash Ba$, then:

\[ (L + \Gamma) \cap (L + \Gamma') = L + \{ \Box_{\alpha} A \lor \Box_{\beta} B \mid A \in \Gamma, B \in \Gamma'; \alpha, \beta \in I_N^\infty \}. \]

Similarly, to (i), there exist simpler versions for the cases $L \ni \mathbf{QK4} + Ba, \mathbf{QK4} + Ba$ or $\mathbf{QS4} + Ba$.

Therefore we have
Proposition 2.10.4 The complete lattices of predicate logics (superintuitionistic or modal) are well-distributive, i.e. they are Heyting algebras. Moreover, the intersection of two recursively axiomatisable logics is recursively axiomatisable. The intersection of finitely axiomatisable logics is finitely axiomatisable for superintuitionistic logics $\supseteq \text{QH} + \text{CD}$ and for 1-modal logics $\supseteq \text{QK4} + \text{Ba}$.

Later in this book we will show that the intersection of finitely axiomatisable superintuitionistic predicate logics may be not finitely axiomatisable, if one of them does not contain CD.

2.11 G"odel–Tarski translation

Definition 2.11.1 G"odel–Tarski translation for predicate formulas is the map $(-)^T : IF= \rightarrow MF=$ defined by the following clauses:

- $A^T = \Box A$ for $A$ atomic;
- $(A \land B)^T = A^T \land B^T$;
- $(A \lor B)^T = A^T \lor B^T$;
- $(A \supset B)^T = \Box(A^T \supset B^T)$;
- $(\exists x A)^T = \exists x A^T$;
- $(\forall x A)^T = \Box \forall x A^T$.

For a set $\Gamma \subseteq IF= $ put $\Gamma^T := \{A^T \mid A \in \Gamma\}$.

Gödel–Tarski translation is obviously extended to formulas with constants.

Since $\text{QS4}^= \vdash (x = y) \equiv \Box(x = y)$, we may also define $(x = y)^T$ just as $x = y$.

Lemma 2.11.2 $\text{QS4}^{(=)} \vdash \Box A^T \equiv A^T$ for any $A \in IF^{(=)}$.

Proof By induction on the length of $A$. $\blacksquare$

Lemma 2.11.3 (1) Let $[x \mapsto y]$ be a (simple) variable transformation. Then for any $A \in IF^= $,

$(A[x \mapsto y])^T = A^T[x \mapsto y]$.

(2) For any formula with constants $A \in IF^= (D)$, for any $D$-transformation $[x \mapsto a]$, $([a/x]A)^T = [a/x]A^T$.

Proof (1) Easy, by induction on $|A|$. We consider only the case $A = \forall z B$.

If $z \neq x$, we have

$(A[x \mapsto y])^T = (\forall z(B[x \mapsto y]))^T = \Box \forall z (B[x \mapsto y])^T$

$= (\Box \forall z B^T)[x \mapsto y]$ (by the induction hypothesis)

If $z = x$, we have

$(A[x \mapsto y])^T = (\forall y(B[x \mapsto y]))^T = \Box \forall y (B[x \mapsto y])^T$

$= (\Box \forall y B^T)[x \mapsto y]$ (by the induction hypothesis)

$(\forall \forall x B^T)[x \mapsto y] = A^T[x \mapsto y]$.

(2) An exercise. $\blacksquare$
Lemma 2.11.4 If $\models A \equiv B$ for $A, B \in IF^=,$ then $A^T \equiv B^T.$

**Proof** We use Proposition 2.3.17. Consider the equivalence relation

$$A \sim B := (A^T \equiv B^T)$$

on $IF^=.$ It is sufficient to show that $\sim$ satisfies the conditions (1)-(3) from 2.3.17.

We check only (1) for $Q = \forall$:

$$(\forall x A)^T \equiv (\forall y (A[x \mapsto y]))^T.$$ 

The remaining (routine) part is left to the reader. In fact, by 2.11.1 and 2.11.3

$$(\forall y (A[x \mapsto y]))^T = \Box \forall y (A^T[x \mapsto y]) = \Box \forall y (A^T[x \mapsto y]),$$

which is congruent to $(\forall x A)^T = \Box \forall x A^T$ by 2.3.17.

Lemma 2.11.5 Let $S = [B(x, y)/P(x)]$ be an IF($=$)-substitution, and consider the MF($=$)-substitution

Then for any $A \in IF(=)$

$$(\forall) \quad QS^4(=) \vdash (SA)^T \equiv S^T A^T.$$ 

**Proof** By induction on $A.$

- If $A = P(z),$ then $(SA)^T = B^T(z, y),$ and

$$S^T A^T = S^T (\Box P(z)) = \Box B^T(z, y),$$

which is equivalent to $B^T(z, y)$ by Lemma 2.11.2.

- If $A$ is atomic, $A \neq P(z),$ the claim is trivial.

- If $A = \forall z C$ is clean and also $BV(A) \cap FV(S) = \emptyset,$ then

$$(SA)^T = (\forall z SC)^T = \Box \forall z (SC)^T,$$

$$S^T A^T = S^T (\Box \forall z C^T) = \Box \forall z S^T C^T.$$ 

But

$QS^4(=) \vdash (SC)^T \equiv S^T C^T$ — by the induction hypothesis, and hence

$QS^4(=) \vdash \Box \forall z (SC)^T \equiv \Box \forall z S^T C^T$ — by replacement. So (*) holds.

\[22\] Otherwise we can consider a congruent formula with this property.
2.1. GÖDEL–TARSKI TRANSLATION

- If $A = \exists z C$, then
  
  \[(S A)^T = (\exists z (SC))^T = \exists z (SC)^T,\]
  
  \[S^T A^T = S^T (\exists z C^T) = \exists z S^T C^T.\]

  Now again we have $QS^4(=) \vdash (SA)^T \equiv S^T A^T$ by the induction hypothesis and replacement.

- If $A = C \lor D$, then
  
  \[(S A)^T = (SC \lor SD)^T = (SC)^T \lor (SD)^T,\]
  
  \[S^T A^T = S^T (C^T \lor D^T) = S^T C^T \lor S^T D^T.\]

  By the induction hypothesis $QS^4 \vdash (SC)^T \equiv S^T C^T$, $(SD)^T \equiv S^T D^T$, so we can apply a replacement.

  The case $A = C \land D$ is almost the same, and we skip it.

- If $A = C \supset D$, then
  
  \[(S A)^T = (SC \supset SD)^T = \square((SC)^T \supset (SD)^T),\]
  
  \[S^T A^T = S^T (\square(C^T \supset D^T)) = \square(S^T C^T \supset S^T D^T).\]

  In this case we can also use $QS^4$-theorems
  
  \[(SC)^T \equiv S^T C^T, \quad (SD)^T \equiv S^T D^T\]

  and replacement.

\[\square\]

**Lemma 2.11.6** $(\text{Sub}(\Gamma))^T \subseteq QS^4(=) + \text{Sub}(\Gamma^T)$ for any $\Gamma \subseteq IF(=)$

**Proof** By Lemma 2.11.5 we have $(SA)^T \in QS^4(=) + S^T A^T$ for any simple substitution $S$. Now recall that every substitution is reducible to simple ones.

**Lemma 2.11.7** For a list of variables $x$ and an $IF(=)$-formula $A(x)$,

\[QS^4(=) \vdash \square \forall x A^T \equiv (\forall x A)^T.\]

**Proof** By induction on the length of $x$. The base: $QS^4 \vdash A^T \equiv \square A^T$ (2.11.2).

For the induction step, assume $QS^4 \vdash A^T \equiv (\forall x A)^T$. Then in $QS^4$ we obtain:

\[\square \forall y \forall x A^T \equiv \square \forall y (\forall x A)^T,\]

(2.6.202) by assumption and replacement.

**Proposition 2.11.8** For any m.p.l.$(=)$ $L$ containing $QS^4$ the set

\[T_L := \{ A \in IF(=) \mid A^T \in L \}\]

is an s.p.l. $(=)$. 

Proof. We check that $^{T}L$ satisfies the conditions (s1)-(s5). By Proposition 1.5.2, $A^{T} \in QS4$ for every propositional intuitionistic axiom $A$. The same holds for predicate axioms and for the axioms of equality, as one can easily see. It is also clear that $^{T}L$ satisfies (s3) from Section 1.2. Corollary 2.11.6 shows that $^{T}L$ is closed under $IF(=)$-substitutions.

Definition 2.11.9 The above defined s.p.l.(=) $^{T}L$ is called the superintuitionistic fragment of the m.p.l.(=) $L$; an m.p.l.(=) $L$ is called a modal counterpart of the s.p.l.(=) $^{T}L$, cf. Definition 1.5.4.

Lemma 2.11.10 For any $A \in IF(=)$,

$$QH(=) \vdash A \Rightarrow QS4(=) \vdash A^{T}.$$ 

Proof. By induction on the length of the proof of $A$. An alternative proof makes use of completeness, see below. □

Lemma 2.11.11 Let $L = QH(=) + \Gamma$ be an s.p.l.(=). Then for any $A \in IF(=)$,

$$L \vdash A \Rightarrow QS4(=) + \Gamma^{T} \vdash A^{T}.$$ 

Proof. Suppose $L \vdash A$. Then by Theorem 2.8.3, $\overline{Sub(\Gamma)} \vdash QH(=) A$, i.e. there exists formulas $A_{1}, \ldots, A_{n} \in \Gamma$ and substitutions $S_{1}, \ldots, S_{n}$ such that $QH(=) \vdash \bigwedge_{i=1}^{n} \overline{\forall S_{i}A_{i}} \supset A$. Then by Lemma 2.11.10,

$$QS4(=) \vdash \left( \bigwedge_{i=1}^{n} \overline{\forall S_{i}A_{i}} \supset A \right)^{T} = \square \left( \bigwedge_{i=1}^{n} \overline{\forall S_{i}A_{i}}^{T} \supset A^{T} \right).$$ 

Hence

$$QS4(=) \vdash \bigwedge_{i=1}^{n} \overline{\forall S_{i}A_{i}}^{T} \supset A^{T}.$$ 

By definition and Lemma 2.11.5, $QS4(=) \vdash (\overline{\forall S_{i}A_{i}})^{T} \equiv \square \overline{\forall S_{i}A_{i}}^{T}$. Therefore $\square \overline{Sub(\Gamma)} \vdash QS4(=) A^{T}$, i.e. $QS4(=) + \Gamma^{T} \vdash A^{T}$ by Theorem 2.8.4. □

As we shall see (cf. Proposition 2.16.17), in many cases the converse of 2.11.11 also holds, i.e.

$$QH(=) + \Gamma = T(QS4(=) + \Gamma^{T}),$$

or

$$QH + \Gamma(=) \vdash A \text{ iff } QS4(=) + \Gamma \vdash A^{T}.$$ 

(\ast)

In particular, it is well known that

$$^{T}QS4(=) = QH(=).$$

[Schütte, 1968], [Rasiowa and Sikorski, 1963].

Let us make another simple observation.
Lemma 2.11.12 If an s.p.l. ($\equiv$) $L = \text{QH}^{(=)} + \Gamma$ has modal counterparts, then $\text{QS4}^{(=)} + \Gamma^T$ is the smallest modal counterpart of $L$.

Proof Suppose $L = T\Lambda$. Then obviously, $\text{QS4}^{(=)} + \Gamma^T \subseteq \Lambda$. By Lemma 2.11.11, $L \vdash A$ only if $\text{QS4}^{(=)} + \Gamma^T \vdash A$. On the other hand, if $L \not\vdash A$, then $\Lambda \not\vdash A^T$, and thus $\text{QS4}^{(=)} + \Gamma^T \not\vdash A^T$. Therefore $\text{QS4}^{(=)} + \Gamma^T$ is a modal counterpart of $L$.

Remark 2.11.13 The paper [Pankratyev, 1989] states that every s.p.l. has modal counterparts. However, the proof in this paper requires some verification, which we postpone until Volume 2.

We do not know if there exist the greatest modal counterparts in the predicate case, and Theorem 1.5.6 has no direct analogue:

Theorem 2.11.14 (1) [Pankratyev, 1989] $\text{QGrz}$ is a modal counterpart of $\text{QH}$.

(2) [Naumov, 1991] There exists a proper extension of $\text{QGrz}$ which is also a modal counterpart of $\text{QH}$.

These matters will also be discussed in Volume 2.

2.12 The Glivenko theorem

Proposition 2.12.1 (Predicate version of the Glivenko theorem) For any intuitionistic predicate formula $A$

$$\text{QCL} \vdash \neg A \text{ iff } \text{QH} + KF \vdash \neg A.$$ 

Proof Let $\text{QCL} \vdash \neg A$. Then by the deduction theorem 2.8.3

$$\text{QH} \vdash \bigwedge_s \neg \neg \neg (A_s \vee \neg A_s) \supset \neg A$$

for some formulas $A_s$. Then by Lemma 1.1.2, we have

(1) $$\text{QH} \vdash \bigwedge_s \neg \neg \neg (A_s \vee \neg A_s) \supset \neg A.$$ 

By Lemma 2.6.20(II),

(2) $$\text{QH} + KF \vdash \forall x \neg \neg B \supset \neg \neg \forall x B.$$ 

By Corollary 1.1.9, $\text{QH} \vdash \neg \neg (A_s \vee \neg A_s)$, so by applying (2), we obtain:

$$\text{QH} + KF \vdash \bigwedge_s \neg \neg \neg (A_s \vee \neg A_s).$$

Hence by (1), $\text{QH} + KF \vdash \neg A$. 

\[ \square \]
Corollary 2.12.2 For any $\neg \land \lor$-formula $A$,

$$QCL \vdash A \iff QH + KF \vdash A.$$  

Proof By induction on $A$ (exercise). $\blacksquare$

We say that an intermediate predicate logic $L$ satisfies the Glivenko property if for any formula $A$,

$$L \vdash \neg A \iff QCL \vdash \neg A$$

and that $L$ has the classical $\neg \land \lor$-fragment if for any $\neg \land \lor$-formula $A$,

$$L \vdash A \iff QCL \vdash A.$$  

Theorem 2.12.3 The following conditions are equivalent (for a superintuitionistic predicate logic $L$):

1. $L$ satisfies the Glivenko property;
2. $L$ has the classical $\neg \land \lor$-fragment;
3. $QH + KF \subseteq L \subseteq QCL$.

In other words, $QH + KF$ is the smallest intermediate predicate logic with the property (i) or (ii).

Proof By Lemma 2.6.20(II),

$$QH + KF = QH + \forall x \neg P(x) \supset \neg \forall x P(x) = QH + \neg (\forall x \neg P(x) \land \forall x P(x)).$$

Thus (i), (ii) imply $QH + KF \subseteq L$. (i) also implies $L \subseteq QCL$. In fact, suppose $A \in (L - QCL)$. Since $QH \vdash A \supset \neg \neg A$ and $QCL \vdash A \equiv \neg \neg A$, we then have $\neg \neg A \in (L - QCL)$, which contradicts the Glivenko property.

Finally, (ii) implies $L \subseteq QCL$ as well. In fact, assume (ii) and suppose $A \in (L - QCL)$ and $QCL \vdash A \equiv B$, where $B$ is a $\neg \land \lor$-formula; then $QCL \vdash \neg \neg (A \equiv B)$, whence $QH + KF \vdash \neg \neg (A \equiv B)$, by Proposition 2.12.1. As we have proved, $L \vdash KF$. So we obtain $L \vdash \neg \neg (A \equiv B)$, whence $L \vdash \neg \neg A \equiv \neg \neg B$, by Lemma 1.1.2. Since $A \in L$, we also have $\neg \neg A \in L$, and thus $\neg \neg B \in (L - QCL)$, which contradicts (ii). $\blacksquare$

The same result holds for the logic with equality $QH^e + KF$.

2.13 $\Delta$-operation

Definition 2.13.1 For a predicate formula $A$ put $\delta A := p \lor (p \supset A)$, where $p$ is a proposition letter that does not occur in $A$. For a superintuitionistic predicate logic $L$ put $\Delta L := QH + \{ \delta A | A \in L \}$ and also $\Delta^0 L := L$, $\Delta^{n+1} L := \Delta \Delta^n L$ by induction.
Lemma 2.13.2 The following formulas are theorems of $QH$:

1. $A \supset \delta A$
2. $\delta(A \supset B) \supset \delta A \supset \delta B$,
3. $\delta \forall A \supset \forall \delta A$.

**Proof** (1) $A \vdash_{QH} p \supset A$ and $p \supset A \vdash_{QH} \delta A$; hence $A \vdash_{QH} \delta A$. Then apply the deduction theorem.

(2) First note that by (Ax2),

$$p \supset (A \supset B), \ p \supset A \vdash_{QH} p \supset B.$$  

Hence by $\forall$-introduction

$$p \supset (p \supset (A \supset B)), \ p \supset (p \supset A) \vdash_{QH} p \supset (p \supset B).$$  

Now (2) follows by the deduction theorem.

(3) We have $QH \vdash \forall A \supset A$ by 2.6.15. Hence $QH \vdash \delta \forall A \supset \delta A$ by (1), and eventually $QH \vdash \delta \forall A \supset \forall \delta A$ by the Bernays rule.

So $\Delta L$ can be axiomatised as follows.

**Lemma 2.13.3** $\Delta L = QH + \{\delta A \ | \ A \in \overline{L}\}$.

**Proof** Let $\overline{\Delta L} := QH + \{\delta A \ | \ A \in \overline{L}\}$; then clearly $\overline{\Delta L} \subseteq \Delta L$. The other way round, if $A \in L$, then $\forall A \in \overline{L}$, so $\delta \forall A \in \overline{\Delta L}$. By Lemma 2.13.2, $\delta \forall A$ implies $\forall \delta A$; so $\delta A \in \overline{\Delta L}$. Therefore $\Delta L \subseteq \overline{\Delta L}$.

In [Komori, 1983] $\Delta L$ was defined as $QH + \{\delta' A \ | \ A \in \overline{L}\}$, where $\delta' A := ((p \supset A) \supset p) \supset p$. As in the propositional case, both definitions are equivalent:

**Lemma 2.13.4** $QH + \delta A = QH + \delta' A$ for any formula $A$.

**Proof** Quite similar to 1.16.2 using the deduction theorem.

**Lemma 2.13.5**

1. $\Delta L \subseteq L$.
2. $L_1 \subseteq L_2 \implies \Delta L_1 \subseteq \Delta L_2$.
3. $\Delta(QH + \bot) = QCL$.
4. $\Delta L \subseteq QCL$ for every consistent s.p.l. $L$.

**Proof** (1) Since $QH \vdash A \supset \delta A$ for any $A$.

(2) Trivial.

(3) $\Delta(QH + \bot) \subseteq QCL$, since $QCL \vdash \delta A$ for every $A$. The other way round, $QCL \subseteq \Delta(QH + \bot)$, since $QCL = QH + EM$ and $EM = p \lor \neg p$ is just $\bot = p \lor (p \lor \bot)$.

(4) follows from (3) by Proposition 2.12.5.
In Chapter 6 we will prove the converse to (2), similarly to 1.16.7 for the propositional case.

The logics

\[ \text{QHP}_n^+ := \Delta^n(\text{QH} + \bot) = \Delta^{n-1}(\text{QCL}) \]

were first introduced in [Komori, 1983] (where they were denoted by \( \Delta^k(W) \), for \( W := \text{QH} + \bot \)).

Let us now turn to a finite axiomatisation of \( \text{QHP}_n^+ \) presented in [Yokota, 1989].

First we generalise \( \delta \)-operation as follows [Yokota, 1989].

**Definition 2.13.6** Let

\[ \delta_{k,P}A := \forall y(P(y) \lor (P(y) \supset A)), \]

where \( P \in PL^k \), \( y \) is a distinct list of variables, \( r(y) \cap FV(A) = \emptyset \). If \( P \) does not occur in \( A \), we use the notation \( \delta_k A \) rather than \( \delta_{k,P}A \).\(^{23}\) Also put

\[ \delta_{k_1\ldots k_n}A := \delta_{k_1} \ldots \delta_{k_n}A, \quad \delta_k^n A := \underbrace{\delta_k \ldots \delta_k}_n A \]

(in particular, \( \delta_0^0 A = A \)). For a set \( \Theta \) of formulas and \( \gamma = \delta_k, \delta_k^n \) etc. put

\[ \gamma \Theta := \{ \gamma A \mid A \in \Theta \}. \]

We also introduce

\[ \delta_{\infty}^n \Theta := \bigcup_{k \in \omega} \delta_k^n \Theta, \]

\[ \delta_\ast \Theta := \underbrace{\delta_{\infty}^1 \ldots \delta_{\infty}^1}_n \Theta = \bigcup_{k_1,\ldots,k_n \in \omega} \delta_{k_1\ldots k_n} \Theta \]

for \( n \in \omega. \)\(^{25}\)

Obviously,

\[ \delta_{\infty}^n \Theta \subseteq \delta_\ast \Theta. \]

Also note that \( \delta_0 A = \delta A \) and \( P_n = \delta_0^n \bot \). Hence

\[ P_n^+ = \delta_1^n \bot. \]

**Lemma 2.13.7** (1) \( \text{QH} + \delta A \vdash \delta_k A. \)

(2) \( \delta_k L \subseteq \Delta L. \)

(3) \( \delta_{k_1\ldots k_n} L \subseteq \Delta^n L. \)

\(^{23}\)Cf. Definitions 1.2.1 and 1.2.4 in [Yokota, 1989].

\(^{24}\)The latter notation is somewhat ambiguous as it means \( \delta_{k,P_1} \ldots \delta_{k,P_n} A \) for different \( P_1,\ldots,P_n \) that do not occur in \( A. \)

\(^{25}\)In [Yokota, 1989] the set \( \delta_\ast^n \{ \bot \} \) was denoted by \( P_n^\ast \).
Proof
(1) \(\delta_{k,p} A = \forall y [P(y)/p] \delta A\), so we can apply substitution and generalisation.
(2) Follows from (1).
(3) Follows from (2) by induction.

The next lemma shows that \(\delta\)-type operators behave as \(\Box\)-modalities.

Lemma 2.13.8 The following are theorems of QH:

(1) \(A \supset \Box A\);
(2) \(\Box (A \supset B) \supset \Box A \supset \Box B\);
(3) \(\Box \left( \bigwedge_{i=1}^{n} A_i \right) \equiv \bigwedge_{i=1}^{n} \Box A_i\);
(4) \(\Box \forall A \supset \forall \Box A\);

where \(\Box = \delta_{k,p}\) or \(\delta_{k_1...k_n}\).

Proof
(1), (2). For \(\Box = \delta_{k,p}\) this follows from 2.13.2 (1), (2) by substitution \([P(y)/p]\) and generalisation. For \(\Box = \delta_{k_1...k_n}\) apply induction.
(3) Similar to 2.13.2 (3). QH \(\vdash \forall A \supset A\) implies QH \(\vdash \Box \forall A \supset \Box A\) by (1), (2); hence QH \(\vdash \Box \forall A \supset \forall \Box A\) by the Bernays rule.
(4) We may suppose \(n = 2\) and use induction for the general case. So let us show

\[
\text{QH} \vdash \Box (A \land B) \equiv \Box A \land \Box B.
\]

Since QH \(\vdash A \land B \supset A\), by (1), (2) and (MP) we obtain

\[
\text{QH} \vdash \Box (A \land B) \supset \Box A.
\]

Similarly

\[
\text{QH} \vdash \Box (A \land B) \supset \Box B,
\]

and thus

\[
\text{QH} \vdash \Box (A \land B) \supset \Box A \land \Box B.
\]

For the converse note that

\[
A \supset (B \supset A \land B)
\]

by (Ax5), hence

\[
\text{QH} \vdash \Box A \supset \Box (B \supset A \land B)
\]

by (1), (2) and next

\[
\text{QH} \vdash \Box A \supset \Box B \supset \Box (A \land B)
\]

again by (1), (2) and the transitivity of \(\supset\). The latter formula is equivalent to

\[
\Box A \land \Box B \supset \Box (A \land B).
\]
Proposition 2.13.9  \( \mathcal{QH} \vdash \delta^m_k A \supset \delta^m_k A \) for \( n \leq m \).

Proof  Readily follows from 2.13.8(1), since \( \delta^m_k = \delta^{m-n}_k \delta^n_k \).

Lemma 2.13.10  (1)  \( \mathcal{QH} \vdash P^+_n \supset \delta^n_k A \), \( P_n \supset \delta^n_k A \) for any formula \( A \).

(2)  \( \mathcal{QH} + \delta^n_k \Theta \subseteq \mathcal{QH} + P^+_n \) and \( \mathcal{QH} + \delta^n_0 \Theta \subseteq \mathcal{QH} + P_n \) for any theory \( \Theta \).

Proof

(1) By 2.13.8 (2), since \( \mathcal{QH} \vdash \bot \supset A \).

(2) Follows from (1).

Lemma 2.13.11  \( \mathcal{QH} + \delta_{k_1 \ldots k_n} A \vdash \delta_{l_1 \ldots l_n} A \) whenever \( l_1 \leq k_1, \ldots, l_n \leq k_n \).

Proof  By substitution and elimination of dummy quantifiers.

The same argument actually proves a stronger claim. Since elimination of dummy quantifiers is legal in \( \mathcal{QH} \), we obtain:

Proposition 2.13.12  \( \delta^n_{\infty} \Theta \subseteq \text{Sub} \delta^n_k \Theta \)

Proof

Lemma 2.13.13  For an s.p.l\((=)\) \( L \) and sets of formulas \( \Theta_1, \Theta_2 \)

(1) If \( L + \Theta_1 \subseteq L + \Theta_2 \), then

(a) \( L + \delta^n_k \Theta_1 \subseteq L + \delta^n_k \Theta_2 \) provided \( \Theta_2 \) is \( \forall \)-perfect in \( L \);

(b) \( L + \delta^n_k (\Theta_1^y_k) \subseteq L + \delta^n_k (\Theta_2^y_k) \).

(2) If \( L + \Theta_1 = L + \Theta_2 \), then

(a) \( L + \delta^n_k \Theta_1 = L + \delta^n_k \Theta_2 \) provided \( \Theta_1, \Theta_2 \) are \( \forall \)-perfect in \( L \);

(b) \( L + \delta^n_k (\Theta_1^y_k) = L + \delta^n_k (\Theta_2^y_k) \).

Proof

(1) (a) If \( A \in \Theta_1 \subseteq L + \Theta_1 \subseteq L + \Theta_2 \), then by 2.13.8 \( \text{Sub}(\Theta_2) \vdash A \), i.e. \( L \vdash B_1 \land \ldots \land B_m \supset A \) for some \( B_1, \ldots, B_m \in \text{Sub}(\Theta_2) \). Then by the monotonicity of \( \delta^n_k \),

\[
L \vdash \delta^n_k (\bigwedge_{i=1}^m B_i) \supset \delta^n_k A;
\]

thus

\[
L \vdash (\bigwedge_{i=1}^k \delta^n_k B_i) \supset \delta^n_k A.
\]

Since \( \delta^n_k B_i \in \text{Sub}(\delta^n_k \Theta_2) \), it follows that \( L + \delta^n_k \Theta_2 \vdash \delta^n_k A \).

Therefore \( \delta^n_k \Theta_1 \subseteq L + \delta^n_k \Theta_2 \).
(b) Readily follows from (a), since $\Theta^\forall$ is $\forall$-perfect in $QH$ and $QH + \Theta = QH + \Theta^\forall$.

(2) Follows from (1).

Lemma 2.13.14 26 If a set $\Theta$ is $\forall$-perfect in an s.p.l. (=) $L$, then $\delta^1_\infty \Theta$ is also $\forall$-perfect in $L$.

Proof For an arbitrary $\delta_k A \in \delta^1_\infty \Theta$, with $A \in \Theta$, let us check the property 2.9.3 ($\forall$-p$^-$). We have

$$(\delta_k A)^m(z) = \forall z \forall x (P(x, z) \lor (P(x, z) \supset A^m(z)))$$

where $xz$ is a distinct list of new variables, $|z| = m$, $|x| = k$, $P$ does not occur in $A^m(z)$. This formula is clearly $QH$-equivalent to $\delta_{k+m} A^m(z)$. Since $\Theta$ is $\forall$-perfect, there exists $B \in \text{Sub}(\Theta)$ such that

$$L \vdash B \supset \forall z A^m(z),$$

and thus

$$L \vdash B \supset A^m(z),$$

We may also assume that $r(xz) \cap FV(B) = \emptyset$. Hence

$$L \vdash \delta_{k+m} B \supset \delta_{k+m} A^m(z),$$

by the monotonicity of $\delta_{k+m}$ and

$$\delta_{k+m} B = \forall z \forall x (P(x, z) \lor (P(x, z) \supset B)) \in \text{Sub}(\delta_{k+m} \Theta),$$

as we can choose $P$ that does not occur in $A^m(z)$ and in $B$.

$$L \vdash \delta_{k+m} B \supset (\delta_k A)^m(z),$$

therefore ($\forall$-p$^-$) holds for $\delta_k A$.

Proposition 2.13.15 If $\Theta$ is $\forall$-perfect in $L$, then $\delta^n_\infty \Theta$ and $\delta^n_\ast \Theta$ are $\forall$-perfect in $L$, for all $n \geq 0$.

Proof For $\delta^n_\ast \Theta = \delta^1_\ast \ldots \delta^1_\ast \Theta$ proceed by induction on $n$. And for $\delta^n_\infty \Theta$ use sub-equivalence: $\delta^n_\infty \Theta \Leftrightarrow_{\text{sub}} \delta^n_\ast \Theta$.

Alternatively, for $\delta^n_\ast \Theta$ one can apply a direct argument generalising the proof of Lemma 2.13.14. In fact,

$$L \vdash B \supset \forall z A^m(z)$$

\[26\text{Cf. [Yokota, 1989, Proposition 1.4.6].}\]
implies

\[ L \vdash B \supset A^m(z) \]

and next

\[ L \vdash \delta_{k+m}^n B \supset \delta_{k+m}^n A^m(z) \]

with

\[ \mathsf{QH} \vdash \delta_{k+m}^n A^m(z) \equiv (\delta_k^n A)^m; \]

again we can present \( \delta_{k+m}^n B, (\delta_k^n A)^m \) as

\[ \delta_{k+m}^n B = \forall z \forall x[P_n(x, z) \lor (P_n(x, z) \supset \forall z \forall x(P_{n-1}(x, z) \lor (P_{n-1}(x, z) \supset \ldots)))], \]

\[ (\delta_k^n A)^m = \forall z \forall x(P_n(x, z) \lor P_n(x, z) \supset \forall x(P_{n-1}(x, z) \lor (P_{n-1}(x, z) \supset \ldots))); \]

with the same set of \((k + m)\)-ary predicate letters in both formulas.

Lemma 2.13.16 If \( L = \mathsf{QH} + \Theta \) and \( \Theta \) is \( \forall \)-perfect in \( \mathsf{QH} \), then

\[ \Delta L = \mathsf{QH} + \delta \Theta = \mathsf{QH} + \delta_\infty \Theta. \]

Proof By 2.13.13(2a)

\[ \Delta L = \mathsf{QH} + \delta L = \mathsf{QH} + \delta \Theta, \]

since \( \mathsf{QH} + L = L = \mathsf{QH} + \Theta \) and both \( \Theta \) and \( L \) are \( \forall \)-perfect in \( \mathsf{QH} \). Thus

\[ \Delta L = \mathsf{QH} + \delta_\Theta \subseteq \mathsf{QH} + \delta_1^1 \Theta. \]

Since by 2.13.7 \( \delta_\infty^1 \Theta \subseteq \Delta L \), it follows that \( \Delta L = \mathsf{QH} + \delta_\infty^1 \Theta. \)

Proposition 2.13.17 (1) If \( L = \mathsf{QH} + \Theta \) and \( \Theta \) is \( \forall \)-perfect in \( \mathsf{QH} \), then

\[ \Delta^n L = \mathsf{QH} + \delta^n \Theta. \]

(2) If \( L = \mathsf{QH} + \Theta \), then

\[ \Delta^n L = \mathsf{QH} + \delta^n (\Theta^\forall). \]

Proof (1) By induction on \( n \). If \( n = 0 \), then \( \Delta L = L \), \( \delta^n \Theta = \Theta \), so the statement is trivial. For the induction step suppose

\[ \Delta^n L = \mathsf{QH} + \delta^n \Theta. \]

Then by 2.13.16 and 2.13.15,

\[ \Delta^{n+1} L = \mathsf{QH} + \delta_\infty^{1} \delta^n \Theta = \mathsf{QH} + \delta^{n+1} \Theta. \]

(2) Follows from (1) and the observation that \( \mathsf{QH} + \Theta = \mathsf{QH} + \Theta^\forall \) and \( \Theta^\forall \) is \( \forall \)-perfect in \( \mathsf{QH} \).

In particular, for \( \Theta = \{ \bot \} \) we have
Corollary 2.13.18 \(^{27}\) \(QHP_n^+ = QH + \delta^n\{\bot\}\).

Now let us show that every \(\delta^m_k A\) for \(k > 1\) (and hence every \(\delta_{k_1,\ldots,k_n}\)), is deducible from \(\delta^n_k A\).

Lemma 2.13.19 \(QH + \delta^k A \vdash (\delta_k A \supset A) \supset A\) for any \(k, n \geq 0\).

Proof By 2.13.8(2),

\[
B \supset A \vdash_{QH} \delta^k_B \supset \delta^k A,
\]

hence

\[
(1) \quad \delta_k A \supset A, \quad B \supset A, \quad \delta_k B \vdash_{QH} A.
\]

On the other hand, by the deduction theorem

\[
(2) \quad \delta_k A \supset A, \quad B \supset (\delta_k A \supset A) \supset A \vdash_{QH} B \supset A.
\]

From (1), (2) we obtain\(^ {28}\)

\[
B \supset (\delta_k A \supset A) \supset A, \quad \delta_k A \supset A, \quad \delta_k B \vdash_{QH} A.
\]

Hence (again by the deduction theorem)

\[
(3) \quad B \supset (\delta_k A \supset A) \supset A \vdash_{QH} \delta_k B \supset (\delta_k A \supset A) \supset A.
\]

Note that \(\delta_k A\) is \(\delta_{k,P} A\) for some \(P\) that does not occur in \(A\), but (3) still holds if \(P\) occurs in \(B\).

Now by induction it follows that

\[
QH \vdash \delta^m_k \supset (\delta_k A \supset A) \supset A
\]

for any \(n\). In fact, for \(n = 0\) this is a substitution instance of an axiom, and for the induction step we can apply (3).

Lemma 2.13.20 \(QH + \delta^m_j \delta_j A \vdash \delta^m_j \delta_{j+1} A\) for \(j > 0, \ m \geq 0\).

Proof Put

\[
L := QH + \delta^m_j \delta_{j,P} A, \quad B := \forall z(Q(y, z) \lor (Q(y, z) \supset A)),
\]

where \(|y| = j, \ z \not\in r(y), \ r(yz) \cap FV(A) = \emptyset\) and \(P, Q\) do not occur in \(A\). Then

\[
B = [Q(y, z)/S(z)] \delta_{1,S} A,
\]

\[
\delta_{j,P} A = \forall y(P(y) \lor (P(y) \supset A)), \quad \delta_{j+1,Q} A = \forall y B.
\]

Hence by substitution \([B/P(y)]\)

\[
(1) \quad L \vdash \delta^m_{1+} \forall y(B \lor (B \supset A)).
\]

\(^{27}\)Cf. [Yokota, 1989, Theorem 1.4.7].

\(^{28}\)Cf. Lemma 1.2.3(4) from [Yokota, 1989].
By Lemma 2.13.19 (for \( k = 1, \ n = m + 1 \)) we have

\[
(2) \quad QH + \delta_{1}^{m+1} A \vdash (\delta_{1,s} A \supset A) \supset A.
\]

By 2.13.11, \( L \vdash \delta_{1}^{m+1} A \), and thus

\[
(3) \quad L \vdash (\delta_{1,s} A \supset A) \supset A,
\]

Hence by substitution \([Q(y, z)/S(z)]\)

\[
(4) \quad L \vdash (B \supset A) \supset A.
\]

Since \( z \not\in FV(A) \), we also obtain \( QH \vdash A \supset B \); thus

\[
(5) \quad L \vdash (B \supset A) \supset B,
\]

which implies

\[
(6) \quad L \vdash B \lor (B \supset A) \supset B,
\]

and therefore

\[
(7) \quad L \vdash \forall y (B \lor (B \supset A)) \supset \forall y B,
\]

by the monotonicity of \( \forall y \). Eventually from (1), (7) and 2.13.8(2) we obtain

\[
L \vdash \delta_{1}^{m} \forall y B = \delta_{1}^{m} \delta_{j+1} A
\]

as required.

Lemma 2.13.21 \( QH + \delta_{1}^{m+1} A \vdash \delta_{1}^{m} \delta_{k} A \) for \( k \geq 1, \ m \geq 0 \).

Proof By induction on \( k \). The case \( k = 1 \) is trivial. If the statement holds for \( k \) we obtain it for \( k + 1 \):

\[
QH + \delta_{1}^{m} \delta_{k+1} A \subseteq QH + \delta_{1}^{m} \delta_{k} A \subseteq QH + \delta_{1}^{m} A
\]

by Lemma 2.13.20 and the induction hypothesis.

Proposition 2.13.22\(^{29}\) For any \( A \in IF(=) \)

(1) \( QH + \delta_{1}^{n} A \vdash \delta_{k}^{n} A \) for \( k, n \geq 0 \);

(2) \( QH + \delta_{1}^{n} A \vdash \delta_{k_{1}} \ldots \delta_{k_{n}} A \) for \( k_{1}, \ldots, k_{n} \geq 0 \).

Proof (1) implies (2), by 2.13.11. The case \( k = 0 \) in (1) also follows from 2.13.11, and the case \( k = 1 \) is trivial. So it remains to prove (1) for \( k > 1 \).

By induction on \( n - m \) let us show

\[
(3) \quad QH + \delta_{1}^{m} A \vdash \delta_{1}^{m} \delta_{k}^{n-m} A
\]

\(^{29}\)Theorem 1.3.5 from [Yokota, 1989] is the particular case of this proposition for \( A = \bot \). Our proof is similar to that paper.
for \( 0 \leq m \leq n \). If \( m = n \), this is trivial. For the step, suppose (3) for \( m \). By 2.13.21 we have
\[
\text{QH} + \delta_i^m \delta_k^{n-m} A \vdash \delta_i^{m-1} \delta_k \delta_k^{n-m} A = \delta_i^{m-1} \delta_k^{n-m+1} A.
\]
So by the induction hypothesis we obtain
\[
\text{QH} + \delta_i^{m-1} \delta_k^{n-m+1} A \subseteq \text{QH} + \delta_i^m \delta_k^{n-m} A \subseteq \text{QH} + \delta_i^n A,
\]
i.e. (3) holds for \( m - 1 \).

Finally note that (1) is (3) for \( m = 0 \).

From 2.13.22 it follows that
\[
\text{QH} + \delta_i^n A = \text{QH} + \delta_k^n A
\]
for any \( k > 1 \). We also have

**Lemma 2.13.23** \( \text{QH} + \delta_k A = \text{QH} + \delta_0 A \).

**Proof** In fact
\[
\text{QH} + \delta_0 A = \text{QH} + p \lor (p \supset A) \vdash \forall y (P(y) \lor (P(y) \supset A)) = \delta_1 A.
\]
by substitution \([P(y)/p]\) and generalisation.

However it may happen that \( \text{QH} + \delta_0^n A \subseteq \text{QH} + \delta^n A \) for some \( A \) and \( n > 1 \). E.g. \( \text{QH} + P_n \not\vdash P_n^+ \) for \( n > 1 \) (cf. [Ono, 1983]; recall that \( P_n = \delta_0^n \perp \), \( P_n^+ = \delta^n_1 \perp \)).

Now we can strengthen 2.13.17:

**Proposition 2.13.24**

1. If \( L = \text{QH} + \Theta \) and \( \Theta \) is \( \forall \)-perfect in \( \text{QH} \), then
\[
\Delta^n L = \text{QH} + \delta^n_1 \Theta.
\]

2. If \( L = \text{QH} + \Theta \), then
\[
\Delta^n L = \text{QH} + \delta^n_1 (\Theta^\forall).
\]

**Proof** Follows from 2.13.17 and 2.13.22.

By applying 2.13.24 to \( \Theta = \{ \perp \} \) we obtain

**Theorem 2.13.25** \( \text{QHP}_n^+ = \text{QH} + P_n^+ \) for \( n > 0 \).

Now let us show perfection for finite axiomatisations of \( \text{QHP}_n^+ \) and obtain an alternative proof of Theorem 2.13.25. This proof is not so straightforward, but does not use the infinite axiomatisation described in 2.13.18. However we still need 2.13.22 for this proof.
Lemma 2.13.26 $H \vdash (C \supset q) \supset C$, where $C = \bigwedge_{i=1}^{n} (p_i \lor (p_i \supset q)), n \geq 0$.

Proof  By induction on $n$. The case $n = 0$ is trivial.
Consider the inductive step from $n$ to $n + 1$. By the deduction theorem it suffices to prove

$$C \supset q \vdash_H C,$$

for $C = \bigwedge_{i=1}^{n+1} (p_i \lor (p_i \supset q))$, which obviously follows from

$$C \supset q \vdash_H p_i \supset q$$

for $1 \leq i \leq n + 1$. Put

$$B := \bigwedge \{ p_j \lor (p_j \supset q) \mid 1 \leq j \leq n+1, j \neq i \}.$$

Then

$$H \vdash C \equiv B \land (p_i \lor (p_i \supset q)),$$

and thus

$$C \supset q, B, p_i \vdash_H q.$$

Hence by the deduction theorem,

$$C \supset q, p_i \vdash_H B \supset q.$$

Together with the induction hypothesis

$$H \vdash (B \supset q) \supset q$$

(2) implies

$$C \supset q, p_i \vdash_q q.$$

Hence (1) follows by the deduction theorem again. ■

Lemma 2.13.27 Let $L$ be a predicate superintuitionistic logic, $\Theta$ a set of formulas, $k \geq 0$. Then

(1) If $\Theta$ is $\land$-perfect in $L$, then $\delta_k \Theta$ is $\land$-perfect in $L$.

(2) If $\Theta$ is $\forall$-perfect in $L$ and $L \supseteq \delta_1^n \Theta$ for some $n > 0$, then $\delta_k \Theta$ is $\forall$-perfect in $L$.

(3) If $\Theta$ is $\land\forall$-perfect in $L$ and $L \supseteq \delta_1^n \Theta$ for some $n > 0$, then $\delta_k \Theta$ is $\land\forall$-perfect in $L$. 
Proof

(1) Consider arbitrary formulas

\[ D_i := \forall y (A_i(y, x) \lor (A_i(y, x) \supset B_i(x))) \in Sub(\delta_k \Theta) \]

for \( i = 1, \ldots, m \), where \( y \) and \( x \) are disjoint lists of variables, \( |y| = k \),

\[ B_i(x) \in Sub(\Theta), \ FV(B_i(x)) \subseteq x, \ FV(A_i(y, x)) \subseteq xy. \]

By \( \land \)-perfection there exists \( B \in Sub(\Theta) \) such that

\[ L \vdash B \supset \bigwedge_i B_i(x); \]

we may assume that \( r(y) \cap FV(B) = \emptyset \). Put

\[ C := \bigwedge_i (A_i(y, x) \lor (A_i(y, x) \supset B_i(x))). \]

By Lemma 2.13.26, \( QH \vdash (C \supset B) \supset C \), hence

\[ L \vdash C \lor (C \supset B) \supset \bigwedge_i (A_i(y, x) \lor (A_i(y, x) \supset B_i(x))), \]

and so

\[ L \vdash \forall y (C \lor (C \supset B)) \supset \bigwedge_i \forall y (A_i(y, x) \lor (A_i(y, x) \supset B_i(x))) \]

by standard properties of quantifiers (Lemma 2.6.15). Since

\[ \forall y (C \lor (C \supset B)) \in Sub(\delta_k \Theta), \]

we obtain the required property for \( \bigwedge_i D_i \).

(2) Suppose \( A \in Sub(\Theta) \),

\[ \delta_k A = \forall y (P(y) \lor (P(y) \supset A)) \]

for \( |y| = k \), where \( P \) does not occur in \( A \),

\[ (\delta_k A)^m = \forall z \forall y (Q(z, y) \lor (Q(z, y) \supset A^m(z))), \]

for \( |z| = m \), \( r(y) \cap r(z) = \emptyset, r(yz) \cap FV(A) = \emptyset \), \( Q \in PL^{k+m} \), where \( Q \) does not occur in \( A^m \).

By \( \forall \)-perfection there is \( B \in Sub(\Theta) \) such that \( L \vdash B \supset \forall z A^m(z) \); we may assume that \( FV(B) \cap r(yz) = \emptyset \) and \( Q \) does not occur in \( B \). Put

\[ C := \delta_{k+m} B = \forall z \forall y (Q(z, y) \lor (Q(z, y) \supset B)). \]
Since \( L \vdash B \supset A^m(z) \), it follows that

\[(3) \quad L \vdash C \supset (\delta_k A)^m.\]

Since by the assumption (2) of the lemma \( \delta_1^n B \in \text{Sub}(\delta_1^n \Theta) \subseteq L \), Proposition 2.13.22 implies \( L \vdash \delta_{t+m}^n B \). Thus by Lemma 2.13.19, \( L \vdash (C \supset B) \supset B \). Also \( L \vdash B \supset C \) by 2.13.8, hence

\[(4) \quad L \vdash (C \supset B) \supset C,\]

and so

\[(5) \quad L \vdash C \lor (C \supset B) \supset (\delta^k A)^m,\]

by (3) and (4); eventually,

\[(6) \quad L \vdash C' \supset (\delta^k A)^m,\]

where

\[C' := \forall y (C \lor (C \supset B)) \in [C/Q(x)] \delta_{k,Q} B, \ Q \in PL^k\]

(note that here \( \forall y \) is a dummy quantifier). Since \( C' \in \text{Sub}(\delta_k \Theta) \), this completes the proof.

Similarly to 2.13.27(1) one can show

Proposition 2.13.28

(1) If \( \Theta \) is \( \land \)-perfect in \( L \), then \( \delta_1^n \Theta \) is \( \land \)-perfect in \( L \) (and thus \( \delta_\infty^n \Theta, \delta_*^n \Theta \) are also \( \land \)-perfect in \( L \)).

(2) If \( \Theta \) is \( \forall \)-perfect in \( L \), then \( \delta_1^n \Theta \) is \( \forall \)-perfect in \( L \) (and \( \delta_\infty^n \Theta, \delta_*^n \Theta \) as well).

Lemma 2.13.27(2) allows us to prove 2.13.25 without applying 2.13.17; cf. the proof of 2.13.16.

Since \( \{\bot\} \) is \( \forall \)-perfect in \( \text{QH} \), we obtain

Lemma 2.13.29

(1) The sets \( \{P_n\} \) and \( \{P_n^+\} \) are \( \land \)-perfect in \( \text{QH} \) for \( n > 0 \).

(2) The sets \( \{P_n\} \) and \( \{P_n^+\} \) are \( \forall \)-perfect in \( \text{QH} + P_r^+ \) for all \( n, r > 0 \) (note that the case \( n \geq r \) is trivial).

Remark 2.13.30 Moreover, these sets are arity-perfect in the sense of [Yokota, 1989].

Proposition 2.13.31 \( \text{QH}^{+}_{P_n} = \text{QH} + P_r^+ + P_n \) for \( r \geq n \geq 0 \).
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So \(P_n\) and \(P_n^+\) are deductively equivalent in \(QHP_r^+\) for a sufficiently large \(r\).

**Proof** By induction on \(n\). Consider the induction step. We have

\[
QHP_{n+1}^+ = \Delta(QHP_n^+) = QH + \delta(QHP_n^+) \subseteq QH + P_r^+ + P_{n+1},
\]

by Lemma 2.13.13 (1a) applied to

\[
L = QH + P_r^+, \quad \Theta_1 = QHP_n^+, \quad \Theta_2 := \{P_n\},
\]

note that \(L + \Theta_1 = QHP_n^+ = L + \Theta_2\) by the induction hypothesis, and \(\Theta_2\) is \(\forall\)-perfect in \(L\) by Lemma 2.13.29 (ii).

Theorem 2.13.25 is clearly a particular case of this statement for \(n = r\). Therefore we obtain an alternative proof of Theorem 2.13.25 that does not use 2.13.18.

So we have

**Corollary 2.13.32** \(QH + P_n^+ = QH + P_r^+ + P_n\) for \(n < r\).

Similarly we obtain

**Proposition 2.13.33** If \(L = L_0 + \Theta, \Theta\) is \(\forall\)-perfect in \(L_0\) and \(\delta_1 \Theta \subseteq L_0\) for some \(r > 0\), then

\[
L_0 + \Delta^n L = L_0 + \delta_n \Theta
\]

and \(\delta_k \Theta\) is \(\forall\)-perfect in \(L_0\) for all \(k, n \geq 0\).

**Proof** By induction on \(n\). The case \(n = 0\) is trivial. Consider the induction step from \(n\) to \(n + 1\). First note that by 2.9.9,

\[
QH \vdash \delta_1 A \supseteq \delta_1 \delta_n A.
\]

Hence \(\delta_1 \delta_n \Theta \subseteq L\), and so \(\delta_n \Theta = \delta_k(\delta_n \Theta)\) is \(\forall\)-perfect in \(L_0\) by the induction hypothesis and Lemma 2.13.27.

Next, by the induction hypothesis and Lemma 2.13.13 (1a)

\[
L_0 + \Delta^{n+1} L = L_0 + \delta(\Delta^n L) = L_0 + \delta(L_0 + \delta_n \Theta) \subseteq L_0 + \delta(\delta_n \Theta) = L_0 + \delta_{n+1} \Theta.
\]

The converse inclusion

\[
L_0 + \delta_{n+1} \Theta \subseteq L_0 + \Delta^{n+1} L
\]

is trivial.

So \(\Delta^n L = L_0 + \delta_n \Theta\) in particular, \(\Delta^n L = L_0 + \delta_n \Theta\), whenever \(L_0 \subseteq \Delta^n L\).

Note that \(\delta_n \Theta \subseteq \Delta^n L\) for any \(L\) and \(\Theta \subseteq L\) and \(n \geq 0\), by Lemma 2.13.7.

Also recall that \(\delta_1 \Theta \subseteq QHP_r^+\) for any \(\Theta\) by 2.13.10.

**Corollary 2.13.34** If \(L = L_0 + \Theta\) and \(\delta_1 \Theta \subseteq L_0\) for some \(r > 0\), then \(L_0 + \Delta^n L = L_0 + \delta_k(\Theta^r)\) for all \(k, n \geq 0\).

Actually we need only the case \(k = 0\) of this statement, because the case \(k = 1\) (and thus, the case \(k > 0\)) readily follows from 2.13.24, without any restriction on \(L_0\) (recall that \(\delta_k L_0 \subseteq L_0\) by 2.13.8 (1)). And 2.13.33 shows that if \(L_0\) contains \(\delta_1 \Theta\) for some (sufficiently large) \(r > 0\), then one can reduce the case \(k = 1\) to \(k = 0\) in an axiomatisation of \(\Delta^n L\) above \(L_0\).
2.14 Adding equality

In this section we study correlation between a logic without equality $L$ and its minimal extension with equality $L^\equiv$. There exists an obvious translation from formulas with equality to formulas without equality just replacing equality with an ordinary predicate letter. In some cases specific equality axioms can be reduced to finitely many formulas — then we obtain a reduction for the corresponding decision problems.

Definition 2.14.1 For an $N$-m.p.l. $L$, let $L^\equiv := \text{QK}^\equiv_N + L$, and for an s.p.l. $L$, let $L^\equiv := \text{QH}^\equiv + L$. $L^\equiv$ is called the equality-expansion of $L$.

The other way round, for an $m.p.l. = (s.p.l.) L$, we define the equality-free fragment as $L^0 := L \cap MF_N$ (respectively, $L \cap IF$).

For any modal formula with equality $A$ and $Q \in PL^2$ that does not occur in $A$, we define $A^Q$ as the formula obtained from $A$ by replacing all occurrences of ‘$=$’ with $Q$. For a set of $N$-modal formulas $\Gamma$ put

$$\Gamma^Q := \{A^Q \mid A \in \Gamma\}, \Box^\infty \Gamma := \{\Box_\alpha B \mid B \in \Gamma, \alpha \in I^\infty_N\}.$$  

Consider the following sets of formulas (for $N \geq 0$)

$$\mathcal{E}_N := \{\forall x \forall y(x = y \supset y = x), \forall x \forall y \forall z(x = y \land y = z \supset x = z), \forall x \ x = x\}$$

$$\cup \{\exists x \forall y(x = y \supset \Box_i(x = y)) \mid 1 \leq i \leq N\}$$

$$\cup \{\forall \left(\bigwedge_{i=1}^n x_i = y_i \supset \bullet P^0_k(x_1, \ldots, x_n) \equiv P^0_k(y_1, \ldots, y_n)\right) \mid n, k \geq 1, P^0_k \neq Q\},$$

where all $x_j$, $y_j$ are different.

For a formula $A$, let $MF_{N,A}$ be the set of all $N$-modal formulas built from predicate letters occurring in $A$:

$$\mathcal{E}_{N,A} := \mathcal{E}_N \cap MF_{N,A}.$$  

We shall omit $N$ if it is clear from the context and use the notation $\Box^\infty \mathcal{E}$, $\Box^\infty \mathcal{E}^Q$, $MF_A$, $\mathcal{E}_A$, $\mathcal{E}_A^Q$.

Lemma 2.14.2 If $A \in \text{Sub}(\mathcal{E}_N)$, then $\Box^\infty \mathcal{E}_N \vdash \text{QK}_N \Box A$.

Proof The only nontrivial case is when

$$A = [B/P(x)] \forall \left(\bigwedge_{i=1}^n x_i = y_i \supset \bullet P(x) \equiv P(y)\right)$$

for $P \in PL^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

Put

$$x \equiv y := \bigwedge_{i=1}^n x_i = y_i.$$  

Then

$$\Box A = \Box(x \equiv y \supset \bullet B \equiv [y/x]B).$$

We prove $\Box^\infty \mathcal{E}_N \vdash \text{QK}_N \Box A$ by induction on the complexity of $B$. 
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- If $B$ is atomic and does not contain $P$, the claim is trivial.
- If $B = P(z)$ for some list $z$ (maybe not distinct), then $[y/x]B = P(t)$, where $t = [y/x]z$, i.e. every $x_i$ is replaced with $y_i$, whenever it occurs in $z$. Then obviously,

$$\forall x(x = x) \vdash_{\text{QK}_N} x \equiv y \supset z \equiv t. \tag{1}$$

By Lemma 2.6.10 (xxvii) we also have

$$\forall(x \equiv y \supset P(x) \equiv P(y)) \vdash_{\text{QK}_N} z \equiv t \supset P(z) \equiv P(t). \tag{2}$$

Now (1) and (2) imply

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} x \equiv y \supset P(z) \equiv P(t),$$

and it remains to apply generalisation.

- If $B = B_1 @ B_2$ for a propositional connective $\otimes$ and

$$\text{QK}_N \vdash x \equiv y \supset B_i \equiv [y/x]B_i$$

by the induction hypothesis, then the claim follows by an argument in classical propositional logic – note that the rule

$$A \equiv A', \ B \equiv B' \\ \Rightarrow \ \ \ \ \ \ A \otimes B \equiv A' \otimes B',$$

is admissible in the $\text{QK}_N$-theory $\{x \equiv y\}$ (with $x$, $y$ considered as constants) and apply the deduction theorem.

- If $B = \Box_i C$ and

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} x \equiv y \supset C \equiv [y/x]C,$$

we obtain by Lemma 2.7.10

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} \Box_i(x \equiv y) \supset \Box_i(C \equiv [y/x]C)). \tag{3}$$

Since

$$K_N \vdash \Box_i(p \equiv q) \supset \Box_i p \equiv \Box_i q$$

(Lemma 1.1.1), it follows that

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} \Box_i(x \equiv y) \supset \Box_i C \equiv \Box_i[y/x]C. \tag{4}$$

Now

$$\mathcal{E}_N \vdash_{\text{QK}_N} x_j = y_j \supset \Box_i(x_j = y_j).$$

implies

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} x \equiv y \supset \Box_i(x \equiv y). \tag{5}$$

From (4) and (5) we have

$$\square^\omega \mathcal{E}_N \vdash_{\text{QK}_N} x \equiv y \supset \Box_i C \equiv \Box_i[y/x]C$$

and thus the claim holds for $B$. 
• If \( B = \forall z C \), we may assume \( z \notin \mathbf{xy} \) (otherwise, after renaming \( z \), \( B \) and \( [y/x]B \) change to equivalent congruent formulas).

Suppose

\[ \square^\infty E_N \vdash_{\mathbf{QK}_N} x \equiv y \supset C \equiv [y/x]C. \]

Then by \( \forall \)-introduction we deduce (in the same theory)

\[ \forall z(x \equiv y \supset C \equiv [y/x]C), \]

and hence

\[ x \equiv y \supset \forall z(C \equiv [y/x]C) \quad (6) \]

by Lemma 2.6.10(xi).

Finally

\[ \forall z(C \equiv [y/x]C) \supset \forall zC \equiv \forall z[y/x]C. \quad (7) \]

by Lemma 2.6.10(xxviii).

So

\[ \forall(x \equiv y \supset \forall zC \equiv \forall z[y/x]C) \]

follows from (6) and (7) by transitivity and \( \forall \)-introduction.

• If \( B = \exists z C \), we also have (6) by the induction hypothesis. Then, instead of (7) we use

\[ \forall z(C \equiv [y/x]C) \supset \exists zC \equiv \exists z[y/x]C, \quad (8) \]

which follows from 2.6.10(xxvii).

Lemma 2.14.2 has an intuitionistic version:

**Lemma 2.14.3** If \( A \in \text{Sub}(E_0) \), then \( E_0 \vdash_{\mathbf{QH}} A \).

**Proof** Very similar to 2.14.2. Again we prove

\[ E_0 \vdash_{\mathbf{QH}} \forall(x \equiv y \supset B \equiv [y/x]B) \]

by induction. The reader can check that in all cases the argument is based only on intuitionistic logic.

**Proposition 2.14.4** Let \( L \) be an \( N \)-m.p.l. Then for any \( N \)-modal formula \( A \) with equality that does not contain \( Q \)

\[ L^= \vdash A \iff \square^\infty E^Q \vdash L A^Q \iff \square^\infty E^Q_A \vdash L A^Q. \]

If \( L \) is conically expressive, then

\[ L^= \vdash A \iff \Box^* E^Q_A \vdash L A^Q. \]
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Proof First note that $\mathcal{E}_N$ is a set of axioms for $L^=$ above $L$ (they are all $L^=$-theorems and the usual axioms follow from $\mathcal{E}_N$). So by the deduction theorem 2.8.4,

$$L^= \vdash A \iff \mathcal{E}_N \vdash L A.$$

All formulas from $\mathcal{E}_N$ are of the form $\Box \forall B$, where $B \in \text{Sub}(\mathcal{E}_N)$, so they are $\mathcal{QK}_N$-provable in $\mathcal{E}_N$ by 2.14.2, generalisation and $\Box$-introduction. Thus

$$L^= \vdash A \iff \mathcal{E}_N \vdash L A.$$

Replacing '=' with '=' does not affect the $L$-inference (more precisely, the equivalence $\Gamma \vdash L A$ iff $\Gamma^Q \vdash L A^Q$ is checked by induction), so

$$L^= \vdash A \iff \mathcal{E}_N^Q \vdash L A^Q,$$
and thus

$$\mathcal{E}_N^Q \vdash L A^Q \Rightarrow L^= \vdash A.$$

The other way round, suppose $L^= \vdash A$. Then $\mathcal{E}_N^Q \vdash L A^Q$. Let $S$ be a formula substitution replacing every atomic formula $P(x)$ with $T$ for any $P \neq Q$ that does not occur in $A$. We claim that

$$\mathcal{E}_N^Q \vdash L B \Rightarrow \mathcal{E}_N^Q \vdash L SB.$$ (10)

In fact,

$$S(Q(x, y) \supset P(x) \equiv P(y)) = (x = y \supset \top \equiv \top)$$
is obviously $L$-provable, so (10) holds for any $B \in \mathcal{E}_N^Q - \mathcal{E}_A^Q$.

Then we can argue by induction, since $S$ distributes over propositional connectives and quantifiers. Eventually

$$\mathcal{E}_N^Q \vdash L SA^Q (= A^Q),$$

which completes the proof.

The observation about conical expressiveness follows from Lemma 2.8.7. ■

Proposition 2.14.5 Let $L$ be an s.p.l. Then for any $A \in IF^=$ without occurrences of $Q$

$$L^= \vdash A \iff \mathcal{E}_0^Q \vdash L A^Q \iff \mathcal{E}_A^Q \vdash L A^Q.$$

Proof Follows the same lines as 2.14.4. By 2.8.3,

$$L^= \vdash A \iff \text{Sub}(\mathcal{E}_0) \vdash L A.$$

Hence by Lemma 2.14.3 and generalisation,

$$L^= \vdash A \iff \mathcal{E}_0 \vdash L A,$$

which is equivalent to

$$\mathcal{E}_0^Q \vdash L A^Q.$$

The implication

$$\mathcal{E}_0^Q \vdash L A^Q \Rightarrow \mathcal{E}_A^Q \vdash L A^Q$$
is again proved by replacing redundant predicate letters with $\top$. ■
In many cases the equality-expansion is conservative. To show this, for logics without equality we define a weak analogue of equality – indiscernibility. Namely, for \( P \in PL^n \) and variables \( x, y \) we put

\[
In_P(x, y) := \forall z_1 \ldots \forall z_n \bigwedge_{j=1}^n ([x/z_j]P(z_1, \ldots, z_n) \equiv [y/z_j]P(z_1, \ldots, z_n)),
\]

where of course, all \( z_j \)'s are different and \( x, y \neq z_j \). For a predicate formula \( A \), put

\[
In_A(x, y) := \bigwedge \{In_P(x, y) \mid P \text{ occurs in } A\}.
\]

**Lemma 2.14.6** For \( P \in PL^n \)

\[
\mathbf{QH} \vdash \bigwedge_{i=1}^n In_P(x_i, y_i) \supset P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n).
\]

**Proof** Almost the same as for 2.6.16(iv).

We show by induction that

\[
\bigwedge_{i=1}^n In_P(x_i, y_i) \vdash \mathbf{QH} [x_1, \ldots, x_m/z_1, \ldots, z_m] P(z) \equiv [y_1, \ldots, y_m/z_1, \ldots, z_m] P(z)
\]

for a list of new variables \( z = (z_1, \ldots, z_n) \). For the induction step we again consider

\[
A_m := [x_1, \ldots, x_m/z_1, \ldots, z_m] P(z),
\]

\[
B_m := [y_1, \ldots, y_m/z_1, \ldots, z_m] P(z).
\]

By the induction hypothesis, \( \forall z_{m+1} \)-introduction and 2.6.10(ii) we obtain

(1) \([x_{m+1}/z_{m+1}]A_m \equiv [x_{m+1}/z_{m+1}]B_m\).

We also have

(2) \([y_{m+1}/z_{m+1}]B_m \equiv [y_{m+1}/z_{m+1}]B_m\).

In fact, \( B_m = P(y_1, \ldots, y_m, z_{m+1}, \ldots, z_n) \), so

\[
In_P(x_{m+1}, y_{m+1}) = \forall z \bigwedge_{j=1}^n ([x_{m+1}/z_j]P(z) \equiv [y_{m+1}/z_j]P(z))
\]

\[
\vdash \mathbf{QH} [y_1, \ldots, y_m/z_1, \ldots, z_m] \bigwedge_{j=1}^n ([x_{m+1}/z_j]P(z) \equiv [y_{m+1}/z_j]P(z))
\]

by 2.6.10(xxv), and (2) is a conjunct in the latter formula. From (1), (2) by transitivity it follows that

\[
A_{m+1} = [x_{m+1}/z_{m+1}]A_m \equiv [y_{m+1}/z_{m+1}]B_{m+1} = B_{m+1}.
\]
Lemma 2.14.7

(1) If $L$ is a conically expressive $N$-m.p.l, then for any $N$-modal predicate formula $A$ that does not contain $Q$

$$[\square^* In_A(x, y)/Q(x, y)]\square^* \mathcal{E}_A^Q \subseteq L.$$ 

(2) If $L$ is an $s.p.l.$, then for any $A \in \text{IF}$

$$[\text{In}_A(x, y)/Q(x, y)]\mathcal{E}_A^Q \subseteq L.$$ 

Proof  We have to prove the corresponding substitution instances of the 'axioms' from $(\square^*)\mathcal{E}_A^Q$ in $L$.

(1) Reflexivity of $Q$.
We have $[x/z_j]P(z) \equiv [x/z_j]P(z)$ by $H$, hence $\text{In}_P(x, x) \vdash L$ by $\wedge$- and $\forall z$-introduction. Thus $\forall x \text{In}_A(x, x)$ in the intuitionistic case, $\forall x \square^* \text{In}_A(x, x)$ in the modal case — by $\wedge$, $\square^*$- and $\forall$-introduction.

(2) Symmetry of $Q$.
By $H$ we have

$$[x/z_j]P(z) \equiv [y/z_j]P(z) \vdash L [y/z_j]P(z) \equiv [x/z_j]P(z),$$

hence $\text{In}_P(x, x) \vdash L \text{In}_P(x, y)$ and next $\text{In}_A(x, y) \vdash L \text{In}_A(y, x)$ by applying $\forall z B \supset B, \wedge$- and $\forall$-introduction. Then $L \vdash \text{In}_A(x, y) \supset \text{In}_A(y, x)$ by the deduction theorem. In the modal case this implies $L \vdash \square^* \text{In}_A(x, y) \supset \square^* \text{In}_A(x, y)$ by the monotonicity of $\square^*$. So it remains to generalise over $x, y$.

(3) Transitivity of $Q$.
Since $\equiv$ is transitive in $H$, we have

$$[x/z_j]P(z) \equiv [y/z_j]P(z), [y/z_j]P(z) \equiv [t/z_j]P(z) \vdash L [t/z_j]P(z), [x/z_j]P(z) \equiv [t/z_j]P(z),$$

which implies

$$\text{In}_P(x, y), \text{In}_P(y, t) \vdash L \text{In}_P(x, t)$$

again by standard arguments with $\wedge$ and $\forall$, and hence

$$\text{In}_A(x, y), \text{In}_A(y, t) \vdash L \text{In}_A(x, t).$$

Since $x, y, t$ are fixed in this proof, in the modal case we may apply Lemma 2.7.10 for $\square^*$:

$$\square^* \text{In}_A(x, y), \square^* \text{In}_A(y, t) \dashv L \square^* \text{In}_A(x, t).$$

Thus

$$L \vdash \forall y(\square^* \text{In}_A(x, y) \land \square^* \text{In}_A(y, t) \supset \square^* \text{In}_A(x, t))$$

by the deduction theorem and $\forall$-introduction.
(4) \( \forall x \forall y (Q(x, y) \supset \Box_t Q(x, y)) \).

By 1.3.46 \( L \vdash \Box^* P \supset \Box_t \Box^* P \), hence \( L \vdash \forall x \forall y (\Box^* In_A(x, y) \supset \Box_t \Box^* In_A(x, y)) \)

by substitution and \( \forall \)-introduction.

(5) \( \forall (\bigwedge_{i=1}^n (\Box^* In_A(x_i, y_i)) \supset P(x) \equiv P(y)) \)

for \( P \in PL^n \cap MF_{N, A}, \; x = (x_1, \ldots, x_n), \; y = (y_1, \ldots, y_n) \).

By 2.14.6,

\( In_A(x_1, y_1), \ldots, In_A(x_n, y_n) \vdash L P(x) \equiv P(y), \)

hence

\( \Box^* In_A(x_1, y_1), \ldots, \Box^* In_A(x_n, y_n) \vdash L P(x) \equiv P(y) \)

by the reflexivity of \( \Box^* \). Now we can apply the deduction theorem and \( \forall \)-introduction.

\[ \text{Theorem 2.14.8} \]

\( L^= \) is a conservative extension of \( L \), i.e. \( (L^=)^\circ = L \) for any superintuitionistic predicate logic \( L \) and for any conically expressive m.p.l. \( L \).

\[ \text{Proof} \]

Let \( A \) be a formula without equality of the corresponding kind and assume that \( Q \) does not occur in \( A \). Then \( A^Q = A \). \( L^= \vdash A \) implies \( \Box^* E^Q_A \vdash L A \)

in the modal case (by 2.14.4) and \( E^Q_A \vdash L A \) in the intuitionistic case (by 2.14.5). Hence \( L \vdash \bigwedge \Box^* E^Q_A \supset A \) and respectively \( L \vdash \bigwedge E^Q_A \supset A \)

by the deduction theorem.

By applying the substitution \([\Box^* In_A(x, y)/Q(x, y)]\) in the modal case, and \([In_A(x, y)/Q(x, y)]\) in the intuitionistic case, we obtain

\( L \vdash \bigwedge [\Box^* In_A(x, y)/Q(x, y)] \Box^* E^Q_A \supset A \)

or respectively,

\( L \vdash \bigwedge [In_A(x, y)/Q(x, y)] E^Q_A \supset A. \)

Therefore \( L \vdash A \) by Lemma 2.14.7.

\[ \text{Problem 2.14.9} \]

Does Theorem 2.14.8 hold for an arbitrary m.p.l. \( L \)?

\[ \text{Remark 2.14.10} \]

T. Shimura and N.-Y. Suzuki obtained the following stronger result for the superintuitionistic case. Fix a binary predicate letter \( P \), and let

\( E^*_p(x, y) := \forall z (P(x, z) \equiv P(y, z)) \)

Then for any \( L \in \mathcal{S} \), and for any \( \Gamma \subseteq IF^= \) consisting of pure equality formulas with only positive occurrences of ‘\( = \)’,

\( (L^= + \Gamma)^\circ = L + [E^*_p(x, y)/Q(x, y)] \Gamma^Q \)

(cf. [Shimura and Suzuki, 1993, Theorem 3]).
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Now recall a well-known definition from recursion theory [Rogers, 1987].

**Definition 2.14.11** For sets of words (in a finite alphabet \( A \)) \( X, Y \) we say that \( X \) is \( m \)-reducible to \( Y \) (notation \( X \leq_m Y \)) if there exists a recursive function \( f : A^\infty \to A^\infty \) such that \( X = f^{-1}[Y] \); \( X \) is \( m \)-equivalent to \( Y \) (notation: \( X \equiv_m Y \)) if \( X \leq_m Y \) and \( Y \leq_m X \).

**Theorem 2.14.12** If \( L \) is an s.p.l. or a conically expressive m.p.l., then \( L \equiv_m L^= \).

**Proof** \( L \leq_m L^= \), since \( L^= \) is conservative over \( L \) (2.14.8); the corresponding function \( f \) sends every formula without equality to itself (and all other words to \( \bot \), say). \( L^= \leq_m L \), since for a formula \( A \) without \( Q \),

\[
L^= \vdash A \text{ iff } L \vdash \bigwedge ((\forall^*)E_A^Q \cdot \supset A^Q)
\]

by 2.14.4, 2.14.5. So the reducing function sends every formula \( A \) without \( Q \) to \( \bigwedge ((\forall^*)E_A^Q \cdot \supset A^Q \) and every nonformula to \( \bot \). If a formula \( A \) contains \( Q \), first replace \( Q \) with another binary predicate letter that does not occur in \( A \).

Let us now consider some specific axioms of equality.

**Definition 2.14.13** A logic (with equality) \( L \) is said to have stable (respectively, decidable, closed) equality if it contains the corresponding formula (\( SE \), \( DE \), or \( CE \), see Section 2.4). For a superintuitionistic predicate logic \( L \) (without equality), let

\[
L^d := L^= + DE, \quad L^s := L^= + SE.
\]

Similarly, for an \( N \)-m.p.l. \( L \), we define

\[
L^e := L^= + \{CE_1, \ldots, CE_N\},
\]

where \( CE_i \) denotes \( CE \) for \( \square_i \).

Obviously, \( L^= \subseteq L^s \subseteq L^d \).

Soon we will show that \( QH^d \) is conservative over \( QH \). However, \( L^d \) is not always conservative over \( L \); the corresponding example will be given later on.

As we do not substitute formulas for `\( = \)` in predicate logics with equality, the following lemma is an easy consequence of the deduction theorem.

**Lemma 2.14.14**

(1) \( L + DE \vdash A \text{ iff } L \vdash DE \supset A \)

for a superintuitionistic logic with equality \( L \) and a formula \( A \); similarly for \( SE \).
(2) \[ L + \bigwedge_{i=1}^{N} CE_i \vdash A \text{ iff } L \vdash \Box \leq^k \bigwedge_{i=1}^{N} CE_i \supset A \text{ for some } k \in \omega, \] where \( L \) is an \( N \)-modal logic with equality.

For a 1-modal logic \( L \supseteq QT \) this can be simplified:

\[ L + CE \vdash A \text{ iff } L \vdash \Box^r CE \supset A \text{ for some } r \in \omega; \]

similarly, for \( L \supseteq QK4, QS4 \), cf. Section 1.1.

## 2.15 Propositional parts

Let us explicitly describe the construction of the propositional part \( L_\pi \) for a predicate logic \( L \) without equality, cf. [Ono, 1972/73] for the case of intermediate logics.

Since all the sets \( PL^n \) are countable, we can consider a bijection

\[ \pi_0 : \left( \bigcup_{n \geq 0} PL^n \right) \rightarrow PL^0. \]

Then for any formula \( A \), let \( \pi(A) \) be the result of replacing each atomic subformula of the form \( P(x) \) with \( \pi_0(P) \) and erasing all occurrences of quantifiers. For a set of formulas \( \Gamma \) let

\[ \pi(\Gamma) = \{ \pi(A) \mid A \in \Gamma \}. \]

Obviously \( \pi(A) \) is equivalent (in \( QH \) or in \( QK_N \)) to a substitution instance of \( A \) (cf. Lemma 2.6.15(1)(iv)). On the other hand, a propositional formula \( A \) is a substitution instance of \( \pi(A) \). Thus we obtain

**Proposition 2.15.1** \( L_\pi = K_N + \pi(L) \) or \( L_\pi = H + \pi(L) \) for a predicate logic \( L \) without equality (respectively, \( N \)-modal or superintuitionistic).

**Proposition 2.15.2**

(1) \( (QK_N)_\pi = K_N \);

(2) \( (QH)_\pi = H \).

**Proof** By induction over a proof of \( A \) in \( QK_N \) (or in \( QH \)) we show that \( \pi(A) \in K_N \) (resp. \( \pi(A) \in H \)). It is easily checked that \( \pi(A) \in K_N \) if \( A \) is a substitution instance of a \( QK_N \)-axiom (for example, \( \pi(A) = (B \supset B) \) for predicate axioms). If \( A \) is obtained by (MP) or (necessitation), then \( \pi(A) \) is also obtained by the same rule. If \( A = \forall x B \), then \( \pi(A) = \pi(B) \).

**Lemma 2.15.3** \( \pi(Sub(\Gamma)) = Sub(\pi(\Gamma)) \) for any \( \Gamma \subseteq IF \) (or \( \Gamma \subseteq MF_N \)), where \( Sub_\pi \) denotes closure under propositional substitutions (of the corresponding type).
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Proof Since every substitution is a composition of simple substitutions, we can consider only simple substitutions. It is easily proved that

$$\pi([C/P(x)]B) = [\pi(C)/\pi_0(P)]\pi(B)$$

(by induction on $B$). On the other hand, every propositional formula can be presented as $\pi(C)$ for some predicate formula $C$.

Proposition 2.15.4 $(L + \Gamma)_\pi = L_\pi + \pi(\Gamma)$ for a modal (or superintuitionistic) logic $L + \Gamma$.

In particular, $(QK_N + \Gamma)_\pi = K_N + \pi(\Gamma)$, $(QH + \Gamma)_\pi = H + \pi(\Gamma)$.

Proof [Modal case.] Let $A \in (L + \Gamma)$. By the deduction theorem,

$$\left( \bigwedge_{s=1}^k \square \leq k \forall A_s \supset A \right) \in L$$

for some formulas $A_s \in \text{Sub}(\Gamma)$. Then

$$\left( \bigwedge_{s=1}^k \square \leq k \pi(A_s) \supset \pi(A) \right) \in L_\pi,$$

while $\pi(A_s) \in \text{Sub}_\pi(\pi(\Gamma))$ (by Lemma 2.15.3). Thus $\pi(A) \in (L_\pi + \pi(\Gamma))$.

The converse inclusion is obvious by Proposition 2.15.1.

Definition 2.15.5 The quantified version of a modal (respectively, superintuitionistic) propositional logic $\Lambda$ is:

$$QA := QK_N + \Lambda \quad (\text{respectively, } QA := QH + \Lambda).$$

Lemma 2.15.6

1. For any propositional logic $\Lambda$ and a set of propositional formulas $\Gamma$,

$$Q(\Lambda + \Gamma) = QA + \Gamma.$$

2. For a propositional S4-logic $\Lambda$, $T(QA) \supset Q^T \Lambda$.

Proof

1. Consider the intuitionistic case. $Q(\Lambda + \Gamma)$ is the smallest s.p.l. containing $\Lambda + \Gamma$, while $QA + \Gamma$ is the smallest s.p.l. containing $QA \cup \Gamma$. These two logics coincide, since every s.p.l. containing $\Lambda$ also contains $QA$.

2. By 2.11.11, $Q^T \Lambda \vdash A$ implies $QS4 + (T\Lambda)^T \vdash A^T$, hence $QA \vdash A^T$. Thus $Q^T \Lambda \subseteq T(QA)$. 


Remark 2.15.7 We do not know if the equality $^T(Q\Lambda) = Q^T\Lambda$ holds for any propositional S4-logic. For example, this is unknown for $\Lambda = \text{S4.2Grz}$.

Definition 2.15.8 A predicate logic $L$ is called a predicate extension of a propositional logic $\Lambda$ if $L$ is called a conservative extension of $\Lambda$ (i.e. if $L_\pi = \Lambda$).

Proposition 2.15.9 For any modal or superintuitionistic propositional logic $\Lambda$, $Q\Lambda$ is a predicate extension of $\Lambda$.

Proof Follows readily from Proposition 2.15.4 since $\pi(\Lambda) \subseteq \Lambda$.

Obviously, $Q\Lambda$ is the weakest predicate extension of $\Lambda$.

To describe the greatest predicate extensions of propositional logics, we use the formula

$$AU_1 = \forall x \forall y (P(x) \supset P(y))$$

from Section 2.6; recall that

$$L + AU_1 = L + \exists x P(x) \supset \forall P(x)$$

for any predicate logic $L$ and

$$L + AU_1 = L + \forall x \forall y (x = y)$$

for any predicate logic $L$ with equality.

For a formula $A$ with equality, let $A_T$ be the result of replacing each occurrence of $(x = y)$ in $A$ with $\top$. Then $(A \equiv A_T) \in (QK^+_{N} + AU_1)$ (respectively, $QH^+ + AU_1$).

Lemma 2.15.10 Let $A$ be a predicate formula without equality. Then $A \in QK^+_{N} + AU_1 + \pi(A)$ (in the modal case) or $A \in QH + AU_1 + \pi(A)$ (in the intuitionistic case).

Proof First, given $\pi(A)$, we can restore all occurrences of quantifiers from $A$, because they are dummy in $\pi(A)$. Next, every occurrence of $\pi_0(P)$ in $\pi(A)$ coming from some $P(x)$ in $A$, can be replaced with $\forall P(x)$, which is equivalent to $P(x)$ in $QK^+_{N} + AU_1$.

Proposition 2.15.11 Let $L$ be a predicate logic without equality, modal or superintuitionistic, and let $\Lambda$ be a propositional logic of the corresponding kind. Then

$$L_\pi = \Lambda \iff Q\Lambda \subseteq L \subseteq Q\Lambda + AU_1.$$

Proof $(Q\Lambda + AU_1)_{\pi} = \Lambda$ by Proposition 2.15.4 since

$$\pi(AU_1') = (\pi_0(P) \supset \pi_0(P)) \in \Lambda.$$

On the other hand, if $L_\pi = \Lambda$ then $L \subseteq Q\Lambda + AU_1$. In fact, $A \in L$ implies $\pi(A) \in L_\pi = \Lambda$ and $A \in Q\Lambda + AU_1$ by Lemma 2.15.10.

\textit{Cf. [Ono, 1973].}
2.15. PROPOSITIONAL PARTS

Therefore $QA + AU_1$ is the greatest predicate extension of a propositional logic $\Lambda$.

Corollary 2.15.12 The greatest intermediate predicate extension of an intermediate propositional logic $\Lambda$ is

$$(QA + AU_1) \cap QCL = QA + AU_1 \lor q \lor \neg q.$$ 

Proof By Proposition 2.10.1 (2); recall that $QCL = QH + q \lor \neg q$ and $CD \in QH + AU_1 \lor q \lor \neg q$. 

Let us also mention the following result on the number of predicate extensions.

Theorem 2.15.13

1. Every nonclassical superintuitionistic propositional logic has uncountably many predicate extensions.

2. Every modal propositional logic which does not contain S5, has uncountably many predicate extensions.

Now let us give another description of the greatest predicate extensions.

Proposition 2.15.14

(I) Let $L$ be a predicate logic (of any kind). Then the following conditions are equivalent:

(a) $L + A = L + \pi(A)$ for any predicate formula $A$ without equality;

(b) $QH^{(=)} + AU_1 \subseteq L$ or $QK_N^{(=)} + AU_1 \subseteq L$
   (for the superintuitionistic or the modal case, respectively);

(c) $L = QA^{(=)} + AU_1$ for a propositional logic $\Lambda$ (superintuitionistic or modal, respectively).

Moreover, these conditions imply

(d) for any predicate formula $A$ (in the language of $L$) there exists a propositional formula $A'$ such that $L + A = L + A'$.

(II) If $L$ is a logic without equality, all conditions (a)-(d) are equivalent.

Proof

(I) $(a) \Rightarrow (b)$. $L + AU_1 = L$ since $\pi(AU_1) = (\pi_0(P) \cup \pi_0(P)) \in H$ (or $K_N$).

$(b) \Rightarrow (a)$. $\pi(A) \in L + A$ by Proposition 2.15.1.

On the other hand, $A \in QH + AU_1 + \pi(A)$ or $A \in QK_N + AU_1 + \pi(A)$, by Lemma 2.15.10.

\[31\text{[Suzuki, 1995].}\]
(a) ⇒ (d). If $L$ is a logic with equality, we take $A' = \pi(A\uparrow)$.

(b) ⇒ (c). We can replace any $A$ from $L$ with $A'$ because (b) implies (d).

(II) (d) ⇒ (a). Let $L + A = L + A'$ for a propositional formula $A'$. Then

$$A' \in (L + A)\pi = L\pi + \pi(A) \subseteq L + \pi(A).$$

On the other hand, $\pi(A) \in (L + A)$.

We say that $L$ is a predicate logic with degenerate predicates if $L$ satisfies the condition (I)(d) from Proposition 2.15.14. Thus, $\mathcal{QH} + AU_1$ and $\mathcal{QK}_N + AU_1$ are the weakest logics with degenerate predicates. On the other hand, there exist logics with equality incomparable with $\mathcal{QH}^= + AU_1$ and satisfying (I)(d), e.g.

$$L = \mathcal{QH}^= + \exists x \exists y (x \neq y \land \forall z (z = x \lor z = y)).$$

It is clear how to describe the propositional fragments for these logics with equality.

**Lemma 2.15.15** Let $L$ and $L'$ be predicate logics containing $AU_1$ such that $L$ is without equality and $L'$ is with equality. Then the following conditions are equivalent:

1. $L'\pi = L\pi$;
2. $(L')^0 = L$;
3. $L' = L^\pi$.

**Proof**

(1) ⇒ (2). By Proposition 2.15.14, (b) ⇒ (a).

(3) ⇒ (1). Let $A$ be a propositional formula, $A \in L^\pi$. Then by the deduction theorem, the following formula is a theorem of $L$:

$$B := \Box^{\leq k}(\forall x(x = x) \land \bigwedge_{s=1}^{k} \neg(x = y \supset ([x/z]A_s \supset [y/z]A_s))) \supset A$$

for some $k \geq 0$ and formulas $A_1, \ldots , A_k \in MF_N^\pi$. Then its substitution instance $B\uparrow$ belongs to $L$ and $\pi(B\uparrow) \in L\pi$. Since $\pi((\forall x(x = x)\uparrow) = \top$ and $\pi(([x/z]A_s)\uparrow) = \pi(([y/z]A_s)\uparrow)$, we obtain $\pi(A) \in L$, i.e. $A \in L\pi$.

(2) ⇒ (3). If $(L')^0 = (L')^0 = L$ then $L^\pi \subseteq L'$, and $L^\pi = L'$ — because $(A \equiv A\uparrow) \in L^\pi$ for any formula with equality $A$ (recall that $A\uparrow$ is a formula without equality).

---

The implication (3) ⇒ (2) is proved in Proposition 2.14.8 for all superintuitionistic logics and for 1-modal logics above $\mathcal{QK}_4$. Here we give a slightly simpler proof that fits for any modal predicate language.
Corollary 2.15.16 Let $L$ be a predicate logic with equality, $\Lambda$ a propositional logic. If
\[ QA^= \subseteq L \subseteq QA^+ AU_1 \]
then $L_\pi = \Lambda$.

**Proof**
\[ QA^= \subseteq L \subseteq QA^+ AU_1 \]
implies
\[ (QA^=)_\pi \subseteq L_\pi \subseteq (QA^+ AU_1)_\pi. \]
Next,
\[ (QA^+ AU_1)_\pi = (QA + AU_1)^\pi = (QA + AU_1)_\pi, \]
by Lemma 2.15.15, and $(QA + AU_1)_\pi = \Lambda$, by Proposition 2.15.11.

Corollary 2.15.17 Let $L$ be a predicate logic without equality. Then
\begin{enumerate}
  \item $(L^=)_\pi = (L^{=d})_\pi = (L^{=s})_\pi = L_\pi$ for a superintuitionistic $L$;
  \item $(L^=)_\pi = (L^{=c})_\pi = L_\pi$ for modal $L$.
\end{enumerate}

**Proof** (Intuitionistic case.) Let $L_\pi = \Lambda, L \subseteq QA + AU_1$. Then
\[ QA^= \subseteq L^= \subseteq L^{=s} \subseteq L^{=d} \subseteq QA^+ AU_1, \]
and we can apply Corollary 2.15.16. The modal case is analogous.

Proposition 2.15.14 and Lemma 2.15.15 show that there exists a natural bijection between intermediate propositional logics and extensions of the logics $QH^{(=)} + AU_1$ (with or without equality), and similarly for the modal case. The logic $QA^+ + AU_1$ is a maximal predicate extension (with equality) of a propositional logic $\Lambda$. Nevertheless, for any propositional logic $\Lambda$ the greatest predicate extension with equality does not exist. Indeed, later on we will show that $L_\pi = \Lambda$ for the logic
\[ L = QA^+ + \neg \forall x \forall y(x = y), \]
which is obviously incompatible with $QA^+ AU_1$.

### 2.16 Semantics from an abstract viewpoint

Consider a language with the set of well-formed formulas $\Phi$, and a set of its subsets $\Lambda \subseteq 2^\Phi$ closed under arbitrary intersections. Elements of $\Lambda$ are called $\Lambda$-logics. We define a semantics for the set $\Lambda$ as a quadruple $S = (\Phi, \Lambda, U, \vdash)$, in which $U$ is a class (whose elements are called $U$-frames), and $\vdash$ is a binary relation between $U$ and $\Phi$ (called the validity relation) such that the set
\[ L_\Phi(F) = \{ A \in \Phi \mid F \vdash A \} \]
is a $\Lambda$-logic for any $F \in \mathcal{U}$. $L_\Sigma(F)$ is called the $\Lambda$-logic of the frame $F$ (in $\Sigma$). Sometimes we write $F \in \Sigma$ instead of $F \in \mathcal{U}$.

Since $\Lambda$ is intersection closed, we obtain that for any class $\Sigma \subseteq \mathcal{U}$, the set $L_\Sigma(\Sigma) = \bigcap \{ L_\Sigma(F) \mid F \in \Sigma \}$ is also a $\Lambda$-logic; it is called the $\Lambda$-logic of $\Sigma$. Logics of this kind are called $\Sigma$-complete, and logics of the form $L_\Sigma(F)$ are called simply $\Sigma$-complete. As stated in [Ono, 1973, Theorem 2.2], completeness does not always imply simple completeness.

A semantics $\Sigma$ gives rise to the logical consequence relation (between $\Gamma \subseteq \Phi$ and $A \in \Phi$):

$$\Gamma \models_\Sigma A \text{ iff } \forall F \in \mathcal{U} (\Gamma \subseteq L_\Sigma(F) \Rightarrow A \in L_\Sigma(F)).$$

We say that $F$ is a $\Gamma$-frame (in $\Sigma$) if $\Gamma \subseteq L_\Sigma(F)$. Thus $\Gamma \models_\Sigma A$ means that $A$ is valid in every $\Gamma$-frame. If $F$ is a $\Gamma$-frame such that $F \not\models A$, we say that $F$ separates $A$ from a set $\Gamma$.

One can recognise a particular case of Galois correspondence here. This correspondence is derived from the relation $\models$ in the standard way; cf. [Chapter 5, Theorem 19] [Birkhoff, 1979]. So

$$C_\Sigma(\Gamma) = \{ A \in \Phi \mid \Gamma \models_\Sigma A \}$$

is a closure operation on $2^\Phi$, and the closed sets of formulas are just the $\Sigma$-complete logics.

Hence we have

**Lemma 2.16.1** $C_\Sigma(\Gamma)$ is the smallest $\Sigma$-complete logic containing $\Gamma$. $C_\Sigma(\Gamma)$ is called the $\Sigma$-completion of $\Gamma$.

Here is a simple criterion of completeness:

**Lemma 2.16.2** A logic $L$ is complete in a semantics $\Sigma$ iff every formula $A \in (\Phi - L)$ is separated from $L$ by a frame in $\Sigma$.

**Proof** In fact, by 2.16.1, $L$ is complete iff $C_S(L) = L$. 

We can compare different semantics for the same set of logics. A semantics $S_1$ is reducible to $S_2$ (notation: $S_1 \preceq S_2$) iff $C_{S_2}(L) \subseteq C_{S_1}(L)$ for any $L \in \Lambda$.

**Lemma 2.16.3**

1. $S_1 \preceq S_2$ iff every $S_1$-complete logic is $S_2$-complete.

2. $S_1 \preceq S_2$ iff for any $S_1$-frame $F$, $L_{S_1}(F)$ is $S_2$-complete.

**Proof**

1. Suppose $S_1 \preceq S_2$. Then for any logic $L$,

$$L \subseteq C_{S_2}(L) \subseteq C_{S_1}(L).$$

If $L$ is $S_1$-complete then $L = C_{S_1}(L)$ (by Lemma 2.16.1), and hence $L = C_{S_2}(L)$ which implies $S_2$-completeness of $\Lambda$ (again by Lemma 2.16.1).

Conversely, assume that $S_1$-completeness implies $S_2$-completeness. Then $C_{S_1}(L)$ is $S_2$-complete, and thus $C_{S_2}(L) \subseteq C_{S_1}(L)$ by Lemma 2.16.1.
2.16. SEMANTICS FROM AN ABSTRACT VIEWPOINT

(2) 'Only if' is an immediate consequence of (i) and Lemma 2.16.1.
To prove 'if' consider an arbitrary $S_1$-complete logic

$$L = L_{S_1}(\Sigma) = \bigcap \{L_{S_1}(F) \mid F \in \Sigma\}.$$ 

If each $L_{S_1}(F)$ is $S_2$-complete, we have that $L_{S_1}(F) = L_{S_2}(\Psi_F)$ for some class $\Psi_F$. Then

$$L = L_{S_1}(\Sigma) = \bigcap \{L_{S_2}(\Psi_F) \mid F \in \Sigma\} = L_{S_2}(\bigcup \{\Psi_F \mid F \in \Sigma\}),$$

showing that $\Lambda$ is $S_2$-complete.

Definition 2.16.4 Semantics $S_1$ and $S_2$ are called equivalent ($S_1 \simeq S_2$) if $S_1 \preceq S_2$ and $S_2 \preceq S_1$; we say that $S_1$ is weaker than $S_2$ (notation: $S_1 \prec S_2$) if $S_1 \preceq S_2$ but not $S_2 \preceq S_1$.

Lemma 2.16.3 readily implies:

Lemma 2.16.5 $S_1 \simeq S_2$ iff $S_1$-completeness is equivalent to $S_2$-completeness.

Definition 2.16.6 A semantics $S_1$ is simply reducible to $S_2$ (notation: $S_1 \subseteq S_2$) if simple $S_1$-completeness implies simple $S_2$-completeness, i.e.

$$\forall F_1 \in S_1 \exists F_2 \in S_2 (L_{S_1}(F_1) = L_{S_2}(F_2)).$$

Semantics $S_1$ and $S_2$ are called simply equivalent (notation: $S_1 \equiv S_2$) if $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$.

We do not usually distinguish between simply equivalent semantics; from this viewpoint a semantics $S = (\Phi, \Lambda, \mathcal{U}, \models)$ can be identified with the corresponding set of simply complete logics $\{L_S(F) \mid F \in \mathcal{U}\}$.

Lemma 2.16.7 $S_1 \subseteq S_2$ implies $S_1 \preceq S_2$.

Proof Obvious, since complete logics are intersections of simply complete ones.

Example 2.16.8 For any set of logics $\Lambda$ there exists a 'trivial semantics' $T$, in which $\mathcal{U} = \Lambda$ (i.e. 'frames' are just logics) and $L \models A$ iff $A \in L$, i.e. $L_T(L) = L$.

Definition 2.16.9 We say that a semantics $S$ has the collection property (CP) if every $S$-complete consistent logic is simply $S$-complete.

By Lemma 2.16.7, for semantics with the (CP), reducibility is equivalent to simple reducibility, so their equivalence implies simple equivalence. But in general equivalent semantics may be not simply equivalent; some counterexamples will be given later on.
Example 2.16.10 In modal propositional logic we have semantics with the (CP):

\[ \text{finite} < \text{Kripke} < \text{topological} < \text{algebraic} \cong \text{trivial}. \]

'Finite semantics' consists of all finite Kripke frames, with the usual definition of validity. So in this semantics simple completeness means tabularity and completeness means the f.m.p.

Algebraic semantics is simply equivalent to trivial, since every logic is algebraically complete, by the Lindenbaum theorem.

For superintuitionistic propositional logics we have the following diagram:

\[ \text{finite} < \text{Kripke} < \text{topological} \preceq \text{algebraic} \cong \text{trivial}. \]

The question, whether topological semantics is trivial, is Kuznetsov’s problem mentioned in Section 1.17.

Recall that in this book we consider four types of predicate logics — modal or intuitionistic, with or without equality. So we alter the notation \( L_S(F) \) respectively. For example, if \( F \) is a frame in a semantics \( S \) for \( N \)-modal predicate logics with equality, for \( \{ A \in MF_N \mid F \models A \} \) we use the notation \( ML^=(F) \) rather than \( L_S(F) \); \( ML(F) \) refers to semantics without equality. Similarly in the intuitionistic case we use the notations \( IL^=(F), IL(F) \).

Every semantics \( S \) for logics with equality generates a semantics \( S^o \) for the same kind of logics without equality, with the same frames and validity. So \( L_S(F) = L_S(F)^o \). Loosely speaking, we call \( S^o \)-complete logics '\( S \)-complete'.

Remark 2.16.11 However it may happen that there exists a semantics \( S' \), with the same frames as \( S \) (and \( S^o \)), but with a slightly different notion of validity. In these cases we define \( ML(F) \) (or \( IL(F) \)) as \( L_{S'}(F) \) and then prove that \( S' = S^o \). We will encounter such a situation in Chapter 5.

Definition 2.16.12 Let \( S \) be a semantics for \( N \)-m.p.l. We say that \( S \) admits equality if there exists a semantics \( S_1 \) for \( N \)-m.p.l.= such that \( S \subseteq S_1 \), i.e.

\[ \forall F \in S \exists F_1 \in S_1 \ L_S(F) = L_{S_1}(F_1)^o. \]

Proposition 2.16.13 If \( S \) admits equality and an \( N \)-m.p.l. \( L \) is \( S \)-complete, then \( L^= \) is conservative over \( L \).

Proof Almost obvious. Suppose \( L \) is \( N \)-modal, \( L = ML(C) \) for a class of \( S \)-frames \( C \) and \( ML(F) = ML^=(F_1)^o \) for any \( F \in C \). Then

\[ L = \bigcap_{F \in C} ML(F) = \bigcap_{F \in C} ML^=(F_1)^o. \]

Put \( C_1 := \{ F_1 \mid F \in C \} \), then for any \( A \in MF_N \)

\[ A \in L \iff \forall F \in C \ F_1 \models A \iff A \in ML^=(C_1). \]
Thus $\mathbf{ML}^\omega(C_1)^\circ = L$, i.e. $\mathbf{ML}^\omega(C_1)$ is conservative over $L = \mathbf{ML}(C)$. Since $\mathbf{ML}^\omega(C_1)$ is a logic with equality containing $L$, it follows that $L^\omega \subseteq \mathbf{ML}^\omega(C_1)$ and thus $L^\omega$ is also conservative over $L$.

Now let us prove a simple result on correlation between completeness of $L$ and $L^\omega$.

**Proposition 2.16.14** Let $L$ be an m.p.l. or an s.p.l., $S$ a semantics for the corresponding logics with equality. If $L^\omega$ is conservative over $L$ and $S$-complete, then $L$ is also $S$-complete.

**Proof** Consider the modal case only. Suppose $L^\omega = \mathbf{ML}^\omega(C)$ for a class of $S$-frames $C$. Then by conservativity, $L = (L^\omega)^\circ = \mathbf{ML}^\omega(C)^\circ$, and obviously, $\mathbf{ML}^\omega(C)^\circ = \mathbf{ML}(C)$. Thus $L$ is $S$-complete.

**Corollary 2.16.15** If $L$ is an s.p.l. or a conically expressive m.p.l. and $L^\omega$ is complete in a semantics $S$, then $L$ is also $S$-complete.

**Proof** By 2.16.14 and 2.14.8.

Finally let us show the existence of modal counterparts using completeness.

**Definition 2.16.16** Let $M$ be a semantics for m.p.l.(=) extending $\mathbf{QS4}(=)$. Then we define its intuitionistic version, the semantics $\mathcal{M}$ for s.p.l.(=), with the same frames as $M$ such that for any $M$-frame $F$,

$$L_{\mathcal{M}}(F) := T_{\mathcal{M}}(F).$$

**Proposition 2.16.17** Every $\mathcal{M}$-complete s.p.l.(=) has a modal counterpart:

$$\mathbf{QH}(=) + \Gamma = T(\mathbf{QS4}(=) + \Gamma^T).$$

**Proof** The inclusion $\subseteq$ is proved in Lemma 2.11.11. For the converse suppose $\mathbf{QH} + \Gamma \not\vdash A$. By assumption $\mathbf{QH} + \Gamma$ is $\mathcal{M}$-complete, so there exists an $M$-frame $F$ such that $\Gamma \subseteq L_{\mathcal{M}}(F)$, but $A \not\in L_{\mathcal{M}}(F)$. By Definition 2.16.16, $L_{\mathcal{M}}(F) = T_{\mathcal{M}}(F)$, hence $\Gamma^T \subseteq L_{\mathcal{M}}(F)$, $A^T \not\in L_{\mathcal{M}}(F)$. Since $M$ is a semantics for modal logics above $\mathbf{QS4}$, we have $L_{\mathcal{M}}(F) \supseteq \mathbf{QS4} + \Gamma^T$. Consequently $\mathbf{QS4} + \Gamma^T \not\vdash A$, i.e. $T(\mathbf{QS4} + \Gamma^T) \not\vdash A$.

As we shall see in Volume 2, the smallest modal counterpart of an $\mathcal{M}$-complete logic may be $\mathcal{M}$-incomplete. But completeness always transfers in the other direction:

**Proposition 2.16.18** If $L \supseteq \mathbf{QS4}(=)$ is $\mathcal{M}$-complete, then $T_L$ is $\mathcal{M}$-complete.

**Proof** Suppose $L = L_{\mathcal{M}}(C)$ for a class of $\mathcal{M}$-frames $C$. Then

$$L_{\mathcal{M}}(C) = \bigcap_{F \in C} L_{\mathcal{M}}(F) = \bigcap_{F \in C} T_{\mathcal{M}}(F) = \{ A \mid \forall F \in C \ A^T \in L_{\mathcal{M}}(F) \} = T_L(C),$$

so $T_L$ is $\mathcal{M}$-complete.
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Part II

Semantics
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Introduction: What is semantics?

In Chapter 2 we proposed a general approach to semantics, which is very formal and does not help understand the meaning of language expressions.

For example, this approach allows for a degenerate semantics, where 'frames' are just logics and $L \models A$ iff $L \vdash A$. In this case the completeness theorem is a triviality, but of course such a semantics does not explain anything.\(^1\)

To describe more plausible semantics we need to define notions of a model and of the truth in a model that could tell us 'how the logic works'. This is not a serious problem for classical first-order logic: in this case model theory can be developed within the standard semantics based on the well-known Tarski truth definition. Due to Gödel's completeness theorem (GCT), the standard semantics works properly, and thus alternative types of semantics (such as sheaves, forcing, polyadic algebras) are of less importance in the classical case.

The situation in nonclassical first-order logic is quite different. GCT has no direct analogues, and incompleteness phenomena enable us to consider various semantics, without any obvious preference between them. Nonclassical model theory is still rather miscellaneous, and this book is aimed at organising some part of it.

The book does not cover all the semantics in equal proportion. So we begin this part with a brief survey of important results in the area and some references for further reading.

Gödel's completeness theorem: discussion

First let us explain why GCT is not always transferred to nonclassical logics. Actually there exist two forms of Gödel's theorem:

(GCT1) A formula $A$ is a theorem of classical predicate calculus iff $A$ is valid in any domain.

(GCT2) Every consistent classical theory has a model.

These two statements are more or less equivalent: (GCT1) is equivalent to (GCT2) for finitely axiomatisable theories.

However in the nonclassical area it is essential to distinguish between logics and theories. A theory is an extension of some basic calculus by additional axioms (and perhaps inference rules) postulating specific properties of objects being studied; a typical example is Heyting arithmetic, which formalises the intuitionistic viewpoint on natural numbers. A logic is a theory whose theorems may be considered as 'logical laws' common for a certain class of theories and not depending on specific 'application domains'. We can treat logical laws as

\(^1\)Of course in general we may have a new logic and several syntactical translations into well-known logics. Such translations can be viewed as semantics and can be very illuminating. See [Gabbay, 1996, Chapter 1].
schemata for producing theorems, and define a logic just as a substitution-
closed theory.\textsuperscript{2} So we can say that (GCT2) is a property of classical theories,
and (GCT1) is a property of classical logic.

Some results similar to (GCT2) can be proved for nonclassical theories. In
fact, it is well known that every consistent first-order normal modal theory \( S \)
is satisfied in some Kripke model, i.e. there exists a Kripke model \( M \) and a
possible world \( w \in M \) such that \( (M, w) \models A \) for any \( A \in S \). Moreover, \( M \) can
be chosen uniformly for all consistent theories [Gabbay, 1976]. Analogous claims
are true for superintuitionistic theories considered as pairs of sets of formulas,
cf. [Dragalin, 1988].

However finding nonclassical analogues of (GCT1) is more problematic. These
are completeness theorems of the following form:

\[(GCT1') \text{ a formula } A \text{ is a theorem of a logic } L \iff A \text{ is } L\text{-valid.}\]

The main problem is in finding 'reasonable' notions of validity, for which (GCT1')
may be true. In classical logic this is validity in a domain. In a general case,
because we require logics to be substitution-closed, we also need 'substitution-

closed' notions of validity. So for a nonclassical logic \( L \), analogues of domains
are frames (or model structures), from which we can obtain models of theories
based on \( A \) if we specify interpretations of basic predicates; a formula is valid
in a frame iff it is true in every model over this frame.

It is usually required that the set of all formulas true in a model is a theory,
and thus the set \( L(F) \) of all formulas valid in a frame \( F \) is also a theory.\textsuperscript{3}
The difference between models and frames is the following substitution property
implying that \( L(F) \) is a logic:

\[(SP) \text{ The set of all formulas valid in a frame } F \text{ is substitution closed.}\]

Generally speaking, there may be many kinds of 'frames' and 'validity'. We
say that a class of frames \( C \) generates a semantics

\[S(C) = \{L(F) \mid F \in C\}.\]

A logic \( L \) is called complete in \( S(C) \) iff for every formula \( A \not\in L \) there exists
a frame \( F \in C \), such that formulas from \( L \) are valid in \( F \), whereas \( A \) is not.
Equivalently, a logic is complete iff it can be presented as an intersection of
logics of the form \( L(F) \) for some frames \( F \).

For example, in the standard classical semantics 'frames' are just sets, and
the notion of validity is well known. So (GCT1) means that the classical predic-
icate logic is complete in this semantics.

\textsuperscript{2}Of course this definition is rather conventional, and there exist examples of 'logics' which
are not substitution-closed.

\textsuperscript{3}This set depends also on the language, where formulas are constructed.
Examples of incompleteness in first-order logic

In Chapter 1 we gave a picture of semantics in nonclassical propositional logic and pointed out some rather strong completeness results. But in first-order logic the situation becomes worse: there are very few completeness theorems known, and incompleteness is very frequent.

Incompleteness already appears for logics extending the classical predicate logic QCL. In fact, consider any formula $A$ valid in all finite domains, but refutable in some infinite domain (and thus refutable in any infinite domain, by the Löwenheim–Skolem theorem). The logic QCL+$A$ is incomplete, because the set of all finitely valid formulas is not recursively axiomatisable [Trachtenbrot, 1950].

This argument seems to be just a trick with definitions. But in nonclassical logic there exist more natural examples of incompleteness.

The first example of this kind was discovered by H. Ono [Ono, 1973], who proved that the intermediate logic of the strong Markov principle

$$QHE = QH + \neg\neg\exists xP(x) \supset \exists x\neg\neg P(x)$$

is incomplete in the standard Kripke semantics ($K$).

Recall that $K$ is generated by predicate Kripke frames; such a frame is a triple $F = (W, R, D)$, in which $(W, R)$ is a propositional Kripke frame; $D$ is a system of ‘expanding individual domains’, i.e. a family of non-empty sets indexed by possible worlds $(D_w)_{w \in W}$, such that

$$\forall u, v (uRv \Rightarrow D_u \subseteq D_v).$$

The logics QS4 and QH are known to be $K$-complete. But their minimal equality extensions ($QS4^\equiv, QH^\equiv$) are incomplete if the symbol ‘$=$’ is interpreted in every domain just as the identity. This is due to the formulas

$$DE = \forall x\forall y (x = y \lor \neg(x = y))$$

(decidable equality principle)

and

$$CE = \forall x\forall y (\circ(x = y) \supset x = y)$$

(closed equality principle)

which are valid in all Kripke frames, but nonprovable respectively in $QH^\equiv$ and in $QS4^\equiv$.

But we can also interpret equality in a world $w$ in another natural way, as an equivalence relation in $D_w$. Then we come to the semantics $KE$ of Kripke frames with equality (KFEs), in which $QS4^\equiv, QH^\equiv$ become complete; so $KE$ is stronger than $K$. The latter observation and the definition of $KE$ for the intuitionistic case first appeared in [Dragalin, 1973] and then in [Dragalin, 1988].

Note that in the classical case these two approaches are equivalent: every ‘non-normal’ model, in which equality is an arbitrary equivalence relation can be ‘normalised’ by identifying equal elements, so that its elementary theory does not change.

For a Kripke frame with equality a similar construction is possible: one can factorise every individual domain through the corresponding equivalence relation.
relation. Thus we obtain a Kripke sheaf; it can be defined as a propositional Kripke frame \((W, R)\) together with a system of individual domains \((D_w)_{w \in W}\) and transition maps \(\rho_{uv} : D_u \rightarrow D_v\) for every \(R\)-related pair \(u, v\), satisfying the natural transitivity condition:

\[ uRvRw \Rightarrow \rho_{uw} = \rho_{uv} \circ \rho_{vw} \]

and such that \(\rho_{uu}\) is the identity map for every \(R\)-reflexive \(u\). So every KFE corresponds to a Kripke sheaf with the same modal logic. Note that predicate Kripke frames correspond to those Kripke sheaves in which all transition maps are inclusions.

Although the semantics \(K&\) was introduced for dealing with equality, it happens to be stronger than \(K\) also for logics without equality. This can be seen again by analysing Ono's counterexample QHE.

### Algebraic and Kripke-type semantics

The semantics \(K&\) also is inadequate in many cases, in particular for quantified versions of all intermediate propositional logics of finite depths. \(K\)-incompleteness of these logics was proved by H. Ono in [Ono, 1973], and the proof is easily transferred to \(K&\). Another counterexample (found independently by S. Ghilardi and V. Shehtman & D. Skvortsov) is presented by the intermediate logic of 'the weak excluded middle' with constant domains:

\[ \text{QHJD} = \text{QH} + \neg p \lor \neg \neg p + \forall x (P(x) \lor q) \supset (\forall x P(x) \lor q). \]

So one can try to generalise \(K&\) in a 'reasonable way'. Generalisations can be done at least in two directions.

The first direction leads to the algebraic semantics \(A&\). In the intuitionistic case a frame in \(A&\) is a Heyting-valued set [Fourman and Scott, 1979]. This is a set of 'individuals', in which every individual has a 'measure of existence' (an element of some Heyting algebra) and every pair of individuals has a 'measure of equality'. For the modal case Heyting algebras are replaced by modal algebras. The neighbourhood (or topological) semantics \(TE\) is a particular case of \(A&\), involving only algebras of topological spaces (or algebras of propositional neighbourhood frames for the modal case).

Both semantics \(A&\), \(TE\) are not much investigated and seem to be rather strong. We do not know if there exist \(A&\)-incomplete logics. On the other hand, we do not know simple and natural examples of logics that are \(A&\)-complete, but \(K&\)-incomplete.

Another direction leads from \(K&\) to the semantics of Kripke quasi-sheaves \((KQ)\), and further to the semantics of Kripke bundles \((KB)\). In Kripke bundles transition maps \(\rho_{uv} : D_u \rightarrow D_v\) are replaced with transition relations \(\rho_{uv} \subseteq D_u \times D_v\) collected in the unified accessibility (or inheritance, or counterpart) relation between individuals. Every individual has an inheritor (not necessarily unique) in any accessible world.

The idea of counterparts first appeared in [Lewis, 1968], and in a formal setting — in [Shehtman and Skvortsov, 1990]. A Kripke bundle can be defined
as a \( p \)-morphism from a (propositional) frame of individuals \((D^+, \rho)\) onto a frame of possible worlds \((W, R)\). Then Kripke sheaves correspond exactly to etale maps, similarly to the well-known fact in sheaf theory [Godement, 1958].

In Kripke bundles an individual may have several inheritors even in its own possible world. Kripke quasi-sheaves are a subclass of Kripke bundles, in which this is not allowed. Further generalisations of \( KB \) are the functor semantics \((\mathcal{FS})\), in which ‘frames’ are set-valued functors (or presheaves over categories); the metaframe semantics, and finally the hyperdoctrine semantics generalising both metaframe and algebraic semantics. In hyperdoctrine semantics every logic is complete, but this semantics is very abstract, and thus not suitable for large-scale applications.

There are several other options, which will be considered later in this book (Volume 3), such as Kripke-Joyal-Reyes semantics [Goldblatt, 1984; Makkai and Reyes, 1995] or ‘abstract realisability’ [Dragalin, 1988]. Also we will discuss the approaches where the basic language is modified, e.g. by adding the existence predicate [Fine, 1978], various kinds of quantifiers [Garson, 1978] or nonstandard substitutions [Ghilardi and Meloni, 1988]. This may help the reader to find his way through the landscape of first-order logic.

Substitution property: discussion

General Kripke-type semantics have some peculiarities: it turns out that in Kripke bundles (and even in Kripke quasi-sheaves) the above mentioned substitution property fails. In fact, a very simple counterexample shows that the (intuitionistic) validity of \( p \lor \neg p \) does not necessarily imply the validity of \( P(x) \lor \neg P(x) \) (cf. Chapter 5). To avoid this, the ‘straightforward’ notion of validity has to be changed. So we introduce strong validity in a frame: a formula is said to be strongly valid iff all its substitution instances are valid (in the original sense). Then the set \( L(F) \) of all formulas strongly valid in a frame \( F \) is a logic (called the logic of \( F \)).

One may argue that after such a modification the notion of frame becomes almost useless and can be replaced by the notion of model. For, in the same manner we can define a ‘logic’ \( L(M) \) of an arbitrary Kripke model \( M \) as the set of all the formulas ‘strongly verifiable’ in \( M \), i.e. \( L(M) \) contains a formula \( A \) iff all substitution instances of \( A \) are true in \( M \). Then we get a first-order version of general propositional frames mentioned above. In this semantics \((GCT2)\) follows from \((GCT1)\), and thus every modal or superintuitionistic first-order logic is complete.

None the less, there is some advantage in dealing with Kripke bundles (or metaframes) rather than Kripke models. For, let us recall the two kinds of substitutions considered in Chapter 2: a strict substitution does not add new parameters to atomic formulas; a shift adds a certain fixed list of new parameters to every atomic formula. Recall that a simple substitution instance \( A^k \) of a formula \( A \) is obtained by replacing every atom \( P^n_i(x_1, \ldots, x_n) \) by \( P^{i+k}_i(x_1, \ldots, x_n, y_1, \ldots, y_k) \), where \( y_1, \ldots, y_k \) are variables not occurring in \( A \). As we know, every substitution is a composition of some strict and some
simple substitution and a variable renaming. One can prove that the set $L^{-}(F)$ of all formulas valid in a Kripke bundle $F$ is closed under strict substitutions and variable renaming. Therefore the logic of $F$ can be derived from $L^{-}(F)$ as follows:

$$L(F) = \{A \mid \forall k \geq 0 \ A^k \in L^{-}(F)\}.$$ 

So to check strong validity of a formula $A$ there is no need to verify all its substitution instances, but only the $A^k$ are sufficient. In other words, strong validity is nothing but validity with arbitrarily many extra parameters.

Unlike that, the set of formulas true in a Kripke model $M$ is usually not closed under strict substitutions. So in general there is no way to describe the logic of $M$ other than checking the truth of all possible substitution instances of formulas. This may be very difficult even in the propositional case; in algebraic terms, this means describing equations in a subalgebra of the modal algebra of a Kripke frame with a given set of generators. So the semantics of Kripke bundles is in some sense 'more constructive' than the semantics of Kripke models (not frames!). Anyway the algebraic semantics $\mathcal{AE}$ is the strongest semantics with (SP), that we know of. The Kripke-type semantics stronger than the Kripke bundle semantics $\mathcal{KB}$ (such as the functor semantics $\mathcal{FS}$ or the metaframe semantics $\mathcal{MF}$) do not have the (SP) either, but here again only simple substitutions are essential for strong validity.
Chapter 3

Kripke semantics

3.1 Preliminary discussion

There is no remembrance in former things; neither shall there be any remembrance of things that are to come with those that shall come after.
(Ecclesiastes, 1.11.)

In propositional modal logic Kripke semantics is widely used and is very helpful. As pointed out in Chapter 1, natural logics turn out to be Kripke-complete, and moreover, many of them have finite model property. This makes Kripke semantics an efficient model-theoretic instrument in the propositional case.

In the first-order case one can try to generalise the propositional Kripke semantics in a straightforward way.

Consider the first-order language $\mathcal{L}_1$ with a single modal operator $\square$ (Section 2.1). Let $(W, R)$ be a 1-modal propositional Kripke frame (Definition 1.3.1). We can define a ‘first-order Kripke model’ over $(W, R)$ as a collection of classical models parametrised by possible worlds:

$$M = (M_u)_{u \in W},$$

where every $M_u$ is a classical $\mathcal{L}$-structure, i.e. at the world $u$ the basic predicates are interpreted as in $M_u$. This is not yet enough for a formal definition, because we have to answer the following two questions:

- What are the individuals in $M$?
- How are the quantifiers interpreted?

The simplest way is to assume that every individual is an element of some $M_u$; more exactly, if $D_u$ is the domain of $M_u$, then $M$ has the set of individuals

$$D^+ = \bigcup_{u \in W} D_u.$$
We can say that $D_u$ consists of the individuals existing in the world $u$. Thus some individuals may exist in one world, but may not exist in another world. A more realistic example is given by a moving lift. Here $W$ is the set of moments of time, $R$ is the earlier–later relation, $D_u$ is the set of people inside the lift at the moment $u$.

From this viewpoint, it is natural to quantify only over existing individuals; thus $\forall x \varphi(x)$ must be true at the world $u$ iff $\varphi(a)$ is true for every $a \in D_u$.

In our example, if $P(x)$ is interpreted as

$x$ is a child,

then $u \vDash \forall x P(x)$ means that at the moment $u$ (it is true that)

only children are in the lift.

Eventually, we can define the forcing relation $u \vDash \varphi$ between a world $u$ and a $D_u$-sentence (cf. 2.2) $\varphi$ by induction on the complexity of $\varphi$, so that

$$\begin{align*}
  u \vDash P_k^n(a_1, \ldots, a_n) & \text{ iff } M_u \vDash P_k^n(a_1, \ldots, a_n) \text{ (classically)} \\
  u \vDash \forall x \varphi & \text{ iff } \forall a \in D_u \ u \vDash [a/x] \varphi \\
  u \vDash \exists x \varphi & \text{ iff } \exists a \in D_u \ u \vDash [a/x] \varphi \\
  u \vDash \Box \varphi & \text{ iff } \forall v \ (uRv \& \varphi \text{ is a } D_v\text{-sentence} \Rightarrow v \vDash \varphi)
\end{align*}$$

and with the standard clauses for the classical propositional connectives.

Returning again to our example, for a certain individual $a$ (say, Robert Smith) $u \vDash \Box P(a)$ means: at the moment $u$ it is true that

always in the future Robert Smith will use the lift only while he is a child.

This is an essential point: we cannot state $u \vDash P(a)$ for an individual $a$ that is not present in $D_u$; and thus to check $u \vDash \Box P(a)$, we have to consider only those $v$ in $R(u)$, which have $a$ in their domain.

Once the forcing relation is defined, we can say that a formula $\varphi$ is true in $M$ if $u \vDash \forall \varphi$ for any $u \in W$. If $FV(\varphi) = \{x_1, \ldots, x_n\}$, the latter is equivalent to

$$u \vDash [a_1, \ldots, a_n/x_1, \ldots, x_n] \varphi \text{ for any } a_1, \ldots, a_n \in D_u.$$ 

Now what formulas are true in every Kripke model? This set can be axiomatised, but unfortunately, it may not be a modal predicate logic in the sense of Chapter 2. To see what happens, consider the formula

$$\alpha = \Box (p \land q) \supset \Box p$$

which is a theorem of $K$ (propositional) and is obviously true in all models defined above. However take the substitution instance of $\alpha$:

$$\beta = \Box (P(x) \land P(y)) \supset \Box P(x).$$

This formula is not always true. For, consider the model with two possible worlds $u, v$ and two individuals $a, b$, such that
$R = \{(u, u), (u, v)\}, D_u = \{a, b\}, D_v = \{a\},$ $u \models P(a) \land P(b), v \not\models P(a),$ see Fig. 3.1.

Then we have $u \models P(a) \land P(b),$ since $P(a) \land P(b)$ is not a $D_v$-sentence, but $u \not\models \Box P(a)$.

Here is a more natural analogue of this example: if

John and Mary always talk when they are in the lift together,

it may be not the case that

John always talks in the lift.

Now there are several options to choose:

A. If we need semantics for the logics described in the previous chapter, we can try to amend the above definition.

B. Alternatively, we can change (actually, extend) the notion of a logic and try to axiomatise the 'logic' which is complete with respect to the above interpretation.

C. We can accept other definitions of semantics, using different kinds of individuals and different interpretations of quantifiers.

In this Volume we accept OPTION A and keep the same notion of logic. There is not so much to change in the original definition: it is enough to assume that
'individuals are immortal', which in precise terms, means the expanding domain condition:

\[(ED) \quad uRv \Rightarrow D_u \subseteq D_v.\]

Of course this requirement contradicts the viewpoint expressed in the epigraph to this section.\(^1\)

But it saves the logic, and as we will see later on, this semantics is sound and complete for a few well-known logics, such as QK, QS4, etc.

Furthermore, (ED) is essential in semantics of intuitionistic logic. In this case \((W, R)\) is an S4-frame, and we would like to keep the intuitionistic truth-preservation principle (TP) from Lemma 1.4.4 for atomic \(D_u\)-sentences:

\[u \models P^a_k(a_1, \ldots, a_n) \& uRv \& a_1, \ldots, a_n \in D_v \Rightarrow v \models P^a_k(a_1, \ldots, a_n).\]

The inductive definition of forcing from Section 1.3 can be extended to the first-order case so that

\[u \models \exists x \phi \text{ iff } \exists a \in D_u \ u \models [a/x] \phi.\]

Now, without (ED), we cannot guarantee the truth-preservation for all \(D_u\)-sentences. For, take a model based on a two-element chain:

\[W = \{u, v\}, \ R = \{(u, u), (u, v), (v, v)\}, \ D_u = \{a\}, \ D_v = \{b\},\]

\[u \models P(a), \ v \not\models P(b).\]

Figure 3.2.

In this case the truth-preservation holds for \(P(a)\) (because \(a\) exists only in \(u\)), but it fails for \(\exists x P(x)\), because obviously,

\[u \models \exists x P(x), \ v \not\models \exists x P(x).\]

To restore the truth-preservation, we can change the definition, e.g. as follows:

\[u \models \exists x \phi \text{ iff } \forall v \in R(u) \ \exists a \in D_v \ u \models [a/x] \phi,\]

\(^1\)Commonsense understanding of individuals is not easy to formalise, and there were many philosophical debates on that subject. 'What is an individual? — This is a good question!' — Dana Scott writes in [Scott, 1970]. Note that in labelled deductive systems, individuals are labelled and may be deemed to have internal structure. Wait for a later volume on this.
3.1. PRELIMINARY DISCUSSION

cf. the clause for the implication in the intuitionistic case. Another alternative is:

\[ u \models \exists x \varphi \iff \exists a \in D_u \forall v \in R(u) \ u \models [a/x] \varphi, \]

which means that only 'immortal' individuals are considered as existing. But nevertheless without (ED), the semantics is not sound, and some intuitionistic theorems may be false. For example, consider the formula \( P(x) \supset \exists x P(x) \). In the above model we have \( u \models P(a) \), but \( u \not\models \exists x P(x) \), since \( uRv \), and \( v \not\models P(b) \), \( b \) being the only individual in \( v \). Thus \( u \not\models P(a) \supset \exists x P(x) \).

We will return to this subject in Volume 2 of our book. Note that anyway, the principle (ED) is quite natural for intuitionistic logic. It normalises the situation, and the logic \( QH \) becomes sound and complete in an appropriate semantics.

OPTION B is close to Kripke's treatment in [Kripke, 1963]; it will be also considered later on (in Volume 2).

OPTION C includes many different approaches; we mention some of them.

OPTION C1 (modal case). As before, a Kripke model is a collection of classical structures, but quantifiers now range over all possible individuals i.e. over the whole set \( D^+ \). This is so-called possibilist quantification.

As individuals may not exist in some worlds, we have to add the unary existence predicate \( E \) to our language. Now instead of \( D_u \)-sentences we evaluate \( D^+ \)-sentences (where arbitrary individuals are used as extra constants). The truth condition for \( \forall \) becomes

\[ u \models \forall x \varphi \iff \forall a \in D^+ \ u \models [a/x] \varphi. \]

Note that \( u \models [a/x] \varphi \) may also hold if \( a \notin D_u \).\(^2\)

Option B can be realised within this approach as well, if we interpret \( \forall x \varphi(x) \) as \( \forall x (E(x) \supset \varphi(x)) \).

It turns out that the minimal logic \( QK \) is sound, but incomplete in this semantics. This is due to the formula

\[ \forall x \Box P(x) \supset \Box \forall x P(x) \]

found by Ruth Barcan and named after her. It is easily checked that the Barcan formula is true in every model, but is not a theorem of \( QK \), as the following countermodel (of the type A) shows (Fig. 3.3).

Here we have \( u \models \Box P(a) \), and so \( u \models \forall x \Box P(x) \). On the other hand, \( v \not\models \forall x P(x) \), and thus \( u \not\models \Box \forall x P(x) \) since \( uRv \).

Although axiomatising the minimal complete logic for the \( C_1 \)-semantics is simple, this approach is controversial, because quantification over all individuals is rather ambiguous, at least in natural language.\(^3\)

Still option \( C_1 \) is possible and useful by formal reasons.

\(^2\)In natural language a non-existing individual can still have a name in the world \( u \). We can say: Sir Isaac Newton discovered the laws of motion, using the name of a person given after the event had happened.

\(^3\)Quantification in natural language is a complicated subject, not to be discussed here. We remark only that in most cases quantification refers to the actual world, as in All students
**OPTION C2.** This is actualist quantification (see [Fitting and Mendelsohn, 1998]), a combination of B and C1.

Now $u \models \varphi$ is defined when $\varphi$ is a $D^+$-sentence, but $u \models P^u(a_1, \ldots, a_n)$ is put to be false when some of $a_1, \ldots, a_n$ do not exist in the world $u$: only actual individuals are allowed to have atomic properties.\(^4\)

The inductive definition of forcing is the same as in the cases A, B. However, there is a difference in the definition of the truth in a model:

$$M \models \varphi(x_1, \ldots, x_n) \text{ iff } u \models \varphi(a_1, \ldots, a_n) \text{ for any } u, \text{ for any } a_1, \ldots, a_n \text{ in } D^+ \text{ (not only in } D_u).$$

This approach again breaks the logic: the formula $\forall x P(x) \supset P(y)$ may be false, while $\forall y (\forall x P(x) \supset P(y))$ is always true.

**FURTHER OPTIONS.** One may argue that one and the same individual cannot exist in different worlds. In fact, nobody would identify a newly born baby and an old man or woman. This suggests we consider disjoint individual domains, together with transition maps (or relations) between them, as mentioned in the Introduction to Part II.

Another idea is to treat individuals as changing entities and consider individual concepts, that are partial functions from $W$ to the set $D^+$ (of 'individual images').

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\(^4\)Note that in the case C1 this is not required, and basic predicates may be true for non-existing individuals. This happens in natural languages as well. For example, Mike likes Socrates may be evaluated as true in the actual world.
3.2. PREDICATE KRIPKE FRAMES

Now let us turn to precise definitions and statements.

Definition 3.2.1 A system of domains over a set $W \neq \emptyset$ is a family of non-empty sets $D = (D_u)_{u \in W}$.

Definition 3.2.2 Let $F = (W, R_1, \ldots, R_N)$ be a propositional Kripke frame. An expanding system of domains over $F$ is a system of domains $D$ over $W$ such that

$$\forall i \in I_N \forall u, v \in W (u R_i v \Rightarrow D_u \subseteq D_v).$$

A predicate Kripke frame over $F$ is a pair $F = (F, D)$, in which $D$ is an expanding system of domains over $F$.

The set $D_u$ (sometimes denoted by $D(u)$) is called the individual domain of the world $u$; the set

$$D^+ := \bigcup_{u \in W} D_u$$

is the total domain of $F$; the frame $F$ (also denoted by $F_\pi$) is called the propositional base of $F$.

The following observation is an easy consequence of the definition.

Lemma 3.2.3 Let $F = (F, D)$ be a predicate Kripke frame, $v \in F \upharpoonright u$. Then $D_u \subseteq D_v$.

Proof By Lemma 1.3.19, $v \in R_\alpha(u)$ for some $\alpha \in I_N^\infty$. Then we can apply induction on $|\alpha|$. In fact, for $\alpha = \lambda$ we have $v = u$, so $D_u = D_v$; if $\alpha = \beta i$ and $u R_\beta w R_i v$ for some $w$, then $u R_\beta w R_i v$ by the induction hypothesis and 3.2.2. ■

By default, we denote an arbitrary propositional Kripke frame by $F = (W, R_1, \ldots, R_N)$ and an arbitrary predicate Kripke frame by $F = (F, D)$.

For an individual $a \in D^+$ the set

$$E(a) := \{ u \in W \mid a \in D_u \}$$

is called the measure of existence (or the extent). Since the system of domain $D$ is expanding, $E(a)$ is a stable subset of $F$. 

In this case the logic grows much larger than $QK$ and even becomes not recursively axiomatisable. We shall return to this topic in Volume 2.
For a tuple \( a \in (D^*)^n \) we also introduce the measure of existence

\[
E(a) := \bigcap_{i=1}^n E(a_i) = \{ u \in W \mid r(a) \subseteq D_u \}.
\]

The set is also stable in \( F \). In particular, for an empty \( a \) we define \( E(\lambda) \) as \( W \) (since \( r(\lambda) = \emptyset \)).

**Definition 3.2.4** A (modal) valuation \( \xi \) in a predicate Kripke frame \( F \) is a function associating with every predicate letter \( P^m_k \) a member of the set \( \prod_{u \in W} 2^{D_u^m} \), i.e. a family of \( m \)-ary relations on individual domains:

\[
\xi(P^m_k) = (\xi_u(P^m_k))_{u \in W},
\]

where \( \xi_u(P^m_k) \subseteq D^m_u \). To include the case \( m = 0 \), we assume that \( D^0_u = \{ u \} \); so \( \xi_u(P^0_k) \) is either \( \{ u \} \) or \( \emptyset \). The pair \( M = (F, \xi) \) is called a (predicate) Kripke model over \( F \).

We may call the function \( \xi_u \) sending every \( n \)-ary predicate letter to an \( n \)-ary relation on \( D_u \), a local valuation in \( F \) at \( u \); this is nothing but a classical valuation in \( D_u \). So we can say that \( \xi \) is a family of local valuations \( (\xi_u)_{u \in W} \).

**Definition 3.2.5** Let \( M = (F, \xi) \) be a Kripke model. For \( u \in M \), the classical model structure \( M_u := (D_u, \xi_u) \) is called the stalk (or the fibre) of \( M \) at \( u \).

On the other hand, we may call a function sending every \( n \)-ary predicate letter to an \( n \)-ary relation on \( D^+ \), a global valuation in \( F \).

Let us also define another kind of valuation in predicate Kripke frames.

**Definition 3.2.6** A global valuation in a predicate Kripke frame \( F \) is a function \( \gamma \) sending every \( n \)-ary predicate letter \( P(n > 0) \) to an \( n \)-ary \( 2^W \)-valued predicate on \( D^+ \), i.e. to a function \( \gamma(P) : (D^+)^n \to 2^W \) such that for any \( a \in (D^+)^n \), \( \gamma(P)(a) \subseteq E(a) \), and every \( q \in PL^0 \) to a subset of \( W \).

So \( \gamma \upharpoonright PL^0 \) is a propositional valuation in the propositional frame \( F \). We can also regard it as an 0-ary predicate, i.e. as a function \( (D^+)^0 \to 2^W \), where \( (D^+)^0 = \{ \lambda \} \). In this case the condition

\[
\gamma(q)(a) \subseteq E(a)
\]

trivially holds for any \( a \in (D^+)^0 \) (i.e. for \( \lambda \)), since \( E(\lambda) = W \).

**Lemma 3.2.7** Let \( F = (F, D) \) be a predicate Kripke frame.

1. For any valuation \( \theta \) in \( F \) there exists a global valuation \( \theta^+ \) such that for any \( P \in PL^m \), \( m > 0 \), for any \( a \in (D^+)^m \)

\[
\theta^+(P)(a) = \{ u \in W \mid a \in \theta_u(P) \}
\]
and for \( q \in PL^0 \)

\[
\theta^+(q) = \{ u \mid \theta_u(P) = \{ u \} \},
\]
3.2. PREDICATE KIRPEKE FRAMES

(2) Every global valuation $\gamma$ equals $\xi^+$ for a unique valuation $\xi$.

Proof

(1) From the definition it is clear that

$$\theta^*(P)(a) \subseteq E(a)$$

for any $P \in PL^n$, $n > 0$, $a \in (D^+)^n$.

(2) Put

$$\xi_u(P^m) := \{a \mid u \in \gamma(P^m)(a)\}$$

for $m > 0$ and

$$\theta_u(P^0) := \begin{cases} \{u\} & \text{if } u \in \gamma(P^0), \\ \emptyset & \text{otherwise.} \end{cases}$$

Then for $P \in PL^m$, $m > 0$

$$u \in \gamma(P)(a) \iff a \in \xi_u(P) \iff u \in \xi^+(P)(a),$$

and similarly for $m = 0$.

The uniqueness of $\xi$ is also clear; $\gamma = \xi^+$ means that for $P \in PL^m$ ($m > 0$), $a \in (D^+)^m$, $u \in W$

$$u \in \gamma(P)(a) \iff a \in \xi_u(P),$$

and for $q \in PL^0$

$$u \in \gamma(q) \iff u \in \xi_u(q).$$

Definition 3.2.8 A 1-modal predicate Kripke frame $F$ is called S4-based, or intuitionistic if its propositional base $F_\pi$ is an S4-frame. In this case a modal valuation $\xi$ in $F$ (and the corresponding Kripke model) is called intuitionistic if it has the truth preservation (or monotonicity) property:5

$$uRv \Rightarrow \xi_u(P^m) \subseteq \xi_v(P^m) \ (\text{if } m > 0);$$

$$uRv \land u \in \xi_u(P^0) \Rightarrow v \in \xi_v(P^0).$$

To define forcing, we use the notion of a $D$-sentence introduced in Section 2.4.

Definition 3.2.9 For a Kripke model $M = (F, \xi)$, $F = (F, D)$, $F = (W, R_1, \ldots, R_N)$ we define the (modal) forcing relation $M, u \vDash A$ (in another notation: $\xi, u \vDash A$, or briefly: $u \vDash A$) between worlds $u \in W^6$ and $D_u$-sentences $A$. The definition is inductive:

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5Recall that we omit the subscript '1' in the 1-modal case.
6Sometimes we write $u \in F$ or $u \in M$ instead of $u \in W$. 
• \( M, u \models P_k^0 \) iff \( u \in \xi_u(P_k^0) \);
• \( M, u \models P_k^m(a) \) iff \( a \in \xi_u(P_k^m) \) (for \( m > 0 \));
• \( M, u \models a = b \) iff \( a \) equals \( b \);
• \( M, u \not\models \perp \);
• \( M, u \models B \lor C \) iff \( (M, u \models B \) or \( M, u \models C \));
• \( M, u \models B \land C \) iff \( (M, u \models B \) and \( M, u \models C \));
• \( M, u \models B \supset C \) iff \( (M, u \not\models B \) or \( M, u \models C \));
• \( M, u \models \Box_i B \) iff \( \forall v \in R_i(u) \) \( M, v \models B \);\(^7\)
• \( M, u \models \exists x B \) iff \( \exists a \in D_u \) \( M, u \models [a/x]B \);
• \( M, u \models \forall x B \) iff \( \forall a \in D_u \) \( M, u \models [a/x]B \).

**Remark 3.2.10** The above truth definition for propositional letters seems peculiar. It is more natural to use truth values 1 and 0 rather than \( \{u\} \) and \( \varnothing \). But our definition will be convenient later on, in Chapter 5.

Now we readily obtain an analogue of Lemma 1.3.3:

**Lemma 3.2.11** For any Kripke model:
\[ u \models \Diamond_i B \iff \exists v \in R_i(u) \ v \models B; \]
\[ u \models \lnot B \iff u \not\models B; \]
\[ u \models \Box_\alpha B \iff \forall v \in R_\alpha(u) \ v \models B; \]
\[ u \models \diamond_\alpha B \iff \exists v \in R_\alpha(u) \ v \models B. \]

Similarly to the propositional case, we give the following inductive definition of intuitionistic forcing.

**Definition 3.2.12** Let \( M = (F, \xi) \) be an intuitionistic Kripke model. We define the intuitionistic forcing relation \( M, u \models A \) between a world \( u \in F \) and an intuitionistic \( D_u \)-sentence \( A \) (also denoted by \( \xi, u \models A \) or just by \( u \models A \)) by induction:

• \( M, u \models A \) iff \( M, u \models A \) for \( A \) atomic;
• \( M, u \models B \land C \) iff \( M, u \models B \) & \( M, u \models C \);\(^8\)
• \( M, u \models B \lor C \) iff \( (M, u \models B \) or \( M, u \models C \));
• \( M, u \models B \supset C \) iff \( \forall v \in R(u) \) \( (M, v \models B \Rightarrow M, v \models C) \);

\(^7\)Note that \( B \) is a \( D_v \)-sentence, since \( D \) is expanding.

\(^8\)Here \& is an abbreviation of 'and', and further on in this definition \( \forall v \in R(u) \) abbreviates 'for any \( v \) in \( R(u) \)', etc.
3.2. PREDICATE Kripke Frames

- \( M, u \vDash \exists x B \iff \exists a \in D_u \ M, u \vDash [a/x]B; \)
- \( M, u \vDash \forall x B \iff \forall v \in R(u) \forall a \in D_v \ M, v \vDash [a/x]B. \)

Hence we easily obtain

**Lemma 3.2.13** The intuitionistic forcing has the following properties

- \( M, u \vDash \neg B \iff \forall v \in R(u) \ M, v \nvDash B; \)
- \( M, u \vDash a \neq b \iff a, b \in D_u \) are not equal.

**Definition 3.2.14** Let \( F \) be an \( S4\)-based Kripke frame; \( M \) a Kripke model over \( F \). The pattern of \( M \) is the Kripke model \( M_0 \) over \( F \) such that for any \( u \in F \) and any atomic \( D_u\)-sentence without equality \( A \)

\[
M_0, u \vDash A \iff M, u \vDash \Box A.
\]

Obviously, every Kripke model \( M \) over \( F \) has a unique pattern; \( M_0 \) is an intuitionistic Kripke model and \( M_0 = M \) if \( M \) is itself intuitionistic.

**Lemma 3.2.15** If \( M_0 \) is a pattern of \( M \), then for any \( u \in M \), for any intuitionistic \( D_u\)-sentence \( A \)

\[
M_0, u \vDash A \iff M, u \vDash A^T;
\]

**Proof** Easy, by induction on the length of \( A \). The atomic case follows from Definition 3.2.14 (and is trivial for \( A \) of the form \( a = b \)). Let us check only the case \( A = \forall x B; \) other cases are left to the reader.

\[
M_0, u \vDash \forall x B \iff \forall v \in R(u) \forall a \in D_v \ M_0, u \vDash [a/x]B \text{ (by Definition 3.2.12)}
\]

\[
\iff \forall v \in R(u) \forall a \in D_v \ M, u \vDash ([a/x]B)^T \text{ (by the induction hypothesis).}
\]

On the other hand, by Definition 3.2.9

\[
M, u \vDash A^T (= \Box \forall x B^T) \iff \forall v \in R(u) \forall a \in D_v \ M, u \vDash [a/x](B^T),
\]

and it remains to note that \( ([a/x]B)^T = [a/x](B^T) \), by 2.11.3.

**Lemma 3.2.16** Let \( u, v \) be worlds in an intuitionistic Kripke model \( M \). Then for any intuitionistic \( D_u\)-sentence \( A \)

\[
M, u \vDash A \Leftrightarrow u R v \Rightarrow M, v \vDash A.
\]

**Proof** This easily follows by induction. The case when \( A = P(a) \) is atomic follows from the monotonicity property (TP). The cases \( A = B \supset C, A = \forall x B \) follow from the transitivity of \( R \). For \( A = \exists x B \) note that \( M, u \vDash A \) implies \( M, u \vDash [a/x]B \) for some \( a \in D_u \), and hence \( M, v \vDash [a/x]B \) by the induction hypothesis, which yields \( M, v \vDash B. \)

Other cases are trivial.
Lemma 3.2.17 Let $M$ be an $N$-modal predicate Kripke model, $A(x)$ an $N$-modal formula with all its parameters in the list $x$, $|x| = n$. Then for any $u \in M$

$$M, u \models \forall x A(x) \iff \forall a \in D^n_u M, u \models A(a).$$

Proof By induction on $n$. The base is trivial and the step is almost trivial:

$$u \models \forall y \forall x A(y, x) \iff \forall b \in D_u \ u \models \forall x A(b, x)$$

iff $\forall b \in D_u \ \forall c \in D^n_u \ u \models A(b, c)$ (by the induction hypothesis)

iff $\forall a \in D^n_u+1 \ u \models A(a)$. □

In the intuitionistic case we have the following

Lemma 3.2.18 Let $M$ be an intuitionistic Kripke model with the accessibility relation $R$, $u \in M$, $A(x)$ an intuitionistic $D_u$-formula with all its parameters in the list $x$, $|x| = n$. Then

$$M, u \models \forall x A(x) \iff \forall v \in R(u) \forall a \in D^n_v M, v \models A(a).$$

Proof By induction, similar to the previous lemma. The base (with $n = 0$) follows from Lemma 3.2.16. For the induction step we have:

$$u \models \forall y \forall x A(y, x) \iff \forall v \in R(u) \forall b \in D_v \ v \models \forall x A(b, x)$$

iff $\forall v \in R(u) \forall b \in D_v \ \forall w \in R(v) \forall c \in D^n_w \ w \models A(b, c)$ (by 3.2.13)

iff $\forall w \in R(u) \forall a \in D^n_w+1 \ w \models A(a)$. □

In the latter equivalence, ‘if’ follows from the transitivity of $R$ and the inclusion $D_v \subseteq D_w$ for $w \in R(v)$; to prove ‘only if’, for given $u, w, a$, take $v = w$ and $b = a_1$, $c = (a_2, \ldots, a_n)$.

Definition 3.2.19 A modal (respectively, intuitionistic) predicate formula $A$ is said to be true in a Kripke model (respectively, intuitionistic Kripke model) $M$ if its universal closure $\forall A$ is true at every world of $M$.

This is denoted by $M \models A$ in the modal case and by $M \models I A$ in the intuitionistic case.

The set of all modal (respectively, intuitionistic) sentences that are true in $M$ is denoted by $\text{MT}^{(=)}(M)$ (respectively, $\text{IT}^{(=)}(M)$) and called the modal (respectively, the intuitionistic) theory of $M$ (with or without equality).

Lemma 3.2.20 Let $M$ be an $N$-modal or intuitionistic predicate Kripke model with a system of domains $D$, $A(x_1, \ldots, x_n)$ a predicate formula of the corresponding type. Then

$$M \models (\models) A(x_1, \ldots, x_n) \iff \forall u \in M \forall a_1, \ldots, a_n \in D_u \ M, u \models (\models) A(a_1, \ldots, a_n).$$

Proof Follows easily from Lemma 3.2.17, 3.2.18. E.g. in the intuitionistic case we have

$$M \models \forall x A(x) \iff \forall u \forall v \in R(u) \forall a \in D^n_v M, v \models A(a)$$

iff $\forall v \forall a \in D^n_v M, v \models A(a)$. □

In the latter equivalence, ‘if’ is trivial; to show ‘only if’, take $u = v$. □
Definition 3.2.21 A modal (respectively, intuitionistic) predicate formula $A$ is said to be valid in a Kripke frame (respectively, S4-based Kripke frame) $F$ if it is true in all Kripke models (respectively, intuitionistic Kripke models) over $F$.

Validity is denoted by $F \vDash A$ in the modal case and by $F \Vdash A$ in the intuitionistic case.

Definition 3.2.22 Let $\Sigma$ be a set of modal (respectively, intuitionistic) sentences. We say that $\Sigma$ is valid in a Kripke frame $F$ (of the corresponding type) if every formula from $\Sigma$ is valid in $F$; or equivalently, we say that $F$ is a $\Sigma$-frame (or $\Sigma \subseteq ML(=)(F)$, $\Sigma \subseteq IL(=)(F)$).

$F \vDash \Sigma$ (or $F \Vdash \Sigma$) denotes that $\Sigma$ is valid in $F$. The class of all $\Sigma$-frames is denoted by $\mathcal{V}(\Sigma)$ and called modally (respectively, intuitionistically) definable (by $\Sigma$).

Lemma 3.2.23 Let $F$ be an $N$-modal (respectively, intuitionistic) Kripke frame, $A$ an $N$-modal (respectively, intuitionistic) propositional formula. Then $F \vDash A$ iff $F_\pi \vDash A$ (respectively, $F \Vdash A$ iff $F_\pi \Vdash A$), i.e. $L(F)_\pi = L(F)$, where $L$ is $ML(=)$ or $IL(=)$.

Proof Easy from definitions. Valuations in $F_\pi$ are in principle the same as valuations of proposition letters in $F$; more precisely, a valuation $\xi$ in $F$ corresponds to $\xi'$ in $F_\pi$ such that

$$\xi'(q) = \xi^+(q) = \{u \mid u \in \xi_u(q)\},$$

for any $q \in PL^0$, and both $\xi$, $\xi'$ are extended to propositional formulas in the same way. So $(F, \xi), u \vDash A$ iff $(F_\pi, \xi'), u \vDash A$ and similarly for the intuitionistic case. Finally note that every propositional valuation in $F_\pi$ is $\xi'$ for some valuation $\xi$ in $F$.

Lemma 3.2.24 Let $M$ be an S4-based predicate Kripke model, $M_0$ its pattern. Then for any $A \in IP^=\pi$, $M_0 \vDash A$ iff $M \vDash A^T$, and thus

$$A \in IT(M_0) \iff A^T \in MT(M).$$

Proof From 3.2.13, 3.2.18.

Proposition 3.2.25 Let $F$ be an S4-based Kripke frame, $A$ an intuitionistic formula with equality. Then

$$F \Vdash A \iff F \vDash A^T.$$
By Lemma 3.2.20, \( F \Vdash A \) implies \( M_0, u \Vdash A(a) \), and thus we obtain \( F \Vdash A^T \).

(If.) Assume \( F \Vdash A^T \). Then for any intuitionistic model \( M \) and \( u \in F \), \( a \in D_u \) we have \( M, u \Vdash A(a)^T (= A^T(a)) \) by Lemma 3.2.20, i.e. \( M, u \Vdash A(a) \) by Definition 3.2.12. Hence \( F \Vdash A \), by Lemma 3.2.20. ■

**Lemma 3.2.26** Let \( A(x), B(x) \) be congruent \( N \)-modal (respectively, intuitionistic) formulas, \( |x| = n \), and let \( M \) be an \( N \)-modal (respectively, intuitionistic) Kripke model. Then for any \( u \in M \), \( a \in D_u^n \)

\[
M, u \Vdash (\ifcn) A(a) \iff M, u \Vdash (\ifcn) B(a).
\]

**Proof** We begin with the modal case. Consider the following equivalence relation between \( N \)-modal formulas:

\[
A \sim B := \text{FV}(A) = \text{FV}(B) \text{ and for any distinct list of variables } x \text{ such that } \text{FV}(A) = r(x), \text{ for any } u \in M, \ a \in D_u^{|x|}
\]

\[
M, u \Vdash [a/x]A \iff M, u \Vdash [a/x]B.
\]

Our aim is to show that \( A \equiv B \) implies \( A \sim B \).

By Proposition 2.3.17 this implication follows from the properties (1)–(4). So let us check these properties for \( \sim \)

(1) \( QyA \sim Qz(A[y \leftrightarrow z]) \) for \( y \not\in \text{BV}(A), \ z \not\in V(A) \).

We consider only the case \( Q = \exists \). Suppose \( \text{FV}(\exists y A) = r(x) \) for a distinct \( x \). Then there are two subcases.

(i) \( y \not\in \text{FV}(A) \) (and thus \( y \not\in V(A) \)).

In this case \( A[y \mapsto z] = A \), so

\[
\exists z(A[y \mapsto z]) = \exists z A, \ [a/x]\exists y A = \exists y[a/x]A, \ [a/x]\exists z A = \exists z[a/x]A
\]

By 3.2.9 we have \( M, u \Vdash \exists y[a/x]A \iff M, u \Vdash [a/x]A \), since \( y \) does not occur in \( [a/x]A \). Similarly,

\[
M, u \Vdash \exists z[a/x]A \iff M, u \Vdash [a/x]A,
\]

so (1) holds.

(ii) \( y \in \text{FV}(A) \). Then again (since \( y \not\in r(x) \))

\[
[a/x]\exists y A = \exists y[a/x]A, \ [a/x]\exists z(A[y \mapsto z]) = \exists z[a/x](A[y \mapsto z]).
\]

Hence

\[
M, u \Vdash [a/x]\exists y A \iff \exists d \in D_u M, u \Vdash [d/y][a/x]A (= [ad/xy]A = [a/x][d/y]A)
\]

Similarly

\[
M, u \Vdash [a/x]\exists z(A[y \mapsto z]) \iff \exists d \in D_u M, u \Vdash [a/x][d/z](A[y \mapsto z]).
\]
But since $y \not\in BV(A)$, $z \not\in V(A)$, we have
\[ [d/y]A = A[y \mapsto d] = (A[y \mapsto z])[z \mapsto d] = [d/z](A[y \mapsto z]). \]

Thus (1) holds in this case too.

(2) Supposing $A \sim B$, let us show that
\[ QyA \sim QyB \text{ for } Q = \forall; \]
the case $Q = \exists$ is quite similar.

Obviously, $FV(A) = FV(B)$ implies $FV(\forall yA) = FV(\forall yB)$; let $r(x) = FV(\forall yA)$. Then
\[ M, u \models [a/x]\forall yA \iff \forall d \in D_u M, u \models [d/y][a/x]A(= [ad/xy]A), \]
and similarly
\[ M, u \models [a/x]\forall yB \iff \forall d \in D_u M, u \models [ad/xy]B. \]

Now if $y \in FV(A) = FV(B)$, then we use the hypothesis ($\#$) for $A, B, xy, ad$.
If $y \not\in FV(A)$, then $FV(A) = FV(B) = r(x)$,
so we can use ($\#$) for $A, B, x, a$. Anyway
\[ M, u \models [a/x]\forall yA \iff M, u \models [a/x]\forall yB. \]

(3) Supposing $A \sim A'$, $B \sim B'$, we prove that $(A * B) \sim (A' * B')$. Let us consider the case $* = \wedge$.

Obviously
\[ FV(A \land B) = FV(A) \cup FV(B) = FV(A') \cup FV(B') = FV(A' \land B'). \]
Let $r(x) = FV(A \land B)$. Then its subset $FV(A)$ is $r(x \cdot \sigma)$ for some injection $\sigma$ and $FV(B) = r(x \cdot \tau)$ for some injection $\tau$. Obviously
\[ [a/x]A = [a \cdot \sigma/x \cdot \sigma]A, [a/x]B = [a \cdot \tau/x \cdot \tau]B, \]
and similarly for $A', B'$;

\[ M, u \models [a/x](A \land B) \iff M, u \models [a/x]A \land M, u \models [a/x]B \iff M, u \models [a \cdot \sigma/x \cdot \sigma]A \land M, u \models [a \cdot \tau/x \cdot \tau]B. \]

Similarly
\[ M, u \models [a/x](A' \land B') \iff M, u \models [a \cdot \sigma/x \cdot \sigma]A' \land M, u \models [a \cdot \tau/x \cdot \tau]B'. \]
Since $A \sim A'$, $B \sim B'$, this implies $(A \land B) \sim (A' \land B')$. 

(4) Supposing $A \sim B$, let us show $\Box_i A \sim \Box_i B$.

Obviously, $FV(\Box_i A) = FV(A) = FV(B) = FV(\Box_i B)$. Next,

$$M, u \vDash [a/x] \Box_i A = \Box_i [a/x] A \iff \forall v \in R_i(u) M, v \vDash [a/x] A,$$

and similarly for $B$.

Since $A \sim B$, these two conditions are equivalent.

In the intuitionistic case note that $A \sim B$ implies $A^T \sim B^T$, by 2.11.3. So by applying the modal case and Lemma 3.2.15 we obtain:

$$M, u \vDash A(a) \iff M, u \vDash A^T(a) \iff M, u \vDash B^T(a) \iff M, u \vDash B(a).$$

Lemma 3.2.27 Let $A$ be a modal (respectively, intuitionistic) formula valid in a Kripke frame (respectively, an S4-based Kripke frame) $F$. Then every formula congruent to $A$ is valid in $F$.

In other words, the sets $ML (=)(F)$, $IL (=)(F)$ are closed under congruence.

Proof By Lemmas 3.2.26, 3.2.17.

Lemma 3.2.28 For any classical $D_u$-formula $A$

$$M, u \vDash A \iff M_u \vDash A \ (in \ the \ classical \ sense).$$

Proof By induction. The classical and modal truth definitions coincide in this case.

Theorem 3.2.29 (Soundness theorem)

(I) The set $ML (=)(F)$ of all modal predicate formulas (with equality) valid in a predicate Kripke frame $F$ is a modal predicate logic (with equality).

(II) The set $IL (=)(F)$ of all intuitionistic predicate formulas (with equality) valid in an S4-based Kripke frame $F$ is a superintuitionistic predicate logic (with equality). Moreover, $IL (=)(F) = TML (=)(F)$.

Proof

(I) Let $F$ be a predicate Kripke frame. The axioms of $K_n$ are obviously valid, since they are valid in every propositional Kripke frame and we can apply 3.2.23.

Let $M$ be a Kripke model over $F$. The classical first-order axioms and the axioms of equality are true in every $M_u$; so by Lemma 3.2.28, they are true in $M$.

It is also easy to check that modus ponens preserves the truth in a Kripke model. In fact, suppose $M \vDash A$, $A \supset B$, and let $x$ be a list of parameters of $(A \supset B)$, $|x| = n$. Then by 3.2.20(1), for any $u \in M$, $a \in D^u_n$

$$M, u \vDash [a/x] A, [a/x] A \supset [a/x] B,$$
hence $M, u \vDash [a/x]B$. Thus $M \vDash B$ by 3.2.20(1).

To verify $\Box$-introduction, note that

$$M, u \vDash [a/x]\Box A \iff \forall v \in R(u) \ M, v \vDash [a/x]A,$$

and the right hand side of the equivalence follows from $M \vDash A$ by 3.2.20.

Next, consider the $\forall$-introduction rule. If $M \vDash A(y, x_1, \ldots, x_n)$, then by Lemma 3.2.20,

$$\forall u \forall b, a_1, \ldots, a_n \in D_u \ M, u \vDash A(b, a_1, \ldots, a_n),$$

and thus

$$\forall u \forall a_1, \ldots, a_n \in D_u \ M, u \vDash \forall y A(y, a_1, \ldots, a_n);$$

hence $M \vDash \forall y A(y, x_1, \ldots, x_n)$ again by Lemma 3.2.20. Thus $\forall$-introduction preserves the truth in $M$.

The only nontrivial part of the proof is to show that the substitution rule preserves validity.

So suppose $F \vDash A$, and consider a simple formula substitution $S=[C(x, y)/P(x)]$, where $P$ occurs$^9$ in $A$, $P \in PL^n$, $n > 0^{10}$ and $x$, $y$ are disjoint lists of different variables such that

$$r(y) \subseteq FV(C) \subseteq r(xy).$$

Recall that $SA$ is obtained by appropriate replacements from a clean version $A^\circ$ of $A$ such that $BV(A^\circ) \cap r(y) = \emptyset$. By Lemma 3.2.27, $A$ and $A^\circ$ are equivalent with respect to validity, so we may assume that $A$ is clean and $BV(A) \cap r(y) = \emptyset$. As we know from 2.5.24,

$$r(y) = FV(S) \subseteq FV(SA) \subseteq FV(A) \cup FV(S),$$

so $BV(A) \cap FV(SA) = \emptyset$.

Let us fix a distinct list $z$ such that $r(z) = FV(A) \cup FV(S)$ (for example, put $z = yt$, where $r(t) = FV(A)$); so $r(y) \subseteq FV(SA) \subseteq r(z)$. Due to our assumption, $r(z) \cap BV(A) = \emptyset$.

Now for an arbitrary Kripke model $M = (F, \xi)$, let us show that $M \vDash SA$, i.e. for any $u \in F$ and $c \in D^m_u$

$$M, u \vDash [c/z]SA,$$

where $m = |z|$.

To verify this (for certain fixed $u$ and $c$), consider another model $M_1 = (F, \eta)$ such that

- for any $v \in F|u$, $a \in D^n_v$

$$M_1, v \vDash P(a) \iff M, v \vDash C(a, c'),$$

where $c'$ is the part of $c$ corresponding to $y$ (i.e. if $y = z_1 \ldots z_k$, then $c' = c_1 \ldots c_k$);

---

$^9$The case when $P$ does not occur in $A$ is trivial.

$^{10}$There is a little difference in the case when $n = 0$ (then $x$ is empty), but we leave this to the reader.
• for any other atomic $D_v$-sentence $Q$

$$M_1, v \models Q \text{ iff } M, v \models Q.$$ 

Thus $\eta_v(P) = \{a \in D^n_v \mid M, v \models C(a, c')\}$.

Now consider a subformula $B$ of $A$. Since $FV(B) \subseteq FV(A) \cup BV(A)$, we can present $B$ as $B(z, q)^{11}$, where $q$ is distinct, $r(q) = BV(A)$.

Then by 2.5.24

$$FV(SB) \subseteq FV(S) \cup FV(B) \subseteq r(zq),$$

So we can also present $SB$ as $(SB)(z, q)$.

Certainly we can use the same presentation in the trivial case when $P$ does not occur in $B$ (and $SB = B$). Now let us prove the following

**Claim.** For any $v \in F \uparrow u$ and tuple $a$ in $D_v$ such that $|a| = |q|$,

$$M_1, v \models B(c, a) \text{ iff } M, v \models (SB)(c, a). \quad (2)$$

This is proved by induction. It is essential that $B$ (as a subformula of $A$) is clean, otherwise the argument is inapplicable, because $SB$ can be constructed by induction only for clean formulas.

• The case when $B$ is atomic and does not contain $P$ is trivial, then $SB = B$ and

$$M_1, v \models B(c, a) \iff M, v \models B(c, a)$$

by the definition of $M_1$.

• If $B$ is atomic and contains $P$, it has the form $P(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are variables from the list $zq$, $q$ consists of parameters of $B$ that are not in $z$ (i.e. $r(q) = FV(B) \cap BV(A)$). So we have $B = P((zq) \cdot \sigma)$ for some $\sigma : I_n \rightarrow I_k$, $k = |zq|$. Then

$$B(c, a) = [ca/zq]P((zq) \cdot \sigma) = P((ca) \cdot \sigma),$$

$$(SB)(c, a) = [ca/zq]SP((zq) \cdot \sigma) = [ca/zq]C((zq) \cdot \sigma, y) = C((ca) \cdot \sigma, c').$$

Recall that $y$ transforms to $c'$ when $z$ transforms to $c$. So in this case

$$M_1, v \models B(c, a) \text{ iff } M, v \models (SB)(c, a)$$

by the definition of $M_1$.

• If $B = B_1 \lor B_2$, then

$$M_1, v \models B(c, a) \text{ iff } (M_1, v \models B_1(c, a) \lor M_1, v \models B_2(c, a))\text{ iff } (M, v \models (SB_1)(c, a) \text{ or } M, v \models (SB_2)(c, a)) \text{ iff } M, v \models (SB)(c, a).$$

---

11 Of course, this only means that $FV(B) \subseteq r(zq)$.
If $B = \exists t B_1$, then

$$M_1, v \models B(c, a) \text{ iff } \exists d \in D_v \ M_1, v \models B_1(c, a') \text{ iff } \exists d \in D_v \ M, v \models (SB_1)(c, a')$$

(by the induction hypothesis) iff $M, v \models (\exists t SB_1)(c, a) (= (SB)(c, a))$.

Here $a'$ denotes the tuple obtained from $a$ by putting $d$ in the position corresponding to $t$, i.e. if $t = q_i$ (in $q$), then $b_i = d$ and $b_j = a_j$ for $j \neq i$.

Let us explain the first equivalence in more detail; the third equivalence is checked in the same way. In fact, we have

$B(c, a) = [ca/\tilde{q}q]B = [c\tilde{a}_i/\tilde{z}\tilde{q}_i]B,$

where $\tilde{q}_i$ is obtained by eliminating $q_i = t$ from $q$ and respectively $\tilde{a}_i$ is obtained by eliminating $a_i$ from $a$, since $t$ is bound in $B$. So by definition,

$$M, v \models B(c, a) \text{ iff } \exists d \in D_v \ M, v \models [d/t][c\tilde{a}_i/\tilde{z}\tilde{q}_i]B_1,$$

and the latter $D_v$-sentence is $B_1(c, a').$

If $B = \square_i B_1$, then $M_1, v \models B(c, a)$ iff

$$\forall w \in R_i(v) \ M_1, w \models B_1(c, a) \text{ iff } \forall w \in R_i(v) \ M, w \models (SB_1)(c, a) \text{ (by the induction hypothesis) iff } M, v \models \square_i (SB_1)(c, a) (= (SB)(c, a)).$$

The remaining cases: $B = C \supset D, C \land D, \forall x C(x, z)$ are quite similar.

Now (1) easily follows from (2). In fact, (1) is equivalent to

$$M, u \models [ac/\tilde{q}q]SA$$

for any $a \in D_u$ since $r(q) \cap FV(SA) = \emptyset$. By (2), this is equivalent to

$$M_1, u \models [ac/\tilde{q}q]A (= [c/z]A),$$

which holds since $F \models A$.

(II) By Proposition 3.2.25, $F \models A$ iff $F \models A^T$. This is exactly the same as $A \in IL^{(=)}(F)$ iff $A^T \in ML^{(=)}(F)$. So we obtain: $IL^{(=)}(F) = \{ A \in ML^{(=)}(F) \}$, and thus $IL^{(=)}(F)$ is an s.p.l. (=), by Proposition 2.11.8.

In accordance with Section 2.16, the set $ML^{(=)}(F)$ is called the modal logic of $F$ (respectively, with or without equality). Similarly, for an intuitionistic $F$, its superintuitionistic logic is $IL^{(=)}(F)$.

For a class of $N$-modal frames $C$ we define the modal logic determined by $C$ (or complete w.r.t. $C$)

$$ML^{(=)}(C) := \bigcap \{ ML^{(=)}(F) \mid F \in C \},$$

and similarly for superintuitionistic logics.
Definition 3.2.30 \((N\text{-modal}) Kripke (frame) semantics \(K_N\) (or \(K^=\), for logics with equality) is generated by the class of all \(N\)-modal predicate Kripke frames. Similarly intuitionistic Kripke (frame) semantics \(K^{(=)}_{int}\) is generated by the class of all intuitionistic predicate Kripke frames. \(K^=\)\(-complete or \(K^{(=)}_{int}\)\)-complete logics are called Kripke (frame) complete (or \(K\)-complete if there is no confusion).

Here is another version of soundness.

Lemma 3.2.31 Let \(M\) be a modal (respectively, intuitionistic) Kripke model, \(L\) an m.p.l.(\(=\)) (respectively, s.p.l.\((=)\)), \(\Gamma\) a modal (respectively, intuitionistic) theory without constants, such that \(M \models (\vdash) \overline{L} \cup \Gamma\). Then for any \(A \in IF^{(=)}\),

\[ \Gamma \vdash_L A \Rightarrow M \models (\vdash) \overline{A}. \]

Proof By induction on the length of an \(L\)-inference of \(A\) from \(\Gamma\).

If \(A \in \Gamma\) the claim is trivial.

If \(A \in L\), then \(\overline{A} \in \overline{L}\), so the claim is also trivial.

It is clear that the truth in \(M\) respects \(MP\), cf. the proof of 3.2.29.

Finally, if \(A = \forall x B\) and \(M \models (\vdash) \overline{B}\), then also \(M \models (\vdash) \overline{A}\), since \(QH \vdash \overline{A} \equiv \overline{B}\) and we can apply soundness.

Definition 3.2.32 If \(F\) is a propositional Kripke frame, \(K(F)\) (or \(KF\) ) denotes the class of all predicate Kripke frames based on \(F\). \(ML^{(=)}(K(F))\) is called the modal logic (with equality) determined over \(F\). Similarly we define \(K(C)\) (or \(KC\) ) for a class of \(N\)-modal propositional frames \(C\) and the modal logic determined over \(C\).

We also define the superintuitionistic logic determined over a propositional S4-frame \(F\) as \(IL^{(=)}(KF)\) and similarly for a class of S4-frames \(C\).

We may regard \(KC\) as a semantics for a certain class of logics. In particular, for a propositional modal logic \(\Lambda\), the class \(KV(\Lambda)\) generates the semantics of all Kripke-complete logics containing \(\Lambda\).

Proposition 3.2.33 \(K^{(=)}_{int}\) is the intuitionistic version of \(KV(S4)\). Thus \(IL^{(=)}(C) = T\ ML^{(=)}(C)\) for any class of S4-based predicate frames \(C\).

Proof According to Definition 2.16.16, the first assertion is equivalent to 3.2.25. The second one is checked in the proof of 2.16.18.

Lemma 3.2.34 Let \(F\) be a predicate Kripke frame over a propositional Kripke frame \(F\). Then

\[ ML(F)_\pi = ML(F). \]

Proof Rather trivial. The definition of validity for propositional formulas in \(F\) is the same as in \(F\).
Proposition 3.2.35 For a class of propositional Kripke frames $\mathcal{C}$,

$$(\text{ML}^{(=)}(\mathcal{K}\mathcal{C}))_{\pi} = \text{ML}(\mathcal{C})$$

and

$$(\text{IL}^{(=)}(\mathcal{K}\mathcal{C}))_{\pi} = \text{IL}(\mathcal{C})$$

in the intuitionistic case.

Proof We begin with the modal case. By Lemma 3.2.34, we obtain

$$(\text{ML}^{(=)}(\mathcal{K}\mathcal{C}))_{\pi} = (\bigcap\{\text{ML}^{(=)}(F) \mid F \in \mathcal{K}\mathcal{C}\})_{\pi} = \bigcap\{\text{ML}^{(=)}(F)_{\pi} \mid F \in \mathcal{K}\mathcal{C}\} = \bigcap\{\text{ML}^{(=)}(F) \mid F \in \mathcal{C}\} = \text{ML}^{(=)}(\mathcal{C}).$$

3.3 Morphisms of Kripke frames

In this section we extend the notions of a frame morphism, a generated subframe and a $p$-morphism to the predicate case.

Definition 3.3.1 Let $F = (F, D), F' = (F', D')$ be predicate Kripke frames based on $F = (W, R_1, \ldots, R_N), F' = (W', R'_1, \ldots, R'_N)$ respectively. A morphism from $F$ to $F'$ is a pair $f = (f_0, f_1)$ such that

1. $f_0 : F \rightarrow F'$ is a morphism of propositional frames;
2. $f_1 = (f_{1u})_{u \in W};$
3. every $f_{1u} : D_u \rightarrow D'_{f_0(u)}$ is a surjective map;
4. $uR_i v \Rightarrow f_{1u} = f_{1v} \upharpoonright D_u.$

$f_0$ is called the world component, $f_1$ the individual component of $(f_0, f_1)$.

Definition 3.3.2 Let $f = (f_0, f_1)$ be a morphism from $F$ to $F'$.

- $f$ is called an equality-morphism (briefly, $=$-morphism) if every $f_{1u}$ is a bijection.
- $f$ is called a $p$-morphism if $f_0$ is surjective (i.e. $f_0$ is a $p$-morphism of propositional frames).
- $f$ is called a $p^=$-morphism if it is a $p$-morphism, and an $=$-morphism.
- $f$ is called an isomorphism if it is an $=$-morphism, and $f_0$ is an isomorphism of propositional frames.
Definition 3.3.3 A morphism of predicate Kripke models $M = (F, \xi)$ and $M' = (F', \xi')$ is a morphism $(f_0, f_1)$ of their frames $F$ and $F'$ preserving the truth values of atomic formulas, i.e. such that for any $P \in PL^m$, $m \geq 0$, $u \in F; b_1, \ldots, b_m \in D_u$

$$M, u \vdash P(b_1, \ldots, b_m) \iff M', f_0(u) \vdash P(f_1u(b_1), \ldots, f_1u(b_m)).$$

A $p$-morphism of Kripke models is a morphism of Kripke models, which is a $p$-morphism of their frames. Similarly for $\ldots$, $p^\ominus$-morphisms and isomorphisms.

The notation $(f_0, f_1): M \rightarrow M'$ means that $(f_0, f_1)$ is a morphism from $M$ to $M'$, and similarly for frames. As in the propositional case, $p$-morphisms of predicate Kripke frames (models) are denoted by $\rightarrow$. We also use the notation $\rightarrow^\ominus$ for $=$-morphisms, $\Rightarrow^\ominus$ for $p^\ominus$-morphisms, $\cong$ for isomorphisms.

Lemma 3.3.4

1. For a predicate Kripke frame $F = (F, D)$, the identity morphism $id_F := (id_F, f_1)$, where $f_1u := id_{D_u}$ for any $u$, is an isomorphism.

2. For morphisms of predicate Kripke frames $(f_0, f_1): F \rightarrow F'$ and $(g_0, g_1): F' \rightarrow F''$, consider the composition

$$(f_0, f_1) \circ (g_0, g_1) := (g_0, g_1) \cdot (f_0, f_1) := (f_0 \circ g_0, (f_1u \circ g_1u)_{u \in F}).$$

This is a morphism $F \rightarrow F''$.

3. The composition of $\Rightarrow$-morphisms is an $\Rightarrow$-morphism; similarly for $p(\Rightarrow)$-morphisms and isomorphisms.

4. The statements (1)-(3) also hold for Kripke models, with obvious changes.

Proof An easy exercise; use Lemma 1.3.32. $lacksquare$

Thus we obtain the categories $PKF_{N}^\ominus$ of $N$-modal predicate Kripke frames and $(\Rightarrow)$-morphisms, with the composition $\circ$ and the identity morphism $id_F$. One can easily check the following

Lemma 3.3.5 Isomorphisms in $PKF_{N}^\ominus$ are exactly isomorphisms in the sense of Definition 3.3.2.

In a similar way we can define the categories $PKF_{int}^\ominus$ of intuitionistic Kripke frames, $PKM_{N}^\ominus$ of $N$-modal Kripke models, $PKM_{int}^\ominus$ of intuitionistic Kripke models; the details are left to the reader.

Definition 3.3.6 Let $F = (W, R_1, \ldots, R_N)$ be a propositional Kripke frame. A predicate Kripke frame morphism over $F$ is a morphism of the form $(id_W, f_1): (F, D) \rightarrow (F, D')$. 
The following is obvious

Lemma 3.3.7 The composition of (=)-morphisms over $F$ is an (=)-morphism over $F$.

Thus we can also consider the categories of frames over $F$ and (=)-morphisms. But the case with equality is not so interesting, because all =-morphisms over $F$ are isomorphisms.

Definition 3.3.8 Let $F, F'$ be propositional Kripke frames, $h : F' \rightarrow F$, and let $F = (F, D)$ be a predicate Kripke frame. Then we say that a Kripke frame $F' = (F', D')$ is obtained from $F$ by changing the base along $h$ if $D'_u = D_{h(u)}$ for all $u \in F$. This frame $F'$ is denoted by $h_*F$.

Lemma 3.3.9 Under the conditions of Definition 3.3.8, there exists a 'canonical' -=morphism $(h, g) : h_*F \rightarrow F$; if $h$ is a $p$-morphism, then $(h, g)$ is a $p= -$-morphism.

Proof Put 

$$g_u := \text{id}_{D'_u} : D'_u \rightarrow D_{h(u)}.$$ 

Then $(h,g) : (F', D') \rightarrow (F, D)$ by Definition 3.3.1.

Now let us show that every morphism is represented as a specific composition:

Proposition 3.3.10 Every (=)-morphism of predicate Kripke frames $(f_0, f_1) : F' \rightarrow F$ can be presented as a composition of an (=)-morphism over the propositional base $F'$ of $F$ and the canonical morphism:

$$F' \xrightarrow{(id_{W'}, f_1)} (f_0)_*F \xrightarrow{(f_0, g)} F.$$ 

Moreover, $(id_{W'}, f_1)$ is a unique morphism $(id_{W'}, h_1)$ over $F'$ such that $(f_0, f_1) = (id_{W'}, h_1) \circ (f_0, g)$.

Proof In fact, 

$$(id_{W'}, h_1) \circ (f_0, g) = (f_0, f_1)$$ 

iff $id_{W'} \circ f_0 = f_0$ (which is true) and for any $u$

$$f_{1u} = h_{1u} \circ g_u = h_{1u} \circ id_{D'_u} = h_{1u},$$

i.e. iff $f_1 = h_1$.

Proposition 3.3.11 If $(f_0, f_1) : M \rightarrow^{(=)} M'$, then for any $u \in M$, $B \in MS_N^{(=)}(D_u)$

$$M, u \models B \iff M', f_0(u) \models f_{1u} \cdot B,$$

where $f_{1u} \cdot B$ is obtained from $B$ by replacing occurrences of every $a \in D_u$ with $f_{1u}(a)$.

If the models are intuitionistic, the same holds for the intuitionistic forcing and $B \in IS^{(=)}(D_u)$.
Proof  By induction on the complexity of $B$ we prove the claim for any $u$. Let us check only two cases

1. $B = \Box_j C$. Then

$$M, u \models B \iff \forall v \in R_j(u) M, v \models C \iff \forall v \in R_j(u) M', f_0(v) \models f_1 v \cdot C$$

by the induction hypothesis.

Since $f_0$ is a morphism, we have $R'_j(f_0(u)) = f_0[R_j(u)]$. By Definition 3.3.1, $f_{1u}(a) = f_{1v}(a)$ for any $a \in D_u$, $v \in R_j(u)$; hence $f_{1v} \cdot C = f_{1u} \cdot C$ for any $D_v$-sentence $C$.

So

$$M, u \models B \iff \forall v' \in R'_j(f_0(u)) M', v' \models f_{1u} \cdot C \iff M', f_0(u) \models \Box_j(f_{1u} \cdot C) = f_{1u} \cdot B).$$

2. $B = \exists x C(x, a)$, where $\exists x C(x, y)$ is a generator of $B$, $a$ is a tuple from $D_u$. Then

$$M, u \models B \iff \exists b \in D_u M, u \models C(b, a) \iff \exists b \in D_u M', f_0(u) \models C(f_{1u}(b), f_{1u} \cdot a) \text{ (by induction hypothesis)} \iff \exists c \in D'_{f_0(u)} M', f_0(u) \models C(c, f_{1u} \cdot a) \text{ (since } f_{1u}[D_u] = D'_{f_0(u)}) \iff$$

$$M', f_0(u) \models \exists x C(x, f_{1u} \cdot a).$$

The intuitionistic case now follows easily by Gödel–Tarski translation. ■

Proposition 3.3.12  Let $(f_0, f_1) : F \rightarrow F'$, and let $M'$ be a Kripke model over $F'$. Then there exists a unique model $M$ over $F$ such that $(f_0, f_1) : M \rightarrow M'$, and similarly for $\rightarrow, \rightarrow^-, \rightarrow^-$. If $M'$ is intuitionistic, then $M$ is also intuitionistic.

Proof  In fact, if $M' = (F', \xi')$, then $M = (F, \xi)$ is uniquely determined by the equalities

$$\xi_v(P^m_k) := \{b \in D^m_v \mid f_{1v} \cdot b \in \xi'_{f_0(v)}(P^m_k)\}$$

for any $v \in F$, $m > 0$, where

$$f_{1v} \cdot (b_1, \ldots, b_m) := (f_{1v}(b_1), \ldots, f_{1v}(b_m)),$$

and

$$\xi_v(P^0_k) := \begin{cases} \{v\} & \text{if } \xi'_{f_0(v)}(P^0_k) = \{f_0(v)\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now suppose $\xi'$ is intuitionistic. Then $vRu$ implies $f_0(v)R'f_0(u)$ and next $\xi'_{f_0(v)}(P^m_k) \subseteq \xi'_{f_0(u)}(P^m_k)$ (for $m > 0$), whence by definition

$$\xi_v(P^m_k) \subseteq \xi_u(P^m_k).$$

In the same way, we obtain

$$vRu \& v \in \xi_v(P^0_k) \Rightarrow u \in \xi_u(P^0_k).$$

Thus $\xi$ is intuitionistic. ■
Now let $M$ be a Kripke model over $\mathbf{F}$ and consider changing the base of $\mathbf{F}$.
By applying 3.3.12 to the canonical morphism $(h,g) : h\ast \mathbf{F} \rightarrow \mathbf{F}$, we obtain a unique Kripke model $M'$ over $h\ast \mathbf{F}$ such that $(h,g) : M' \rightarrow M$. We also say that $M'$ is obtained from $M$ by changing the base along $h$ and denote $M'$ by $h\ast M$.

**Proposition 3.3.13** If there exists a $p^{(=)}$-morphism $\mathbf{F} \rightarrow (=)\mathbf{F}'$, then $\mathbf{ML}^{(=)}(\mathbf{F}) \subseteq \mathbf{ML}^{(=)}(\mathbf{F}')$, and similarly, $\mathbf{IL}^{(=)}(\mathbf{F}) \subseteq \mathbf{IL}^{(=)}(\mathbf{F}')$ for intuitionistic Kripke frames.

**Proof** Consider the modal case only. Let $(f_0,f_1) : \mathbf{F} \rightarrow \mathbf{F}'$, and assume that $\mathbf{F}' \not\vdash A(x_1,\ldots,x_n)$. So by Lemma 3.2.20, for some $u' \in \mathbf{F}'$, $a_1,\ldots,a_n \in D'_{u'}$, for some model $M'$ over $\mathbf{F}'$ we have

$$M', u' \not\vdash A(a_1',\ldots,a_n').$$

By Proposition 3.3.12, there exists a model $M$ over $\mathbf{F}$ such that

$$(f_0,f_1) : M \rightarrow M'.$$

Now since $f_0$ and $f_1u$ are surjective, there exist $u,a_1,\ldots,a_n$, such that

$$u' = f_0(u), \quad a_1' = f_1u(a_1), \ldots, \quad a_n' = f_1u(a_n),$$

and thus by Lemma 3.3.11, we obtain $M,u \not\vdash A(a_1,\ldots,a_n)$.

Therefore, $\mathbf{F}' \not\vdash A$ implies $\mathbf{F} \not\vdash A$. \hfill \blacksquare

**Proposition 3.3.14** If there exists a $p$-morphism $\mathbf{F}' \rightarrow \mathbf{F}$, then $\mathbf{ML}^{(=)}(\mathbf{KF}') \subseteq \mathbf{ML}^{(=)}(\mathbf{KF})$

(and $\mathbf{IL}^{(=)}(\mathbf{KF}') \subseteq \mathbf{IL}^{(=)}(\mathbf{KF})$ in the intuitionistic case).

**Proof** Consider the modal case. Let $h : \mathbf{F}' \rightarrow \mathbf{F}$. For any $\mathbf{F} \in \mathbf{KF}$ we have

$$\mathbf{ML}^{(=)}(\mathbf{KF}') \subseteq \mathbf{ML}^{(=)}(\mathbf{KF}) \subseteq \mathbf{ML}^{(=)}(h\ast \mathbf{F}) \subseteq \mathbf{ML}^{(=)}(\mathbf{F})$$

by Lemma 3.3.9 and Proposition 3.3.13. Hence

$$\mathbf{ML}^{(=)}(\mathbf{KF}') \subseteq \mathbf{ML}^{(=)}(\mathbf{KF}).$$

\hfill \blacksquare

**Definition 3.3.15** Let $\mathbf{F} = (W,R_1,\ldots,R_N)$ be a propositional Kripke frame, $\mathbf{F} = (F,D)$ a predicate Kripke frame, $V \subseteq W$. A subframe of $\mathbf{F}$ obtained by restriction to $V$ is defined as

$$\mathbf{F} \upharpoonright V := (\mathbf{F} \upharpoonright V,D \upharpoonright V),$$

where $D \upharpoonright V := (D_u)_{u \in V}$. If $M = (\mathbf{F},\xi)$ is a Kripke model, we define the submodel

$$(\mathbf{F} \upharpoonright V,\xi \upharpoonright V),$$

where

$$(\xi \upharpoonright V)(P^m_k) := (\xi_\upsilon(P^m_k))_{\upsilon \in V}.$$ 

If $V$ is stable, the subframe $\mathbf{F} \upharpoonright V$ and the submodel $M \upharpoonright V$ are called generated.

\[12\text{Cf. [Ono, 1972/73], Theorem 3.4.}\]
The notation $M_1 \subseteq M$ means that $M_1$ is a submodel of $M$.

**Definition 3.3.16** A submodel $M_1 \subseteq M$ is called reliable if for any $u \in M_1$, for any $D_u$-sentence $A$

$$M_1, u \vdash A \iff M, u \vdash A.$$ 

In this case obviously, $MT(M) \subseteq MT(M_1)$.

**Definition 3.3.17** Similarly to the propositional case (Definition 1.3.14), we define cones, rooted frames and rooted models:

$$F \uparrow u := F \uparrow (W \uparrow u), \ M \uparrow u := M \uparrow (W \uparrow u).$$

**Lemma 3.3.18 (Generation lemma)** Let $F$ be a predicate Kripke frame, $M$ a Kripke model over $F$, $V$ a stable set of worlds in $F$. Then

1. there is a morphism $(j, i) : F \uparrow V \rightarrow F$ (and $M \uparrow V \rightarrow M$), in which $j : V \rightarrow W$ is the inclusion map and every $i_u$ is the identity map on the corresponding domain;
2. $M \uparrow V$ is a reliable submodel of $M$;
3. $ML(=)(F) \subseteq ML(=)(F \uparrow V)$; similarly, for the intuitionistic case.

**Proof**

(1) Obvious.

(2) Apply Lemma 3.3.11 and (1).

(3) Every valuation $\xi'$ in $F \uparrow V$ equals $\xi \uparrow V$ for a valuation $\xi$ in $F$ such that $\xi_u = \xi'_u$ whenever $u \in V$. Such a $\xi$ obviously exists, e.g. put

$$\xi'_u(P) := \begin{cases} 
\xi'_u(P) & \text{if } u \in V, \\
\varnothing & \text{otherwise.}
\end{cases}$$

If $\xi'$ is intuitionistic, then $\xi$ is also intuitionistic, since $V$ is stable. Now the claim follows from (2).

**Definition 3.3.19** If $f : F \rightarrow F'$ is a morphism of Kripke frames and $V$ is stable in $F$, then $f \uparrow V := (j, i) \circ f$, where $(j, i)$ is a morphism from 3.3.18(1), is called the restriction of $f$ to $V$. Restrictions of Kripke model morphisms are defined in the same way.

**Lemma 3.3.20** A restriction of an $(\approx)$-morphism (of Kripke frames or models) to a generated subframe (or submodel) is an $(\approx)$-morphism (respectively, of frames or models).
3.3. MORPHISMS OF KRIPKE FRAMES

If \( G = F \upharpoonright V \), we also denote \( f \upharpoonright V \) by \( f \upharpoonright G \).

Lemma 3.3.21

(1) \( ML(\leq)(F) = \bigcap_{u \in F} ML(\leq)(F \upharpoonright u) \)

and analogously in the intuitionistic case.

(2) Every Kripke complete modal or superintuitionistic logic is determined by a class of rooted predicated Kripke frames:

\[ L(\leq)(C) = L(\leq)(C \upharpoonright), \]

where \( L \) is \( ML \) or \( IL \),

\[ C \upharpoonright := \{ F \upharpoonright u \mid F \in C, \ u \in F \}. \]

Proof

(1) Similar to Lemma 1.3.26 (an exercise).

(2) Follows from (1).

\[ \square \]

Definition 3.3.22

The notions 'path', 'connectedness', 'component', 'non-oriented path' are obviously extended to the predicate case. Viz., a predicate Kripke frame is called connected if its propositional base is connected; a non-oriented path in \((F, D)\) is the same as in \(F\); a component in \((F, D)\) is its restriction to a component in \(F\).

Now we have a predicate version of 1.3.38.

Proposition 3.3.23

Let \((F_i \mid i \in I)\) be a family of all different components of a predicate Kripke frame \(F\).

Then

(1) for any morphism \( f : F \rightarrow G \), every \( f \upharpoonright F_i \) is a morphism;

(2) for any family of morphisms \( f_i = (g_i, h_i) : F_i \rightarrow G \) there is a joined morphism \( \bigcup_{i \in I} f_i : F \rightarrow G \) defined as \((g, h)\), with

\[ g := \bigcup_{i \in I} g_i, \ h := (h_u)_{u \in F}, h_u := (h_i)_u \text{ for } u \in F_i; \]

(3) every morphism \( f : F \rightarrow G \) is presented as \( \bigcup_{i \in I} (f \upharpoonright F_i) \).

Proof

(1) Follows from Lemma 3.3.20.
(2) \( g \) is a morphism of propositional bases, by Proposition 1.3.38. Every \( h_u \) is surjective, since it coincides with some \((h_i)_u\). Finally, if \( u \in F_i, uR_jv, \) then \( v \in F_i, \) thus for any \( a \in D_u \)
\[ h_u(a) = (h_i)_u(a) = (h_i)_v(a) = h_v(a), \]
i.e. \( h_u = h_v \upharpoonright D_u. \)

(3) Trivial, by definition.

We write \( F_1 \rightarrow F_2 \) to denote that there is a p-morphism from \( F_1 \) onto \( F_2 \). Similarly to the propositional case, we give

**Definition 3.3.24** A predicate Kripke frame \( F_1 \) is called \((=)-\)reducible to a rooted predicate Kripke frame \( F_2 \) if there exists \( u \in F_1 \) such that
\[ F_1 \uparrow u \rightarrow (=) F_2. \]

**Definition 3.3.25** Let \( C_1, C_2 \) be classes of predicate Kripke frames. We say that \( C_1 \) is \((=)-\)reducible to \( C_2 \) if for any \( F_2 \in C_2 \) for any \( v \in F_2 \) there exists \( F_1 \in C_1 \) that is \((=)-\)reducible to \( F_2 \uparrow v. \)

In the similar way reducibility is defined for classes of propositional Kripke frames.

\( C_1 \text{ red}(=) C_2 \) denotes that \( C_1 \) is \((=)-\)reducible to \( C_2 \). It is clear that \( =-\)reducibility implies reducibility.

**Proposition 3.3.26** If \( C_1 \text{ red}(=) C_2 \) for classes \( C_1, C_2 \) of predicate Kripke frames, then \( L(=)(C_1) \subseteq L(=)(C_2) \) (where \( L \) denotes ML in the modal case and IL in the intuitionistic case).

**Proof** Suppose \( C_1 \text{ red}(=) C_2 \). Then for any \( F_2 \in C_2, v \in F_2 \) there is \( F_1 \in C_1, u \in F_1 \) such that \( F_1 \uparrow u \rightarrow (=) F_2 \uparrow v. \) Hence by 3.3.18 and 3.3.13
\[ L(=)(F_1) \subseteq L(=)(F_1 \uparrow u) \subseteq L(=)(F_2 \uparrow v), \]
and thus \( L(=)(F_1) \subseteq L(=)(F_2) \) by 3.3.21. Therefore \( L(=)(C_1) \subseteq L(=)(C_2) \) for any \( F_2 \in C_2, \) which implies \( L(=)(C_1) \subseteq L(=)(C_2). \)

**Lemma 3.3.27** Let \( F \) be a propositional Kripke frame, \( u \in F. \) Then
\[ \mathcal{K}(F \uparrow u) = \{ F \uparrow u \mid F \in \mathcal{K}F \}. \]
We shall denote the latter class by \((\mathcal{K}F) \uparrow u. \)

**Proof** The inclusion \((\mathcal{K}F) \uparrow u \subseteq \mathcal{K}(F \uparrow u)\) is trivial. The other way round, if \( F' = (F \uparrow u, D'), \) then \( F' = F \uparrow u \) for the frame \( F = (F, D), \) where
\[ D_v := \begin{cases} D'_v & \text{if } v \in F \uparrow u, \\ D_u & \text{otherwise.} \end{cases} \]
The system of domains \( D \) is expanding, since by 3.2.3, \( D'_u \subseteq D'_v \) for any \( v \in F \uparrow u. \) Thus \( F' \in (\mathcal{K}F) \uparrow u. \)
3.3. MORPHISMS OF KRIPKE FRAMES

Proposition 3.3.28 Let \( \mathcal{C} \) be a class of \( N \)-modal propositional Kripke frames,
\[
\mathcal{C} \uparrow := \{ F \uparrow u \mid u \in F, \ F \in \mathcal{C} \}.
\]
Then
\[
(1) \quad \mathcal{K}(\mathcal{C} \uparrow) = (\mathcal{K}\mathcal{C}) \uparrow = \{ F \uparrow u \mid u \in F, \ F \in \mathcal{K}\mathcal{C} \}
\]
\[
(2) \quad \mathbf{ML}(\mathcal{C}) = \mathbf{ML}(\mathcal{K}(\mathcal{C} \uparrow)),
\]
\[
(3) \quad \mathbf{IL}(\mathcal{C}) = \mathbf{IL}(\mathcal{K}(\mathcal{C} \uparrow)) \text{ if } \mathcal{C} \text{ is a class of S4-frames.}
\]
\textbf{Proof} (1) In fact,
\[
\mathcal{K}(\mathcal{C} \uparrow) = \bigcup \{ \mathcal{K}(F \uparrow) \mid u \in F, \ F \in \mathcal{C} \},
\]
\[
(\mathcal{K}\mathcal{C}) \uparrow = \bigcup \{ (\mathcal{K}F) \uparrow u \mid u \in F, \ F \in \mathcal{C} \},
\]
and we can apply 3.3.27.

(2), (3) follow from (1) and 3.3.21(2).

Lemma 3.3.29 Let \( F_1, F_2 \) be propositional Kripke frames such that \( F_1 \) red \( F_2 \).
Then \( (\mathcal{K}F_1) \) red\(\rightarrow\) \( (\mathcal{K}F_2) \).
\textbf{Proof} By assumption, there exists \( h : F_1 \uparrow u \rightarrow F_2 \) for some \( u \). So by 3.3.9, for any \( F_2 \in \mathcal{K}F_2 \) there is a canonical
\[
\gamma : h_*F_2 \rightarrow \rightarrow F_2,
\]
and \( h_*F_2 \in \mathcal{K}(F_1 \uparrow u) = (\mathcal{K}F_1) \uparrow u \) (by 3.3.27). Since \( h \) is surjective, for any \( v \in F_2 \), there is \( w \in F_1 \uparrow u \) such that \( h(w) = v \). Then by 1.3.32, the restriction of \( h \) is a \( \mathbf{p}\)-morphism
\[
F_1 \uparrow w = (F_1 \uparrow u) \uparrow w \rightarrow F_2 \uparrow v,
\]
and it follows that
\[
(h_*F_2) \uparrow w \rightarrow F_2 \uparrow v
\]
by the restriction of \( \gamma \). Since \( (h_*F_2) \uparrow w \in (\mathcal{K}F_1) \uparrow w \), we obtain \( (\mathcal{K}F_1) \) red\(\rightarrow\) \( (\mathcal{K}F_2) \). ■

Proposition 3.3.30 If \( \mathcal{C}_1 \) red \( \mathcal{C}_2 \) for classes \( \mathcal{C}_1, \mathcal{C}_2 \) of propositional Kripke frames, then \( (\mathcal{K}\mathcal{C}_1) \) red\(\rightarrow\) \( (\mathcal{K}\mathcal{C}_2) \).
\textbf{Proof} Let \( F_2 \in \mathcal{K}\mathcal{C}_2 \), i.e. \( F_2 \in \mathcal{K}F_2 \) for some \( F_2 \in \mathcal{C}_2 \); then \( F_2 \uparrow v \in (\mathcal{K}F_2) \uparrow v = \mathcal{K}(F_2 \uparrow v) \). Since \( \mathcal{C}_1 \) red \( \mathcal{C}_2 \), there exists \( F_1 \in \mathcal{C}_1 \) such that \( F_1 \) red \( (F_2 \uparrow v) \).

By Lemma 3.3.29, \( (\mathcal{K}F_1) \) red\(\rightarrow\) \( (\mathcal{K}F_2) \); thus \( F_1 \) red \( (F_2 \uparrow v) \) for some \( F_1 \in \mathcal{K}F_1 \subseteq \mathcal{K}\mathcal{C}_1 \). Therefore \( (\mathcal{K}\mathcal{C}_1) \) red\(\rightarrow\) \( (\mathcal{K}\mathcal{C}_2) \). ■

Corollary 3.3.31 If \( \mathcal{C}_1 \) red \( \mathcal{C}_2 \) for classes \( \mathcal{C}_1, \mathcal{C}_2 \) of propositional Kripke frames, then \( \mathbf{ML}(\mathcal{C}_1) \subseteq \mathbf{ML}(\mathcal{C}_2) \) (and \( \mathbf{IL}(\mathcal{C}_1) \subseteq \mathbf{IL}(\mathcal{C}_2) \) for S4-frames).
Definition 3.3.32 We say that a Kripke frame \((F, D')\) is obtained by domain restriction from \((F, D)\) if for some \(V \subseteq D^+\), \(D'_u = D_u \cap V\).

In this case we denote \((F, D')\) by \((F, D) \cap V\).

Obviously this definition is sound iff \(V \cap D_u \neq \emptyset\) for any \(u \in F\).

Proposition 3.3.33 \((F, D) \text{ red } (F, D) \cap V\).

Proof If \(F\) has a root \(v\), there exists a \(p\)-morphism over \(F\)

\[(id_F, g) : (F, D) \to (F, D') = (F, D) \cap V.\]

In fact, we can define \(g_u\) as an identity map on \(D'_u\) sending every \(a \notin V\) to some fixed element of \(V\). Then obviously, \(g_u \upharpoonright D_u = g_u\) whenever \(uR_iv\).

Definition 3.3.34 Let \((F_i)_{i \in I}\) be a family of predicate Kripke frames, \(F_i = (F_i, D_i)\). The disjoint sum (or disjoint union) of the family \((F_i)_{i \in I}\) is the frame

\[
\biguplus_{i \in I} F_i := \left( \biguplus_{i \in I} F_i, D \right),
\]

where

\[D_{(u,i)} := (D_i)_u \times \{i\}\]

for \(i \in I, u \in F_i\).

If for each \(i \in I, M_i = (F_i, \theta_i)\) is a Kripke model over \(F_i\), the disjoint sum (union) of \((M_i)_{i \in I}\) is the Kripke model

\[
\biguplus_{i \in I} M_i := \left( \biguplus_{i \in I} F_i, \theta \right),
\]

where

\[\theta_{(u,i)}(P) := \{(a_1, i), \ldots , (a_n, i) \mid (a_1, \ldots , a_n) \in (\theta_i)_u(P)\}\]

for \(P \in PL^n, n > 0\) and

\[\theta_{(u,i)}(P) := \{\{(u,i)\}\} \text{ if } (\theta_i)_u(P) = \{u\}, \]

\[\emptyset \text{ otherwise}\]

for \(P \in PL^0\).

Obviously, there exists an isomorphism from \(F_k\) (respectively, \(M_k\)) onto a generated subframe in \(\biguplus_{i \in I} F_i\) (respectively, \(\biguplus_{i \in I} M_i\)), given by the pair \((f_0, f_1)\) such that

\[f_0(u) := (u, k), f_{1u}(a) := (a, k).\]
In particular, the definition of $\theta_{(u,i)}$ yields

$$\bigsqcup_{i \in I} M_i, (u, k) \models P(f_1u(a_1), \ldots, f_1u(a_n)) \text{ iff } M_k, u \models P(a_1, \ldots, a_n).$$

Now we have an analogue of 1.3.37.

**Proposition 3.3.35** Let $(F_i \mid i \in I)$ be a family consisting of all different components of a predicate Kripke frame $F$. Then $F \cong \bigsqcup_{i \in I} F_i$.

**Proof** A required isomorphism is the map $f$ from the proof of 1.3.37 together with the family $(id_{D_u})_{u \in F}$ (as usual, we assume that $F = (F, D)$).

**Proposition 3.3.36**

1. $\mathbb{ML}^{(=)}\left(\bigsqcup_{i \in I} F_i\right) = \bigcap_{i \in I} \mathbb{ML}^{(=)}(F_i)$, $\mathbb{MT}^{(=)}\left(\bigsqcup_{i \in I} M_i\right) = \bigcap_{i \in I} \mathbb{MT}^{(=)}(M_i)$ for the modal case.

2. $\mathbb{IL}^{(=)}\left(\bigsqcup_{i \in I} F_i\right) = \bigcap_{i \in I} \mathbb{IL}^{(=)}(F_i)$, $\mathbb{IT}^{(=)}\left(\bigsqcup_{i \in I} M_i\right) = \bigcap_{i \in I} \mathbb{IT}^{(=)}(M_i)$ for the intuitionistic case.

**Proof** Similar to Lemma 1.3.28 using Lemma 3.3.21; a simple exercise for the reader.

**Corollary 3.3.37** Kripke frame semantics $K^{(=)}_N$, $K^{(=)}_{\text{int}}$ have the Collection Property (cf. Definition 2.16.9).

**Proposition 3.3.38** Every modally or intuitionistically definable class of predicate Kripke frames is closed under generated subframes, disjoint sums and $p^{(=)}$-morphic images.

**Proof** Validity is preserved for generated subframes by 3.3.18, for $p^{(=)}$-morphic images by 3.3.13, for disjoint sums by 3.3.36.

Finally let us prove a predicate analogue of Lemma 1.3.44.

**Lemma 3.3.39** Let $L$ be a conically expressive $N$-m.p.l.$(=)$. Then for any $N$-modal Kripke model $M$ such that $M \models L$ for any $u \in M$, for any $D_u$-sentence $A$

$$M, u \models \Box^* A \iff \forall v \in R^*(u) M, v \models A,$$

where $R^*$ is the same as in Lemma 1.3.19.

---

13$p^{(=)}$-morphic images — if the class is definable by a set of formulas with equality.
Proof. Similar to 1.3.44 and the soundness of the substitution rule in 3.2.29. Let $M_0$ be a propositional model over the same propositional frame as $M$, such that for any $v \in M$

$$M_0, v \models p \iff M, v \models A.$$  

Then by induction we obtain for any propositional formula $X(p)$ for any $v \in M$.

$$M_0, v \models X(p) \iff M, v \models X(A).$$

Hence for any $u \in M$ and $C$ from 1.3.42

$$M_0, u \models C(p) \iff M, u \models C(A) \left(= \Box^* A \right)$$

and so by 1.3.42,

$$M, u \models \Box^* A \iff M_0 \upharpoonright u \models p.$$ But by Lemma 1.3.19 and the choice of $M_0$

$$M_0 \upharpoonright u \models p \iff \forall v \in R^*(u) \; M_0, v \models p \iff \forall v \in R^*(u) \; M, v \models A.$$ Hence the claim follows.

3.4 Constant domains

Definition 3.4.1. A predicate Kripke frame $F$ has a constant domain (or $F$ is a CD-frame) iff $D_u = D_v$ for any $u, v \in F$.

A CD-frame $(F, D)$, in which $V = D_u$ for all $u \in F$, is denoted by $F \otimes V$.

Proposition 3.4.2

(1) A rooted $N$-modal predicate Kripke frame $F$ has a constant domain iff $\forall i \in I_N \; F \models B_{ai}$.

(2) A rooted intuitionistic Kripke frame $F$ has a constant domain iff $F \models CD$.

Proof. (1) (Only if.) Suppose $M$ is a Kripke model over $F$ and $M, u \models \Diamond_i \exists x P(x)$. Then $M, v \models P(a)$ for some $v \in R_i(u), a \in D_v = D_u$. Hence $M, u \models \Diamond_i P(a)$ and thus $M, u \models \exists x \Diamond_i P(x)$.

(If.) Let $u_0$ be a root of $F$. If $F$ does not have a constant domain, then $D_u \neq D_v$ for some $u, v \in F$, and thus $D_u \neq D_{u_0}$ or $D_v \neq D_{u_0}$. Consider the first option. Then there exists a path $u_0 R_{i_1} u_1, \ldots, u_k R_{i_k} u_k = u$, so $D_{u_0} \subseteq D_{u_1} \subseteq \ldots \subseteq D_u$. It follows that for some $k$ $D_{u_k} \subset D_{u_{k+1}}$.

Thus for some $i, u, v$ we have $D_u \subset D_v$, $v \in R_i(u)$. Let $a_0 \in (D_v - D_u)$. Consider a model $M = (F, \xi)$ such that (for a certain $P \in PL^1$)

$$\xi_u(P) := \begin{cases} \{a_0\} & \text{if } a_0 \in D_w, \\ \emptyset & \text{otherwise.} \end{cases}$$
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Then we have $M, v \models \exists x P(x)$, and thus $M, u \models \Diamond \exists x P(x)$; but $M, u \not\models \exists x \Diamond P(x)$, since $a_0 \notin D_u$.

(2) Let $R$ be the relation in $F$.

(Only if.) If $M$ is an intuitionistic Kripke model over $F$ and $M, u \not\models \forall x (P(x) \lor q)$, but $M, u \not\models q$ and $M, u \not\models \forall x P(x)$, then $M, v \not\models P(a)$ for some $v \in R(u)$, $a \in D_v = D_u$. Hence $M, u \not\models \forall x (P(x) \lor q)$, which contradicts $M, u \models \forall x (P(x) \lor q)$ and $u Ru$.

(If.) Let $u$ be root of $F$, and suppose $D_u \neq D_v$ for some $v \in W = R(u)$. Since $u Ru$, we have $D_u \subseteq D_v$, so there exists $a_0 \in (D_v - D_u)$. Consider the valuation $\xi$ in $F$ such that for any $w$

$$\xi_w(P) := D_u,$$

$$\xi_w(q) := \begin{cases} \emptyset & \text{if } w Ru, \\ \{w\} & \text{otherwise,} \end{cases}$$

for a certain $P \in \mathcal{P} L^1$, $q \in \mathcal{P} L^0$ and $\xi_w(Q) = \emptyset$ for all other predicate letters $Q$. It is clear that $\xi$ is intuitionistic. Under this valuation we have $u \not\models q$, $v \not\models P(a_0)$, and thus $u \not\models \forall x P(x) \lor q$; but $u \models \forall x (P(x) \lor q)$.

In fact, suppose $u R w$. If $w Ru$, then obviously $D_w = D_u$, and thus $w \models P(a)$ for any $a \in D_w$; on the other hand, $w \not\models q$.

Therefore $u \not\models CD$.

From Definition 3.4.1 it is clear that the class of CD-frames is closed under generated subframes.

**Definition 3.4.3** Modal and intuitionistic Kripke semantics with constant domains are generated by CD-frames:

$$\mathcal{C}K^{(\equiv)}_N := \{ \mathcal{M}L^{(\equiv)}(\mathcal{X}) \mid \mathcal{X} \text{ is a class of } N\text{-modal CD-frames} \};$$

$$\mathcal{C}K^{(\equiv)}_{int} := \{ \mathcal{I}L^{(\equiv)}(\mathcal{X}) \mid \mathcal{X} \text{ is a class of intuitionistic CD-frames} \}.$$  

The class of CD-frames is not closed under disjoint sums (the reader can easily construct a counterexample), but there is an equivalent semantics generated by a class with this property.

**Definition 3.4.4** A predicate Kripke frame $F = ((W, R_1, \ldots, R_N), D)$ is called a local CD-frame if it satisfies

$$(*) \quad \forall u \forall v \forall u (u R_i v \Rightarrow D_u = D_v).$$

**Lemma 3.4.5** (1) For an $N$-modal frame $F$, $F \models \bigwedge_{i \in I_N} Ba_i$ iff every cone $F \uparrow u$ has a constant domain iff $F$ is a local CD-frame.

(2) Similarly, for an intuitionistic $F$, $F$ is a local CD-frame iff $F \models CD$ iff all cones in $F$ have constant domains.

**Proof** Easily follows from 3.4.2.
Proposition 3.4.6

(1) The class of local CD-frames is closed under disjoint sums, generated subframes and p-morphic images.

(2) CD-frames and local CD-frames generate equivalent semantics.

Proof

(1) By 3.4.5 and 3.3.38 this class is modally (or intuitionistically) definable, so we can apply 3.3.38.

(2) Note that every CD-frame is a local CD-frame; on the other hand, if $\mathcal{X}$ is a class of (N-modal) local CD-frames, then by Lemma 3.3.21, and all the $F^\uparrow u$ are CD-frames.

Exercise 3.4.7 Show that the class of CD-frames is not closed under p-morphic images, but closed under $p^=\text{-morphic images.}$

Let us now describe morphisms of local CD-frames.

Lemma 3.4.8 Every connected local CD-frame is a CD-frame.

Proof In a local CD-frame, if $u$, $v$ are in the same component, then $D_u = D_v$. This easily follows by induction on the length of a non-oriented path from $u$ to $v$.

Proposition 3.4.9

(1) If $(f_0, f_1) : (F, D) \rightarrow (F', D')$ for a connected CD-frame $(F, D)$, then all the maps $f_{1u}$ (and certainly their targets $D'_{f_0(u)}$) for $u \in F$ coincide and $(F', D')$ is a connected CD-frame.

(2) Conversely, assume that $(F, D)$, $(F', D')$ are connected CD-frames, $f : F \rightarrow F'$ and $g : D_u \rightarrow D'_u$ is a surjective map. Then $(f, f_1) : (F, D) \rightarrow (F', D')$ for $f_1 = (g)_{u \in F}$.

Proof

(1) Again we show $f_{1u} = f_{1v}$ by induction on the length of a non-oriented path from $u$ to $v$. It suffices to consider the case when $uR_iv$. Then $f_{1u} = f_{1v} \uparrow D_u$, so $f_{1u} = f_{1v}$ as functions, since $D_u = D_v$. The targets coincide, due to the surjectivity.

We know from Lemma 1.3.39 that $F'$ is connected. By 3.4.6, $(F', D')$ is local CD, so it is CD by 3.4.8.
(2) Trivial by definition.

The p-morphism described in 3.4.9 is briefly denoted by \( f \circ g \).

**Corollary 3.4.10** Let \( F \) be a local CD-frame, \((F_i \mid i \in I)\) a family of all its different components. Then the p-morphisms \( F \to G \) are exactly the maps of the form \( \bigcup_{i \in I} f_i \), where every \( f_i : F_i \to G \) (as a p-morphism onto its image) has the form described in 3.4.9.

**Proof** Note that every \( F_i \) is a CD-frame by 3.4.8 and apply Proposition 3.3.23.

Every Kripke frame over a propositional frame \( F \) is reducible to some Kripke frame \( F \odot V \); for example, with a singleton \( V \). More precisely, the following proposition holds (its intuitionistic version was proved in [Ono, 1972/73]).

**Proposition 3.4.11** Let \( F = (F,D) \) and \( F' = F' \odot V \) be predicate Kripke frames such that \( F \) is reducible to \( F' \) and \( |D_u| \geq |V| \) for any \( u \in F \).

Then \( F \) is reducible to \( F' \) and thus\(^{14}\) \( ML(F) \subseteq ML(F') \) (and \( IL(F) \subseteq IL(F') \) for the intuitionistic case).

**Proof** Assuming \( h : F \uparrow v \to F' \uparrow u \), let us show

\[
F \uparrow v \to F' \uparrow u.
\]

To simplify notation, we assume that \( F = F \uparrow v \) and \( F' = F' \uparrow u \). Let \( a_0 \) be a fixed element of \( V \), \( g_v : D_v \to V \) a surjective map. For every \( u \in F \) consider the following surjective function \( g_u : D_u \to V \).

\[
g_u(a) := \begin{cases} g_v(a) & \text{if } a \in D_v, \\ a_0 & \text{otherwise.} \end{cases}
\]

It is clear that \( uR_i u' \) implies \( g_u = g_{u'} \upharpoonright D_u \), so we obtain \((h, g) : F \to F'\), and thus \( ML(F) \subseteq ML(F') \) by Proposition 3.3.13.

**Remark 3.4.12** Proposition 3.4.11 is not transferred to the case with equality. In fact, 3.4.11 implies that

\[
ML(F \odot V) \subseteq ML(F' \odot V') \text{ if } |V| \geq |V'|.
\]

But a similar assertion does not hold for logics with equality (even for \textit{infinite} \( V' \)). In fact, consider the following formula [Skvortsov, 1989]:

\[
A_0 := \exists x \exists y \ (x \neq y \land (P(x) \equiv P(y))).
\]

\(^{14}\)By Proposition 3.3.26.
Lemma 3.4.13

\[ F \odot V \models A_0 \iff \forall u \in F \left| HA(F \upharpoonright u) \right| \geq |V|. \]

Proof Let \( M \) be an intuitionistic model over \( F \odot V \) and let \( u \in F \). Consider the sets

\[ \Xi_a := \{ v \in F \upharpoonright u \mid M, v \models P(a) \} \]

for \( a \in V \). It is clear that \( \Xi_a \in HA(F \upharpoonright u) \).

(If.) If \( |HA(F \upharpoonright u)| \not\geq |V| \) for every \( u \in F \), then \( \Xi_a = \Xi_b \) for some \( a \neq b \) (from \( V \)) — otherwise the map \( a \mapsto \Xi_a \) embeds \( V \) is \( HA(F \upharpoonright u) \). Then \( M, u \models a \neq b \land (P(a) \equiv P(b)) \), and thus \( M, u \not\models A_0 \).

(Only if.) Suppose \( |V| \leq |HA(F \upharpoonright u)| \) for some \( u \in F \). Let \( h : V \rightarrow HA(F \upharpoonright u) \) be an injection, and consider a model \( M \) over \( F \) such that for any \( w \)

\[ M, w \models P(a) \iff w \in h(a). \]

Then \( \Xi_a = h(a) \), and thus \( \Xi_a \neq \Xi_b \), whenever \( a \neq b \). Hence \( M, u \not\models P(a) \equiv P(b) \) whenever \( a, b \in V, a \neq b \), and thus \( M, u \not\models A_0 \).

Therefore, if a p.o. set \( F \) contains an infinite cone \( F \upharpoonright u \) and \( V' \) is countable, but \( |V| > 2^{\left| F \right|} \), we have \( A_0 \in (IL^= (F \odot V) - IL^= (F \odot V')) \).

3.5 Kripke frames with equality

3.5.1 Introduction

Let us first make some informal comments about interpreting equality. In Kripke frame semantics this interpretation is the simplest (Definition 3.2.9):

\[ u \models a = b \iff a \text{ equals } b. \]

But does this properly correspond to our intuitive understanding of equality?

In fact, we should evaluate \( a = b \) with respect to a certain world, so it might happen that \( a \) and \( b \) are the same in a world \( u \), but different in another world. Examples of this kind are quite popular in literature.

But as mentioned in the Introduction to Part II, there are more formal reasons for modifying the notion of predicate Kripke frame. For example, Lemma 3.10.5 (see below) shows that the principle of decidable equality \( DE \) is valid in any intuitionistic Kripke frame. But from the intuitionistic point of view, equality may be undecidable. In particular, this happens in intuitionistic analysis — for real numbers or functions. So we should choose the interpretations of equality appropriately.

In classical model theory there exist two ways of dealing with equality. The standard way is to interpret equality as coincidence of individuals ('normal

\[ ^{15} \text{Of course, we can replace } \geq \text{ with } < \text{ if we accept Axiom of Choice.} \]

\[ ^{16} \text{Recall that } M, u \not\models a \neq b \iff a, b \text{ are different.} \]
models’). The second way is to interpret equality as an equivalence relation preserving values of basic predicates. It is well-known that in classical logic these two approaches are equivalent, because we can always take the quotient domain modulo the equivalence relation corresponding to equality, and obtain a logically equivalent ‘normal’ interpretation.

Kripke semantics for intuitionistic and modal logics corresponds to classical ‘normal models’, but the second approach also makes sense. So we can interpret equality in a predicate Kripke frame as an equivalence relation on every individual domain $D_u$. Thus we obtain the notion of a predicate Kripke frame with equality ($KFE$) and the corresponding semantics $KE$. This semantics is stronger than the semantics of Kripke frames $K$ for formulas with equality — the crucial formula $DE$ can be refuted in a KFE. Moreover, as we shall see, $KE$ is stronger than $K$ for logics without equality.

We will also describe an equivalent semantics of ‘Kripke sheaves’. Kripke sheaves are obtained from KFEs by taking quotients of individual domains through the corresponding equivalence relation.

### 3.5.2 Kripke frames with equality

**Definition 3.5.1** A predicate Kripke frame with equality ($KFE$) is a triple $(F, D, \succeq)$, in which $(F, D)$ is a predicate Kripke frame and $\succeq$ is a valuation for the binary predicate symbol ‘$\equiv$’ satisfying the standard equality axioms. In more detail, if $F = (W, R_1, \ldots, R_N)$, then $\succeq = (\succeq_u)_{u \in W}$ is a family of equivalence relations ($\succeq_u \subseteq D_u \times D_u$), which is $R_i$-stable for every $i \in I_N$:

$$u R_i v \land a \succeq_u b \Rightarrow a \succeq_v b.$$

The following lemma gives an alternative definition of KFEs presenting them as a kind of $\Omega$-sets (see Chapter 4).

**Lemma 3.5.2** (1) Let $(F, D, \succeq)$ be a KFE, $F = (W, R_1, \ldots, R_N)$. Consider the function $E : D^+ \times D^+ \rightarrow 2^W$, such that for any $a, b \in D^+$

$$E(a, b) = \{u \mid a \succeq_u b\}.$$

Then the following holds:

(E1) $E(a, b) = E(b, a);$  

(E2) $E(a, b) \cap E(b, c) \subseteq E(a, c);$  

(E3) $\bigcup_{a \in D^+} E(a, a) = W;$  

(E4) $R_i(E(a, b)) \subseteq E(a, b)$ for $i \in I_N$.

(2) Given a propositional Kripke frame $F$, a non-empty set $D^+$, and a function $E$ satisfying (E1)–(E4), we can uniquely restore the corresponding KFE as $(F, D, \succeq)$, with

$$D_u := \{a \in D^+ \mid u \in E(a, a)\}, \succeq_u := \{(a, b) \mid u \in E(a, b)\}.$$
Proof

(1) (E1), (E2) follow respectively from the symmetry and the transitivity of $\succsim_u$. The reflexivity of $\succsim_u$ implies (E3) — it follows that $u \in E(a,a)$ for $a \in D_u$, and such an $a$ exists since $D_u \neq \emptyset$. (E4) follows from the $R_i$-stability of $\succsim$.

(2) Note that

$E(a,b) = E(a,b) \cap E(b,a) \subseteq E(a,a)$

by (E1), (E2) and similarly, $E(a,b) \subseteq E(b,b)$. Thus $u \in E(a,b)$ implies $u \in E(a,a) \cap E(b,b)$, i.e. $\succsim_u \subseteq D_u \times D_u$.

(E1) implies the symmetry of $\succsim_u$, and (E2) implies the transitivity.

(E3) implies the non-emptiness of every $D_u$. The reflexivity of $\succsim_u$ is obvious.

The $R_i$-stability of $\succsim$ follows from (E4).

$E(a,b)$ is called the measure of identity of $a$ and $b$. $E(a,a)$ is called the measure of existence (or the extent) of $a$, and is also denoted by $E(a)$.

Definition 3.5.3 A KFE-model over a Kripke frame with equality $F = (F, D, \succsim)$ is a pair $M = (F, J)$, where $J$ is a valuation in $(F, D)$ respecting the relations $\succsim_u$, i.e. such that for every $P \in PL^n, n \geq 1$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in D^+$

$\ (a_1, \ldots, a_n) \in J(P) \land a_1 \succsim_u b_1 \land \ldots \land a_n \succsim u b_n \Rightarrow (b_1, \ldots, b_n) \in J(P).$

Then we define forcing $M, u \models A$ (for $u \in F$ and a $D_u$-sentence $A$) by the same conditions as in Definition 3.2.9, with the following difference:

- $M, u \models a = b$ iff $a \succsim_u b$.

A KFE-model $M$ is called intuitionistic if the corresponding model without equality $(F, \xi)$ is intuitionistic. In this case we define forcing $M, u \models A$ according to Definition 3.2.12 with the only difference:

- $M, u \models a = b$ iff $a \succsim u b$.

A modal predicate formula $A$ is called true in $M$ if $\forall A$ is true at every world of $M$; similarly for the intuitionistic case.

Definition 3.5.4 A modal (respectively, intuitionistic) predicate formula $A$ is called valid in a KFE (respectively, S4-based KFE) $F$ iff it is true in all KFE-models (respectively, intuitionistic KFE-models) over $F$.

Again we use the notation $M \models A$, $F \models A$ for the modal case; $M \models A$, $F \models A$ for the intuitionistic case.

Lemma 3.2.11 obviously transfers to KFE-models. Lemma 3.2.13 has the following analogue.
Lemma 3.5.5

- $M, u \models \neg B \iff \forall v \in R(u) M, v \not\models B$;
- $M, u \models a \neq b \iff \forall v \in R(u) a \neq v b$.

We also have an analogue of 3.2.16:

Lemma 3.5.6 For an intuitionistic model $M$ and $D_u$-sentence $A$

$$M, u \models A \& uRv \Rightarrow M, v \models A$$

Next we obtain analogues of 3.2.17, 3.2.18, 3.2.20 by the same arguments.

Lemma 3.5.7

(1) Let $M$ be an $N$-modal KFE-model, $A(x)$ an $N$-modal formula with $FV(A) = r(x)$, $|x| = n$. Then

(i) for any $u \in M$

$$M, u \models \forall x A(x) \iff \forall a \in D_u^n M, u \models A(a),$$

(ii) $M \models A(x) \iff \forall u \in M \forall a \in D_u^n M, u \models A(a)$.

(2) Let $M$ be an intuitionistic KFE-model with the accessibility relation $R$, $A(x)$ an intuitionistic formula with $FV(A) = r(x)$, $|x| = n$. Then

(i) for any $u \in M$

$$M, u \models \forall x A(x) \iff \forall v \in R(u) \forall a \in D_u^n M, v \models A(a),$$

(ii) $M \models A(x) \iff \forall u \in M \forall a \in D_u^n M, u \models A(a)$.

Definition 3.5.8 Let $\mathbf{F}$ be an $\text{S}_4$-based KFE, $M$ a KFE-model over $\mathbf{F}$. The pattern of $M$ is the intuitionistic KFE-model $M_0$ over $\mathbf{F}$ such that for any $u \in \mathbf{F}$ and any atomic $D_u$-sentence $A$ without equality

$$M_0, u \models A \iff M, u \models \Box A.$$ 

Let us check soundness of this definition:

Lemma 3.5.9 The pattern exists for every $\text{S}_4$-based KFE-model.

Proof According to 3.5.3, we have to show that for any $P \in PL^n$, $a, b \in D_u^n$ such that $\forall i a_i \succ_u b_i$,

$$M, u \models \Box P(a) \iff M, u \models \Box P(b). \quad (*)$$

In fact, by 3.5.1 for any $v \in R(u)$, $a_i \succ v b_i$; hence

$$M, v \models P(a) \iff M, v \models P(b)$$

by 3.5.3. Then

$$\forall v \in R(u) M, v \models P(a) \iff \forall v \in R(u) M, v \models P(b),$$

which implies $(*)$.

Thus $M_0$ always exists; it is intuitionistic by definition.
Now there is an analogue of 3.2.15.

**Lemma 3.5.10** Let $M_0$ be the pattern of $M$; then for any $u \in M$ and for any intuitionistic $D_u$-sentence $A$

$$M_0, u \vDash A \iff M, u \vDash A^T,$$

and for any intuitionistic sentence $A$

$$M_0 \vDash A \iff M \vDash A^T$$

**Proof** The same as for 3.2.15, with a difference in the case when $A$ is atomic of the form $a = b$. Then we have

$$M_0, u \vDash a = b \iff a \succ u b,$$

$$M, u \vDash A^T(= \Box(a = b)) \iff \forall v \in R(u) a \succ v b,$$

which is equivalent to $a \succ u b$ by Definition 3.5.1.

**Proposition 3.5.11** Let $F$ be an S4-based KFE, $A \in IF^=$. Then

$$F \vDash A \iff F \vDash A^T.$$

**Proof** The same as for 3.2.25 based on 3.5.7 and 3.5.10.

**Corollary 3.5.12** For a class $C$ of S4-based KFEs

$$\mathbf{IL}^{=} (C) = T \mathbf{ML}^{=} (C).$$

**Proof** Follows from 3.5.11 as in the proof of 2.16.18.

The set of formulas valid in a KFE $F$ is denoted by $\mathbf{ML}^{=} (F)$ (or $\mathbf{IL}^{=} (F)$). This notation is not quite legal before we show soundness; the proof of soundness is postponed until Section 3.6.

**Lemma 3.5.13** For any KFE $(F, D, \succ)$, $\mathbf{ML}(F, D) \subseteq \mathbf{ML}(F, D, \succ)$.

**Proof** For formulas without equality the definitions of forcing in $(F, D, \succ)$ and $(F, D)$ are the same. Every valuation in $(F, D, \succ)$ is a valuation in $(F, D)$, so refutability of a formula in $(F, D, \succ)$ implies its refutability in $(F, D)$.  

**Remark 3.5.14** Not all valuations in $(F, D)$ are admissible in $(F, D, \succ)$, so it may happen that $\mathbf{ML}(F, D, \succ) \neq \mathbf{ML}(F, D)$. A trivial counterexample is $(F, D, \succ)$ where $F$ is a reflexive singleton, $D$ is two-element and $\succ$ is universal. Then obviously $(F, D) \not\vDash \exists x P(x) \supset \forall x P(x)$, while $(F, D, \succ) \vDash \exists x P(x) \supset \forall x P(x)$.

On the other hand, predicate Kripke frames can be regarded as a particular kind of KFEs.
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Lemma 3.5.15

(1) Every Kripke frame \( F = (F, D) \) is associated with a simple KFE \( F_e = (F, D, \simeq) \), in which \( \simeq_u = \text{id}_{D_u} \) for any \( u \in F \). Then \( \text{ML}(\simeq)(F) = \text{ML}(\simeq)(F_e) \), and \( \text{IL}(\simeq)(F) = \text{IL}(\simeq)(F_e) \) in the intuitionistic case.

(2) A KFE \( (F, D, \simeq) \) is simple iff

\[ \forall a, b \in D^+(a \neq b \Rightarrow E(a, b) = \emptyset), \]

where \( E \) is the measure of identity (3.5.2).

Proof

(1) Valuations and the corresponding forcing relations in \( F \) and \( F_e \) are just the same.

(2) In fact, in a simple KFE, \( u \in E(a, b) \) implies \( a = b \). The other way round, if \( E(a, b) = \emptyset \) whenever \( a \neq b \), then \( a \simeq_u b \) holds only for \( a = b \).

\[ \square \]

Proposition 3.5.16 Let \( C \) be a class of \( N \)-modal propositional frames. Then \( \text{ML}(\mathcal{KE}C) = \text{ML}(\mathcal{KC}). \) Similarly, for a class \( C \) of intuitionistic propositional frames, \( \text{IL}(\mathcal{KE}C) = \text{IL}(\mathcal{KC}). \)

Proof Consider the modal case. If \( A \in \text{ML}(\mathcal{KE}C) \), then \( A \) is valid in every KFE over \( C \). In particular, for any \( F \in \mathcal{KC} \), \( A \) is valid in the associated simple frame \( F_e \). So by Lemma 3.5.15, \( A \in \bigcap \{ \text{ML}(F) \mid F \in \mathcal{KC} \} = \text{ML}(\mathcal{KC}). \)

The other way round, suppose \( A \in \text{ML}(\mathcal{KC}) \). Then for any \( (F, D, \simeq) \) with \( F \in C \), we have \( A \in \text{ML}(F, D) \subseteq \text{ML}(F, D, \simeq) \) by Lemma 3.5.13. Hence \( A \in \text{ML}(\mathcal{KE}C) \).

\[ \square \]

3.5.3 Strong morphisms

In this section we consider strong morphisms defined in an obvious way — as morphisms of Kripke frames preserving equality. A larger class of morphisms will be considered in Section 3.7.

Definition 3.5.17 Let \( F = (F, D, \simeq), F' = (F', D', \simeq') \) be Kripke frames with equality. A strong (p-)morphism from \( F \) to \( F' \) is a (p-)morphism of frames without equality \( f = (f_0, f_1) : (F, D) \rightarrow (F', D') \) (Definition 3.3.1) such that for any \( u \in F, a, b \in D^+ \)

\[ a \simeq_u b \text{ iff } f_{1u}(a) \simeq'_{f_0(u)} f_{1u}(b). \]

A strong isomorphism is a strong morphism which is an isomorphism of frames without equality.\(^{17}\) Strong morphisms of KFE-models are defined as morphisms of their frames satisfying the reliability condition from Definition 3.3.2; similarly for strong p-morphisms and strong isomorphisms.

\(^{17}\)Definition 3.3.2.
Obviously, if the KFEs $F = (F, D, \prec)$, $F' = (F', D', \preccurlyeq)$ are simple, then a strong KFE-morphism from $F$ to $F'$ is nothing but an $\equiv$-morphism from $(F, D)$ to $(F', D')$.

Strong morphisms of KFE-frames and models are denoted by $\rightarrow^\equiv$, strong p-morphisms by $\rightarrow^\equiv$, strong isomorphisms by $\cong$.

Lemma 3.3.4 easily transfers to KFEs:

**Lemma 3.5.18**

(1) For a KFE $F = (F, D, \prec)$ the identity morphism $id_F := id_{(F, D)}$ is a strong isomorphism.

(2) The composition of strong morphisms (in the sense of 3.3.4) is a strong morphism; similarly for p-morphisms and isomorphisms.

This yields the categories of N-modal KFEs and strong morphisms, intuitionistic KFEs and strong morphisms, and similarly, of KFE-models.

**Lemma 3.5.19** Strong isomorphisms are exactly isomorphisms in the category of N-modal KFEs and strong morphisms.

**Proof** An exercise. ■

Now we have an analogue of Proposition 3.3.11

**Lemma 3.5.20** If $(f_0, f_1) : M \rightarrow^\equiv M'$, for KFE-models $M$, $M'$, then for any $u \in F$ and for any $D_u$-sentence $B$

$$M, u \models B \text{ iff } M', f_0(u) \models f_{1u} \cdot B,$$

where $f_{1u} \cdot B$ is obtained from $B$ by replacing occurrences of every $c \in D_u$ with $f_{1u}(c)$.

If the models are intuitionistic, the same holds for any intuitionistic $D_u$-sentence (and the intuitionistic forcing).

**Lemma 3.5.21** Let $(f_0, f_1) : F \rightarrow^\equiv F'$ be a KFE-morphism, and let $M'$ be a KFE-model over $F'$. Then there exists a unique model $M$ over $F$ such that $(f_0, f_1) : M \rightarrow^\equiv M'$, and similarly for $\rightarrow^\equiv$.

**Proof** As in the proof of 3.3.12, we put

$$\xi_u(P) := \{ b \mid M', f_0(u) \models P(f_{1u} \cdot b) \}$$

for any $u \in F$, $P \in PL^m$. Then $M = (F, \xi)$ is a KFE-model. In fact, we have to check that

$$a \in \xi_u(P) \& \forall i a_i \prec_u b_i \Rightarrow b \in \xi_u(P),$$

i.e.

$$M', f_0(u) \models P(f_{1u} \cdot a) \& a \prec_u b \Rightarrow M', f_0(u) \models P(f_{1u} \cdot b), \quad (1)$$
where

\[ a \preceq_u b := \forall i a_i \preceq_u b_i. \]

But (1) holds, since

\[ a \preceq_u b \Rightarrow (f_{1u} \cdot a) \preceq_{f_0(u)} (f_{1u} \cdot b) \]

by the definition of a strong KFE-morphism, and

\[ M', f_0(u) \models P(f_{1u} \cdot a) \& (f_{1u} \cdot a) \preceq_{f_0(u)} (f_{1u} \cdot b) \Rightarrow M', f_0(u) \models P(f_{1u} \cdot b) \]

by the definition of a KFE-model. The claim \((f_0, f_1) : M \rightarrow M'\) and the uniqueness of \(M'\) follow easily.\[ \]

Hence we have an analogue of 3.3.13:

**Proposition 3.5.22** If \(F = F'\) for KFEs \(F, F'\), then \(\text{ML}^= (F) \subseteq \text{ML}^= (F')\), and similarly, \(\text{IL}^= (F) \subseteq \text{IL}^= (F')\) in the intuitionistic case.

**Proof** Along the same lines as 3.3.13, now using 3.5.21, 3.5.7, 3.5.20. \[ \]

### 3.5.4 Main constructions

Now let us extend the definition of subframes and submodels 3.3.15 to the case with equality.

**Definition 3.5.23** Let \(F = (W, R_1, \ldots, R_N)\) be a propositional Kripke frame, \(F = (F, D, \succeq)\) a KFE, \(V \subseteq W\). A subframe of \(F\) obtained by restriction to \(V\) is

\[ F \upharpoonright V := (F \upharpoonright V, D \upharpoonright V, \succeq \upharpoonright V), \]

where \(D \upharpoonright V\) is the same as in 3.3.15, \(\succeq \upharpoonright V := (\succeq_u)_{u \in V}\).

If \(M = (F, \xi)\) is a KFE-model, we define the submodel

\[ M \upharpoonright V := (F \upharpoonright V, \xi \upharpoonright V), \]

where \(\xi \upharpoonright V\) is the same as in 3.3.15. If \(V\) is stable, \(F \upharpoonright V, M \upharpoonright V\) are called generated.

It is obvious that \(M \upharpoonright V\) is a KFE-model, since \(\xi \upharpoonright V\) coincides with \(\xi\) on \(V\).

The definitions of reliability, rooted frames (models) and cones are trivially extended to frames and models with equality.

**Lemma 3.5.24** (Generation lemma) Let \(F\) be a KFE, \(M\) a KFE-model over \(F, V\) a stable set of worlds in \(F\). Then

1. \(M \upharpoonright V\) is a reliable submodel of \(M\);

2. \(\text{ML}^= (F) \subseteq \text{ML}^= (F \upharpoonright V)\); similarly, for the intuitionistic case.
Proof Almost the same as for 3.3.18. We use the same map \((j, i)\) and apply Lemma 3.5.20. To prove (2), we need a valuation \(\xi\) in \(F\) such that \(\xi_u = \xi'_u\) for \(u \in V\), viz.

\[
\xi_u(P) := \begin{cases} 
\xi'_u(P) & \text{if } u \in V, \\
\emptyset & \text{otherwise}.
\end{cases}
\]

This is really a valuation; \(\xi\) respects \(\succsim_u\) for \(u \in V\), since it coincides with the valuation \(\xi'\); and for \(u \notin V\) there is nothing to prove. 

We define cones exactly as in 3.3.17:

Definition 3.5.25

\[
F \uparrow u := F \uparrow (W \uparrow u), \; M \uparrow u := M \uparrow (W \uparrow u).
\]

Then we obtain an analogue of 3.3.21:

Lemma 3.5.26

(1) \(\text{ML} (=)(F) = \bigcap_{u \in F} \text{ML} (=)(F \uparrow u)\)

and similarly for the intuitionistic case.

(2) For a class \(C\) of N-modal or intuitionistic KFEs,

\[
\text{L} (=)(C) = \text{L} (=)(C \uparrow),
\]

where \(\text{L}\) is \(\text{ML}\) or \(\text{IL}\), \(\uparrow\) is the class of all cones of frames from \(C\).

Lemma 3.5.27 Let \(F\) be a propositional Kripke, \(u \in F\), then

\[
\mathcal{KE}(F \uparrow u) = (\mathcal{KE}F) \uparrow u,
\]

where \((\mathcal{KE}F) \uparrow u := \{F \uparrow u \mid F \in \mathcal{KE}F\}\).

Proof Similar to 3.3.27. Given a KFE \(G = (F \uparrow u, D, \succsim)\), we have to show that \(G\) is \(F \uparrow u\) for some KFE \(F\) over \(F\). It suffices to extend the domain function \(D\) to the whole \(F\); \(\succsim\) is extended in the trivial way (as identity at every world). We do this by putting

\[
D'_w := \begin{cases} 
D_u & \text{if } w \notin F \uparrow u, \\
D_w & \text{if } w \in F \uparrow u.
\end{cases}
\]

Then \(D'\) is obviously expanding, since \(F \uparrow u\) is a generated subframe and \(D_u \subseteq D'_w\) for any \(w \in F\).

This implies an analogue of 3.3.28:

Proposition 3.5.28 Let \(C\) be a class of N-modal or intuitionistic propositional Kripke frames. Then
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(1) \( \mathcal{KE}(C \uparrow) = (\mathcal{KE}C) \uparrow \)

(2) \( L(=)(\mathcal{KE}C) = L(=)(\mathcal{KE}(C \uparrow)) \), where \( L \) is ML or IL respectively.

Proof Similar to 3.3.28; apply 3.5.27 and 3.5.26(2).

Let us also define disjoint sums:

Definition 3.5.29 Let \( F_i = (F_i, D_i, \sim_i) \) be predicate Kripke frames with equality. Then

\[
\bigsqcup_{i \in I} F_i := \left( \bigsqcup_{i \in I} F_i, D, \sim \right),
\]

where

\[
D(u, i) := D_i(u) \times \{i\}, \quad (a, i) \sim_{u, i} (b, i) := a \sim_i b
\]

for \( i \in I, \ u \in F_i, \ a, b \in D_u \).

Since \( F_i \) is isomorphic to a generated subframe of \( \bigsqcup_{i \in I} F_i \), we obtain an analogue of 3.3.36:

Lemma 3.5.30 \( ML(=) \left( \bigsqcup_{i \in I} F_i \right) = \bigcap_{i \in I} ML(=)(F_i) \), and similarly

\( IL(=) \left( \bigsqcup_{i \in I} F_i \right) = \bigcap_{i \in I} IL(=)(F_i) \) for the intuitionistic case.

3.6 Kripke Sheaves

Now let us consider semantical equivalents of KFEs called ‘Kripke sheaves’. To define them, let us begin with the S4-case. In this case a Kripke sheaf is a set-valued functor defined on a certain category.

Recall that a category consists of objects (‘points’) and morphisms (‘arrows’) between some of these points. In precise terms, it can be defined as a tuple.

\[
C = (X, Y, o, t, I, \circ)
\]

where \( X, Y \) are non-empty classes, \( o, t, I \) are functions:

\[
o, t : Y \to X;
I : X \to Y;
\]

\( \circ \) is a partial function \( Y \times Y \to Y \).

\( X \) is called the class of objects of \( C \) and is also denoted by \( \text{Ob} C \). \( Y \) is called the class of morphisms of \( C \) and is also denoted by \( \text{Mor} C \). For a morphism \( f \), the object \( o(f) \) is called the origin of \( f \), and \( t(f) \) the target of \( f \).

The notation \( f : a \to b \) or \( a \xrightarrow{f} b \) means that \( o(f) = a \) and \( t(f) = b \); this is read as ‘\( f \) is a morphism from \( a \) to \( b \)’ (or between \( a \) and \( b \)).

\( C(a, b) \) denotes the class of all morphisms in \( C \) between \( a \) and \( b \). \( I(a) \) is called the identity morphism of \( a \) and is also denoted by \( 1_a \).

There are also the following conditions (‘axioms’):
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(1) $1_a : a \rightarrow a$

(2) $\alpha \circ \beta$ is defined iff $o(\beta) = t(\alpha)$ (i.e. iff arrows $\alpha, \beta$ are consecutive);

(3) if $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$, then $x \xrightarrow{\alpha \circ \beta} z$;

(4) if $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \xrightarrow{\gamma} u$, then $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$;

(5) if $x \xrightarrow{\alpha} y$, then $1_x \circ \alpha = \alpha \circ 1_y = \alpha$.

A standard example of a category is SET, the category of sets, in which objects are arbitrary sets, morphisms are maps.

There is a well-known canonical way of associating a category $C = \text{Cat}_F$ with an S4-frame $F = (W, R)$ [Goldblatt, 1984]. Viz., we put

$$\text{Ob}C := W, \text{Mor}C := R, C(u,v) := \{(u,v)\}, 1_u := (u,u).$$

So arrows of $C$ just represent the relation $R$. The composition is defined according to the 'triangle rule':

$$(u,v) \circ (v,w) := (u,w).$$

**Definition 3.6.1** An (S4-based, or intuitionistic) Kripke sheaf over an S4-frame $F$ is a SET-valued co-functor defined on $\text{Cat}_F$. This means that a Kripke sheaf is a triple $\Phi = (F, D, \rho)$ where $(F, D)$ is a system of domains, $\rho = (\rho_{uv} | uRv)$ is a family of functions $\rho_{uv} : D_u \rightarrow D_v$ (transition maps) satisfying the following functoriality conditions:

(1) for every $u \in F$, $\rho_{uu} = \text{id}_{D_u}$ (the identity function on $D_u$);

(2) $uRvRw$ (in $F$) implies $\rho_{uv} \circ \rho_{vw} = \rho_{uw}$.

We call $\rho_{uv}(a)$ the inheritor of the individual $a$ (from $D_u$) in the world $v$. The domains $D_u$ are also called the fibres or the stalks of $\Phi$.

To extend this definition to arbitrary frames, recall that an arbitrary propositional Kripke frame $F = (W, R_1, \ldots, R_N)$ is associated with an S4-frame $F^* := (W, R^*)$, where $R^*$ is the reflexive transitive closure of $R_1 \cup \ldots \cup R_N$.

**Definition 3.6.2** A Kripke sheaf over a propositional Kripke frame $F$ is a triple $\Phi = (F, D, \rho)$ such that $\Phi^* := (F^*, D, \rho)$ is an S4-based Kripke sheaf over $F^*$.

The frame $F$ is called the (propositional) base of $\Phi = (F, D, \rho)$ and denoted by $\Phi_\pi$. $(F, D, \rho)$ is called $N$-modal if $F$ is $N$-modal.

Note that in Definition 3.6.2 $\rho$ is a system of functions parameterised by $R^*$, i.e. $\rho = (\rho_{uv} | uR^*v)$, satisfying the conditions 3.7.1 (1), (2) for $R^*$.

**Lemma 3.6.3** Let $F = (W, R_1, \ldots, R_N)$ be a propositional Kripke frame, $(F, D)$ a predicate Kripke frame, $\rho = (\rho_{iuv} | uR_iv, 1 \leq i \leq N)$ a family of functions $\rho_{iuv} : D_u \rightarrow D_v$ satisfying the following ‘coherence’ conditions:
(1) if $uR_i u_1 R_i u_2 \ldots u_k R_i u$ for some $k \geq 0$, then $\rho_{i_0 u_1} \circ \rho_{i_1 u_1 u_2} \circ \cdots \circ \rho_{i_k u_k u} = id_{D_u}$;

(2) if $uR_i u_1 R_i u_2 \ldots u_k R_i v$ and $uR_j v_1 R_j v_2 \ldots v_m R_j v$, then $\rho_{i_0 u_1} \circ \rho_{i_1 u_1 u_2} \circ \cdots \circ \rho_{i_k u_k v} = \rho_{i_0 u_1} \circ \rho_{j_1 v_1 u_2} \circ \cdots \circ \rho_{j_m v_m v}$.

Then there exists a unique Kripke sheaf $(F, D, \rho^*)$ such that $\rho_{iuv} = \rho^*_{uv}$ whenever $uR_i v$.

Proof  For any pair $(u, v) \in R^*$ we can define a function $\rho^*_{uv} : D_u \rightarrow D_v$ such that $\rho^*_{uv} := id_{D_u}$ and $\rho^*_{uv} := \rho_{i_0 u_1 u_2} \circ \cdots \circ \rho_{i_k u_k v}$ for any path $(u, i_0, u_1, i_1, \ldots, u_k, i_k, v)$. The conditions (1), (2) show that $\rho^*_{uv}$ is well defined. Then it follows that $\rho^*$ determines a Kripke sheaf over $F^*$.

Of course for S4-based Kripke sheaves the above conditions (1), (2) follow from Definition 3.6.1. So every Kripke sheaf can be presented in the form described in 3.6.3. Viz. for $\Phi = (F, D, \rho)$ put $\rho_{iuv} := \rho_{uv}$ whenever $uR_i v$. Then the conditions 3.6.3 (1), (2) hold and the corresponding Kripke sheaf $(F, D, \rho^*)$ is just $\Phi$. So 3.6.3 yields an alternative definition of a Kripke sheaf.

If $(F, D, \rho)$ is a Kripke sheaf, $a \in D_u$, $uR^* v$, we sometimes use the notation $a|v := \rho_{uv}(a)$ and $a|v := (a_1|v, \ldots, a_n|v)$ for $a = (a_1, \ldots, a_n) \in D^n_u$. But this notation should be used carefully, because it is ambiguous if the fibres are not disjoint (then it may happen that $\rho_{u_1 v}(a) \neq \rho_{u_2 v}(a)$ for $a \in D_{u_1} \cap D_{u_2}$).

Definition 3.6.4 A valuation in a Kripke sheaf $\Phi = (F, D, \rho)$ is just a valuation in the frame $(F, D)$. If $\xi$ is a valuation, $M = (\Phi, \xi)$ is called a Kripke sheaf model over $D$.

Forcing relation $M, u \models A$ between a world $u$ in an N-modal Kripke sheaf model and an N-modal $D_u$-sentence. $A$ is defined by the same clauses as in predicate Kripke frames (Definition 3.2.9), with the only difference:

- $M, u \models \Box_i B(a_1, \ldots, a_n)$ iff $\forall v \in R_i(u) M, v \models B(\rho_{uv}(a_1), \ldots, \rho_{uv}(a_n))$.

Definition 3.6.5 A valuation $\xi$ in an S4-based Kripke sheaf (and the corresponding Kripke sheaf model) is called intuitionistic if it satisfies the following conditions:

- $uRv \& (a_1, \ldots, a_n) \in \xi_u(P^n_k) \Rightarrow (\rho_{uv}(a_1), \ldots, \rho_{uv}(a_n)) \in \xi_v(P^n_k)$,

- $uRv \& u \in \xi_u(P^0_k) \Rightarrow v \in \xi_v(P^0_k)$.

Then the intuitionistic forcing in $M$ is defined by the following clauses (cf. Definition 3.2.10):

- $u \models P^n_k(a_1, \ldots, a_n)$ iff $(a_1, \ldots, a_n) \in \xi_u(P^n_k)$ (for $n > 0$);

- $u \models P^0_k$ iff $u \in \xi_u(P^0_k)$;

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18So in particular, $\rho_{iuu} = id_{D_u}$ if $uR_i u$; $\rho_{iuv} = \rho_{juv}$ if $uR_i v$ and $uR_j v$. 

• $u \not\vdash \bot$;

• $u \vdash B \land C$ iff $u \vdash B$ & $u \vdash C$;

• $u \vdash B \lor C$ iff $u \vdash B \lor u \vdash C$;

• $u \vdash (B \supset C)(a_1, \ldots, a_n)$ iff

\[
\forall v \in R(u) \ (v \vdash B(\rho_{uv}(a_1), \ldots, \rho_{uv}(a_n)) \Rightarrow v \vdash C(\rho_{uv}(a_1), \ldots, \rho_{uv}(a_n)));
\]

• $u \vdash \exists x A(x)$ iff $\exists a \in D_u \ u \vdash A(a)$;

• $u \vdash \forall x B(x, a_1, \ldots, a_n)$ iff $\forall v \in R(u) \ \forall c \in D_v \ v \vdash B(c, \rho_{uv}(a_1), \ldots, \rho_{uv}(a_n))$.

Lemma 3.6.6 Let $u, v$ be worlds in an intuitionistic Kripke sheaf model $M$. Then for any intuitionistic $D_u$-sentence $A(a)$

\[
M, u \vdash A(a) \& uRv \Rightarrow M, v \vdash A(\rho_{uv} \cdot a).
\]

Proof Similar to 3.2.16.

Lemma 3.6.7 The intuitionistic forcing relation has the following properties:

\[
M, u \vdash \neg B(a) \iff \forall v \in R(u) \ M, v \not\vdash B(\rho_{uv} \cdot a),
\]

\[
M, u \vdash a \neq b \iff \forall v \in R(u) \ \rho_{uv}(a) \neq \rho_{uv}(b).
\]

Proof Obvious from 3.6.5.

Definition 3.6.8 An $N$-modal predicate formula $A$ is called true in an $N$-modal Kripke sheaf model $M$ if $\forall A$ is true at every world of $M$. $A$ is called valid in a Kripke sheaf $\Phi$ if it is true under any valuation in $\Phi$.

The definitions for the intuitionistic case are similar.

The notation for the truth and validity is the same as in the case of Kripke frames.

Lemma 3.6.9 Let $M$ be an $N$-modal Kripke sheaf model, $A(x)$ an $N$-modal formula with $FV(A(x)) = r(x)$, $|x| = n$. Then for any $u \in M$

\[
M, u \vdash \forall x A(x) \iff \forall a \in D_u^n \ M, u \vdash A(a).
\]

Proof Similar to 3.2.17.

Lemma 3.6.10 Let $M$ be an intuitionistic Kripke sheaf model with the accessibility relation $R$, $u \in M$, $A(x)$ an intuitionistic $D_u$-formula with $FV(A(x)) = r(x)$, $|x| = n$. Then

\[
M, u \vdash \forall x A(x) \iff \forall v \in R(u) \forall a \in D_v^n \ M, v \vdash A(\rho_{uv} \cdot a).
\]

Proof Similar to 3.2.18.
3.6. Kripke Sheaves

Definition 3.6.11 Let $\Phi$ be an S4-based Kripke sheaf; $M$ a model over $F$. The pattern of $M$ is the model $M_0$ over $F$ such that for any $u \in F$ and any atomic $D_u$-sentence without equality $A$

$$M_0, u \models A \iff M, u \models \Box A.$$ 

As in the case of Kripke frames, the pattern is an intuitionistic model and it always exists.

Lemma 3.6.12 If $M_0$ is a pattern of a Kripke sheaf model $M$, then

1. for any $u \in M$, for any intuitionistic $D_u$-sentence $A$
   $$M_0, u \models A \iff M, u \models A^T;$$

2. for any $A \in IF^=\$
   $$M_0 \models A \iff M \models A^T.$$

Proof Similar to 3.2.15 and 3.2.24.

Definition 3.6.13 For a set $\Sigma$ of modal (or intuitionistic) sentences, a $\Sigma$-sheaf is a Kripke sheaf (of the corresponding type) validating every formula from $\Sigma$. $\Phi \models \Sigma$ (or $\Phi \models \Sigma$) denotes that $\Sigma$ is valid in $\Phi$.

The class of all $\Sigma$-sheaves is denoted by $V_{K_S}(\Sigma)$ and called modally (respectively, intuitionistically) definable (by $\Sigma$).

Lemma 3.6.14 Let $\Phi$ be an $N$-modal (respectively, intuitionistic) Kripke sheaf over a propositional frame $F$, $A$ an $N$-modal (respectively, intuitionistic) propositional formula. Then $\Phi \models (\models)A$ iff $\Phi_s \models (\models)A$.

Proof Similarly to the case of Kripke frames (Lemma 3.3.32), validity for propositional formulas in $\Phi$ is the same as in $F$.

Proposition 3.6.15 Let $\Phi$ be an S4-based Kripke sheaf, $A \in IF^=\$. Then

$$\Phi \models A \iff \Phi \models A^T.$$

Proof Along the same lines as 3.2.25, now using 3.6.11.

Now we have an analogue of 3.2.26:

Lemma 3.6.16 Let $A(x), B(x)$ be congruent modal (or intuitionistic) formulas, $|x| = n$, and let $M$ be a modal (respectively, intuitionistic) Kripke sheaf model. Then for any $u \in M, a \in D_u^n$

$$M, u \models (\models) A(a) \iff M, u \models (\models) B(a).$$

Thus the set of formulas valid in a Kripke sheaf is closed under congruence.
Proof Along the same lines as 3.2.26. Again we consider the equivalence relation on modal formulas

\[ A \sim B \text{ iff } FV(A) = FV(B) \]

and for any distinct \( x \) with \( FV(A) = r(x) \), for any \( u \in M, a \in D_u^{[x]} \)

\[ M, u \models [a/x]A \iff M, u \models [a/x]B. \]

We have to check the properties 2.3.17(1)-(4) for this relation. For (1)-(3) the proof is the same as in 3.2.26. For (4) there is a slight difference: now

\[ M, u \models [a/x]\Box_iA \iff \forall v \in R_i(u) \ M, v \models [\rho_{uv} \cdot a/x]A \]

and similarly for \( B \). So \( A \sim B \) implies \( \Box_iA \sim \Box_iB. \)

\[ \Box \]

Theorem 3.6.17 (Soundness theorem)

(I) The set of all modal predicate formulas (with equality) valid in a Kripke sheaf is a modal predicate logic (with equality).

(II) The set of all intuitionistic predicate formulas (with equality) valid in an \( S4 \)-based Kripke sheaf is a superintuitionistic predicate logic (with equality).

Proof Along the same lines as 3.2.29. The main thing is to check that formula substitutions preserve validity.

So we assume that \( \Phi \models A \) for a formula \( A \) and a Kripke sheaf \( \Phi \) and show that \( \Phi \models SA \) for \( S = [C(x, y)/P(x)] \), where \( P \in PL^n \) occurs in \( A \), the list \( xy \) is distinct, and \( r(y) \subseteq FV(C) \subseteq r(xy) \).

By Lemma 3.6.16 we can replace \( A \) with a congruent formula, so we assume that \( A \) is clean, \( BV(A) \cap r(y) = \emptyset \). Again we choose a distinct list \( z \) such that \( FV(A) \cup r(y) = r(z) \); then

\[ r(y) \subseteq FV(A) \subseteq r(z), \quad r(z) \cap BV(A) = \emptyset. \]

Let \( m = |z| \). Given a model \( M = (\Phi, \xi) \), we show that for any \( u \in \Phi, c \in D_u^n \)

\[ M, u \models [c/z]SA. \]

Let \( c' \) be the part of \( c \) corresponding to \( y \). We define \( M_1 = (\Phi, \eta) \) similarly to 3.2.29:

- for any \( v \in \Phi \uparrow u, \ a \in D_v^n \)

\[ M_1, v \models P(a) \text{ iff } M, v \models C(a, c'|v); \]

- for any other atomic \( D_v \)-sentence \( Q \)

\[ M_1, v \models Q \text{ iff } M, v \models Q. \]
Then every subformula of $A$ has the form $B(z,q)$, where $q$ is distinct, $r(q) = BV(A)$; so by 2.5.24 we present $SB$ as $(SB)(z,q)$.

Then we prove the claim:

$$\forall v \in \Phi \upharpoonright u \forall a \in D_{u}^{[q]} (M_{1}, v \models B(c|v, a) \iff M, v \models (SB)(c|v, a))$$

by induction. The only difference with 3.2.29 is in the case $B = \Box_{i} B_{1}$:

- $M_{1}, v \models B(c|v, a)$ iff $\forall w \in R_{i}(v) M_{1}, w \models B_{1}(c|w, a|w)$
- $\forall w \in R_{i}(v) M, w \models (SB_{1})(c|w, a|w)$ (by the induction hypothesis)
- $M, v \models \Box_{i}(SB_{1})(c|v, a)(= (SB)(c|v, a))$.

Here we use the equality $(a|v)|w = a|w$, which follows from Definition 3.6.1.

**Remark 3.6.18** One can similarly define forcing and validity for the case when $p$ does not satisfy the coherence conditions (1), (2) from 3.6.3. But then the set of validities is not necessarily substitution-closed. These 'frames' are a special kind of Kripke bundle considered in Chapter 5; note that validity in Kripke bundles is not substitution closed either, cf. Exercise 5.2.13.

Every predicate Kripke frame corresponds to a Kripke sheaf, in which $u R_{i} v$ always implies $D_{u} \subseteq D_{v}$ and $\rho_{uv}$ is the inclusion map, i.e. $\rho_{uv}(a) = a$ for any $a \in D_{u}$.

More generally, every KFE $F = (W, R_{1}, \ldots, R_{N}, D, \simeq)$ corresponds to a Kripke sheaf $\Theta(F)$ constructed as follows. Let $a_{u}$ be the class of $a \in D_{u}$ modulo $\simeq_{u}$. We define $\Theta(F)$ as the Kripke sheaf with the fibres $D_{u}' := \{a_{u} \mid a \in D_{u}\}$ and the transition maps $\rho_{uv}(a_{u}) := a_{u}$ for $u R_{i} v$; $\Theta(F)$ is well defined, due to Lemma 3.6.3 and since $u R_{i} v$ implies $D_{u} \subseteq D_{i}$ and $\simeq_{i} \subseteq \simeq_{u}$.

The following is almost obvious:

**Lemma 3.6.19** Valuations (both modal and intuitionistic) in a KFE $F$ and in the Kripke sheaf $\Theta(F)$ are associated. Namely, a valuation $\xi$ in $F$ corresponds to the valuation $\Theta(\xi)$ in $\Theta(F)$ such that

$$\Theta(\xi)(P) = \{(a_{1}, \ldots, a_{n}) \in (D_{u}')^{n} \mid (a_{1}, \ldots, a_{n}) \in \xi_u(P)\}$$

for $P \in PL^{n}, n > 0$, and $\Theta(\xi)_u, \xi_u$ coincide on $PL^{0}$. The other way round, every valuation in $\Theta(F)$ has the form $\Theta(\xi)$ for some valuation $\xi$ in $F$. $\xi$ is intuitionistic iff $\Theta(\xi)$ is intuitionistic.

**Proof** The above definition of $\Theta(\xi)$ is sound; recall that $(a_{1}, \ldots, a_{n}) \in \xi_u(P)$ depends only on classes of $a_{1}, \ldots, a_{n}$ modulo $\simeq_{u}$.

Now if $\eta$ is a valuation in $\Theta(F)$, we define $\xi$ by

$$\xi_u(P) := \{a \mid a_u \in \eta_u(P)\}$$

where

$$a_u := ((a_{1}), \ldots, (a_{n})) \text{ for } a = (a_{1}, \ldots, a_{n}), n > 0.$$ 

This definition is sound, because $a_u = b_u$ iff $\forall i \ a_i \simeq_{u} b_i$. Then obviously $\eta = \Theta(\xi)$.

The argument for the intuitionistic case is left to the reader.
Lemma 3.6.20

(1) For any $N$-modal formula $A(x_1, \ldots, x_n)$, a valuation $\xi$ in $F$, for any $u \in F$; $a_1, \ldots, a_n \in D_u$:

$$\xi, u \models A(a_1, \ldots, a_n) \text{ (in } F) \iff \Theta(\xi), u \models A((a_1)_u, \ldots, (a_n)_u) \text{ (in } \Theta(F)),$$

and similarly for the intuitionistic case.

(2) If $F$ is an $N$-modal KFE, $A$ is an $N$-modal formula, then $F \models A$ iff $\Theta(F) \models A$.

(3) If $F$ is an intuitionistic KFE, $A$ is an intuitionistic formula, then $F \vdash A$ iff $\Theta(F) \vdash A$.

Proof (1) By induction on the complexity of $A$. For example,

$$\xi, u \models a = b \iff a \prec_u b \iff a_u = b_u \iff \Theta(\xi), u \models a = b;$$

$$\xi, u \models \Box_i B(a) \text{ iff } \forall v \in R_i(u) \xi, v \models B(a) \text{ iff } \forall v \in R_i(u) \Theta(\xi), v \models B(a_u) \iff \Theta(\xi), u \models \Box_i B(a_u),$$

since $a_v = \rho_{uv} \cdot a_u$.

The remaining cases are left to the reader.

The claims (2), (3) now follow from 3.6.19, 3.6.9, 3.6.10. ■

Due to Theorem 3.6.17 and Lemma 3.6.20, we obtain

Theorem 3.6.21 (Soundness theorem)

(I) The set of all modal predicate formulas (with equality) valid in a KFE is a modal predicate logic (with equality).

(II) The set of all intuitionistic predicate formulas (with equality) valid in an S4-based KFE is a superintuitionistic predicate logic (with equality).

These logics are denoted by $ML(\equiv)(F)$, $IL(\equiv)(F)$ as usual.

According to the general definitions from Section 2.16, the modal predicate logic of a class of Kripke sheaves $F$ is

$$ML(\equiv)(F) := \bigcap \{ML(\equiv)(\Phi) \mid \Phi \in F\}.$$ 

The superintuitionistic logic $IL(\equiv)(F)$ is defined analogously.

The other way round, every Kripke sheaf is equivalent to one of the form $\Theta(F)$. To show this, let us introduce a convenient subclass of Kripke sheaves.

Definition 3.6.22 A Kripke sheaf is said to be disjoint if all its fibres are disjoint.

Definition 3.6.23 An isomorphism between Kripke sheaves $(F, D, \rho)$ and $(F, D', \rho')$ is a family of bijections $f_u : D_u \to D'_u$ such that $f_u \cdot \rho'_{uv} = f_v \cdot \rho_{uv}$ whenever $u R^* v$. 

A more general notion of morphism will be discussed in the next section. It is almost obvious that isomorphic Kripke sheaves have the same modal (or superintuitionistic) logics; for a precise proof, one should check the equivalence

\[ M, u \models B(a) \text{ iff } M', u \models B(f_u \cdot a) \]

for any \( D_u \)-sentence \( B(a) \) if it holds for any atomic \( D_u \)-sentence. We leave this as an exercise for the reader.

**Lemma 3.6.24** Every Kripke sheaf is isomorphic to a disjoint Kripke sheaf over the same propositional frame.

**Proof** In fact, we can replace each \( D_u \) with \( D'_u = \{(a, u) \mid a \in D_u\} \) and change the functions \( \rho_{uv} \) appropriately, viz., put \( \rho'_{uv}(a, u) := (\rho_{uv}(a), v) \).

For a disjoint Kripke sheaf \((F, D, \rho)\) and \( a \in D_u, uR_iv \), we sometimes write \( aRiv \) and say that \( a \) is \( R_i \)-related to \( v \).

Now let \( \Phi = (F, D, \rho) \) be a disjoint Kripke sheaf. Consider the KFE \( G(\Phi) := (F, D', \simeq') \), where

\[ D'_u := \bigcup \{D_w \mid w \in F, \ u \in F^w\} \]

\[ \simeq'_u := \{(a, b) \in (D'_u)^2 \mid (a|u) = (b|u)\} \]

Here is an equivalent presentation of \( G(\Phi) \) in the form described in Lemma 3.5.2:

\[ D'^+ := \bigcup_{u \in F} D_u, \]

\[ E'(a, b) := \{w \in (F^u \cap F^v) \mid (a|w) = (b|w)\} \]

for \( a \in D_u, b \in D_v; \ u, v \in F \).

It follows easily that \( D'_u \subseteq D'_v \) and \( \simeq'_u \subseteq \simeq'_v \), whenever \( uR_iv \); thus \( G(\Phi) \) is really a KFE.

Speaking informally, \( D'_u \) absorbs \( D_u \) and the domains of all \( R^*_i \)-predecessors of \( u ; \simeq'_u \) makes every individual from \( D_u \) equivalent to all its predecessors. So there is a natural bijection between \( D_u \) and \( D'_u / \simeq'_u \). This observation is used in the proof of the following

**Lemma 3.6.25** The Kripke sheaves \( \Phi \) and \( \Theta(G(\Phi)) \) are isomorphic.

**Proof** We have \( \Theta(G(\Phi)) = (F, D'', \rho'') \), where \( D'' = (D'_u / \simeq'_u)_{u \in F} \), and the transition map \( \rho''_{uv} \) sends every equivalence class \( a/ \simeq'_u \) to \( a/ \simeq'_v \) (for \( a \in D'_u \)).

Now, every class \( a'' = (b/ \simeq'_u) \in D'' \) contains a single element \( a \) from \( D_u \), namely, \( a = (b|u) \). So there exists a well-defined bijection \( \theta_u : D''_u \to D_u \) such that \( \theta_u(b/ \simeq'_u) = b|u \). (For the surjectivity, note that \( (a|u) = a \) for \( a \in D_u \).) Finally, \( uR_iv \) implies

\[ \rho_{uv}(\theta_u(b/ \simeq'_u)) = \rho_{uv}(b|u) = b|v, \]
and
\[ \theta_v(\rho''(b/ \sim'_u)) = \theta_v(b/ \sim'_v) = b|v. \]
Thus \( \rho_{uv} \cdot \theta_u = \theta_v \cdot \rho''(b) \), which means that the family of functions \( (\theta_u : u \in F) \) is an isomorphism between the Kripke sheaves \( \Theta(G(\Phi)) \) and \( \Phi \).

So we can introduce semantics generated by predicate Kripke frames with equality. Due to Lemmas 3.6.20 and 3.6.25, the same semantics are generated by Kripke sheaves.

**Definition 3.6.26** The N-modal Kripke sheaf semantics \( \mathcal{K}^={}_N \) is generated by the class of all N-modal predicate Kripke frames with equality (or Kripke sheaves). Similarly the intuitionistic Kripke sheaf semantics \( \mathcal{K}^=_\text{int} \) is generated by the class of all intuitionistic KFEs (or Kripke sheaves). Logics complete in these semantics are called Kripke sheaf complete, or just \( \mathcal{K}^= \)-complete.

As we pointed out at the end of Subsection 3.4.2, every predicate Kripke frame is equivalent a simple KFE. Thus \( \mathcal{K}^={}_N \succeq \mathcal{K}^= \) and actually \( \mathcal{K}^={}_N \preceq \mathcal{K}^=_N \), as we shall see later on; the same is true for the intuitionistic semantics.

Similarly to Corollary 3.3.37 we obtain

**Proposition 3.6.27** Kripke sheaf semantics has the collection property.

**Proof** Obvious, by Lemma 3.5.30.

For Kripke and Kripke sheaf semantics there also exists a stronger version of completeness.

**Definition 3.6.28** An N-modal theory \( \Gamma \) is called satisfiable in an N-modal Kripke frame (respectively, KFE) \( F \) if there exists a model (respectively, KFE-model) \( M \) over \( F \) and a world \( u \in M \) such that \( M, u \vdash \Gamma, \) i.e. \( M, u \vdash A \) for any \( A \in \Gamma \).

An intuitionistic theory \( (\Gamma, \Delta) \) is called satisfiable in an intuitionistic Kripke frame (or KFE) \( F \) if there exists an intuitionistic model (or KFE-model) \( M \) over \( F \) and a world \( u \in M \) such that \( M, u \models (\Gamma, \Delta), \) i.e. \( M, u \models A \) for any \( A \in \Gamma \) and \( M, u \not\models B \) for any \( B \in \Delta \).

**Lemma 3.6.29** A theory (modal or intuitionistic) is satisfiable in an L-KFE iff it is satisfiable in a Kripke sheaf validating L.

**Proof** In fact, by Lemma 3.6.20, \( F \) and \( \Theta(F) \) have the same logic (of the corresponding kind); a sentence \( A \) is satisfiable at \( F, u \) iff it is satisfiable at \( \Theta(F), u \).

On the other hand, by Lemma 3.6.25, every Kripke sheaf \( \Phi \) is isomorphic to \( \Theta(G(\Phi)) \), so satisfiability in \( \Phi \) and \( G(\Phi) \) is the same.
Definition 3.6.30 An $N$-modal predicate logic $L$ (with or without equality) is called strongly Kripke complete (respectively, strongly Kripke sheaf complete) if every $L$-consistent $N$-modal theory (respectively, with or without equality) is satisfiable in some Kripke $L$-frame (respectively, $L$-KFE). Similarly, an intuitionistic predicate logic $L$ (with/without equality) is called strongly Kripke (Kripke sheaf) complete if every $L$-consistent intuitionistic theory (with/without equality) is satisfiable in some Kripke $L$-frame ($L$-KFE).

Thanks to Lemma 3.6.29, the notions of strong completeness are the same for Kripke sheaves and KFEs.

Lemma 3.6.31 Every strongly complete m.p.l.(=) or s.p.l. (\(=\)) is complete (in the corresponding semantics of Kripke frames or Kripke sheaves).

Proof In fact, in the modal case, if $A \not\in L$, then \(\{\neg A\}\) is $L$-consistent. So if $L$ is strongly complete, then $\neg A$ is satisfiable in an $L$-frame $F$, and thus $F$ separates $A$ from $L$. Therefore $L$ is complete by Lemma 2.16.2.

In the intuitionistic case, if $A \not\in L$, then $(\emptyset, \{A\})$ is $L$-consistent; the remaining argument is the same as in the modal case. ■

3.7 Morphisms of Kripke sheaves

Kripke sheaf morphisms are a natural generalisation of predicate Kripke frame morphisms defined in 3.3.1:

Definition 3.7.1 Let \(\Phi = (F, D, \rho), \Phi' = (F', D', \rho')\) be Kripke sheaves over the frames $F = (W, R_1, \ldots, R_N), F' = (W', R'_1, \ldots, R'_N)$. A morphism from $\Phi$ to $\Phi$ is a pair $f = (f_0, f_1)$ satisfying the conditions (1)–(3) from Definition 3.3.1 and also

(5) the following diagram commutes whenever $uR_iv$:

\[
\begin{array}{ccc}
D_u & \xrightarrow{f_1v} & D'_{f_0(v)} \\
\rho_{uv} & & \rho'_{f_0(u)f_0(v)} \\
D_u & \xrightarrow{f_{1u}} & D'_{f_0(u)}
\end{array}
\]

The latter condition can be briefly written as

\[f_{1u}(a|v) = f_{1u}(a)|f_0(v)\]

(for $a \in D_u, uR_iv$), but strictly speaking, this makes sense only for disjoint sheaves.
Exercise 3.7.2 Show that under the conditions of 3.7.1 the diagram (5) commutes whenever \( uR^*v \).

Definition 3.7.3 A morphism of Kripke sheaf models \( M = (\Phi, \xi) \) and \( M' = (\Phi', \xi') \) is a morphism \((f_0, f_1): \Phi \to \Phi'\) such that for any \( P \in PL^m, \ m \geq 0\), \( u \in \Phi, \ a \in D^m_u\)

\[ M, u \models P(a) \text{ iff } M', f_0(u) \models P(f_{1u} \cdot a). \]

The notions ‘\(-\)-morphism’, ‘\(p(-)-morphism\)’, ‘isomorphism’ are transferred to Kripke sheaves in an obvious way.

Now we easily obtain an analogue to Lemma 3.3.4:

Lemma 3.7.4

1. The identity morphism \( \text{id}_\Phi := (\text{id}_W, (\text{id}_{D_u})_{u \in W}) \) is an isomorphism of Kripke sheaves.

2. The composition of morphisms \((f_0, f_1): \Phi \to \Phi'\) and \((g_0, g_1): \Phi' \to \Phi''\) defined as \((f_0 \circ g_0, f_{1u} \circ g_{1u})_{u \in W}\) is a morphism \( \Phi \to \Phi' \), similarly for all other kinds of morphism.

This yields us different categories of Kripke sheaves as in the case of Kripke frames.

Next, we have analogues to 3.3.6, 3.3.8–3.3.10:

Definition 3.7.5 A Kripke sheaf morphism over a propositional Kripke frame \( F \) is a morphism of Kripke sheaves over \( F \), in which the world component is the identity map.

Definition 3.7.6 Let \( \Phi = (F, D, \rho), \Phi' = (F', D', \rho') \) be Kripke sheaves, \( h: F' \to F \) a morphism of propositional frames. We say that \( \Phi' \) is obtained by changing the base along \( h \) if \( D'_u = D_{h(u)} \) for any \( u \in F \) and \( \rho'_{uv} = \rho_{h(u)h(v)} \) for any pair \((u, v) \in R_i\). We use the notation \( h^*_\Phi \) for \( \Phi' \).

Remark 3.7.7 This definition is sound in the case when \( h \) is only monotonic, i.e., for a ‘more traditional category of sheaves’.

Proposition 3.7.8 Under the conditions of the Definition 3.7.6, there exists a ‘canonical’ \(-\)-morphism \((h, g): h^*_\Phi \to \Phi\). Every morphism \((h, f_1): \Phi'' \to \Phi, \text{ where } \Phi'' = F'\), can be uniquely presented as a composition \( \Phi'' \to h^*_\Phi \to \Phi \) of a morphism over \( F' \) and the canonical morphism.

The next claim is analogous to 3.3.11:

Proposition 3.7.9 If \((f_0, f_1): M \to M'\) for \( N\)-modal Kripke sheaf models \( M, M' \), then for any \( u \in M, \ A \in MS_N^{(m)}(D_u)\)

\[ M, u \models A \text{ iff } M', f_0(u) \models f_{1u} \cdot A, \]

where \( f_{1u} \cdot A \) is obtained by replacing every \( a \in D_u \) with \( f_{1u}(a) \).

The same holds in the intuitionistic case.
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Proof We check only the □-case. Let $R_i, R'_i$ be the accessibility relations in $M, M'$. If $A = □_i B(a)$ (for a formula $B(x)$), then

$M, u ⊨ A$ iff $∀v ∈ R_i(u) M, v ⊨ B(ρ_{uv} \cdot a)$ iff $∀v ∈ R_i(u) M', f_0(v) ⊨ B(f_{1v} \cdot (ρ \cdot a))$

by the induction hypothesis. By 3.7.1 (5), the latter is equivalent to

$∀v ∈ R_i(u) M', f_0(v) ⊨ B(ρ_{f_0(u)f_0(v)} \cdot (f_{1u} \cdot a))$,

which is the same as

$∀w ∈ f_0[R_i(u)] M', w ⊨ B(ρ_{f_0(u)w} \cdot (f_{1u} \cdot a))$.

Since $f$ satisfies the conditions 3.3.1 (1)-(3), we have $f_0[R_i(u)] = R'_i(f_0(u))$.

Eventually

$M, u ⊨ A$ iff $M', f_0(u) ⊨ □_i B(f_{1u} \cdot a)$

as required.

Lemma 3.7.10 Let $f : Φ → Φ'$ be a morphism of Kripke sheaves, $M'$ a Kripke sheaf model over $Φ'$. Then there exists a unique model $M$ over $Φ$ such that $f : M → M'$. The same holds for all kinds of morphism. If $M'$ is intuitionistic, then $M$ is also intuitionistic.

Proof Similar to 3.3.12; an exercise for the reader.

Proposition 3.7.11 Let $Φ_1$ and $Φ_2$ be Kripke sheaves. If there exists a $p(=)$-morphism from $Φ_1$ onto $Φ_2$, then $ML^=(Φ_1) ⊆ ML^=(Φ_2)$ (or $IL^=(Φ_1) ⊆ IL^=(Φ_2)$ in the intuitionistic case).

Proof Similar to 3.3.13. Use Lemmas 3.6.9, 3.6.10 and 3.7.10.

In the case of disjoint Kripke sheaves there exists an equivalent definition of a morphism. Let us first present disjoint Kripke sheaves in an equivalent form.

Definition 3.7.12 A morphism of propositional Kripke frames

$f : F = (W, R_1, \ldots, R_N) → F' = (W', R'_1, \ldots, R'_N)$

is called etale if it has the unique lift property

$∀u ∈ W ∀v' ∈ W' (f(u)R'^*v' ⇒ ∃!v (f(v) = v' & uR^*v))$.

In this case every restriction $f \upharpoonright (F \upharpoonright u)$ is an isomorphism to $F' \upharpoonright f(u)$. In fact, this is a bijection, due to the monotonicity and the unique lift property. Its converse is also monotonic, due to the lift property.

Proposition 3.7.13
(1) Let $\Phi = (F, D, \rho)$ be a disjoint Kripke sheaf over a frame $F = (W, R_1, \ldots, R_N)$. Consider the frame of individuals $F^+ := (D^+, \rho_1^+, \ldots, \rho_N^+)$, where

$$D^+ = \bigcup_{u \in W} D_u, \quad \rho_i^+ := \bigcup_{u R_i v} \rho_{uv}.$$ 

Let $\tau : D^+ \to W$ be the map sending every individual to its world (i.e. $\tau(a) = u \iff a \in D_u$). Then $\tau : F^+ \to F$ is etale.

(2) Conversely, for any etale $p$-morphism of propositional frames $\tau : F' \to F$ there exists a disjoint Kripke sheaf $\Phi = (F, D, \rho)$, in which $F'$ is the frame of individuals and $D_u = \tau^{-1}(u)$.

Proof

(1) The monotonicity and the lift property of $\tau$ hold, since $\rho_{uv}$ is a function from $D_u$ to $D_v$ whenever $u R_i v$. The unique lift property

$$u R^* v \& a \in D_u \Rightarrow \exists! b \in D_v a(\rho^+)^* b$$

holds, since $(\rho^+)^*$ induces a function $\rho_{uv} : D_u \to D_v$ for any pair $(u, v) \in R^*$, according to 3.6.1, 3.6.2.

(2) In fact, for $u R^* v$ we can define the function $\rho_{uv} : D_u \to D_v$ by the condition $\rho_{uv}(a) = b \iff \tau(b) = v \& a(\rho')^* b$. Since $\tau$ is etale, this $b$ is always unique. The unique lift property also implies the properties 3.6.1 (1), (2) for $(F^*, D, \rho)$.

Proposition 3.7.14

(1) Let $\Phi = (F, D, \rho)$, $\Phi' = (F', D', \rho')$ be disjoint Kripke sheaves over $N$-modal frames, and let $\tau : F^+ \to F$, $\tau' : F'^+ \to F'$ be the corresponding etale morphisms. Also let $(f_0, f_1) : \Phi \to \Phi'$ be a morphism (in the sense of 3.7.1). Consider the map $f_1^+ : D^+ \to D'^+$ of total domains such that $f_1^+(a) = f_{1u}(a)$ for $a \in D_u$ (i.e., $f_1^+ = \bigcup_{u \in F} f_{1u}$). Then $f_1^+ : F^+ \to F'^+$ and the following diagram commutes:

$$
\begin{array}{ccc}
F^+ & \xrightarrow{f_1^+} & F'^+
\\
\downarrow \tau & & \downarrow \tau'
\\
F & \xrightarrow{f_0} & F'
\end{array}
$$
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\( f_1^+ \) is surjective on all fibres; it is a \( p \)-morphism whenever \( f_0 \) is a \( p \)-morphism.

(2) Conversely, every morphism \( g : F^+ \to F'^+ \), for which every \( g \upharpoonright D_u \) is a surjective map onto \( D_{f_0(u)} \), equals \( f_1^+ \) for some morphism \( (f_0, f_1) : \Phi \to \Phi' \).

**Proof** (1) Let us show that \( f_1^+ \) is monotonic. Suppose \( a \rho^+_i b, a \in D_u, b \in D_v \). Then by definition, \( b = a \rho_i v, f_1^+(a) = f_{1u}(a), f_1^+(b) = f_{1v}(b), u R_i v \). By 3.7.1,

\[
 f_{1v}(b) = f_{1u}(a)|f_0(v), \quad f_0(u) R_i f_0(v),
\]

and thus \( f_1^+(a) \rho_i^+ f_1^+(b) \).

The commutativity of the diagram is almost obvious: for \( a \in D_u \),

\[
 \tau'(f_1^+(a)) = \tau'(f_{1u}(a)) = f_0(u),
\]
since \( f_{1u} : D_u \to D_{f_0(u)} \). But \( f_0(u) = f_0(\tau'(a)) \).

Now we can check the lift property for \( f_1^+ \). In fact, suppose \( a \in D_u, f_1^+(a) = f_{1u}(a) \rho_i^+ b' \). Then \( b' = f_{1u}(a)|v' \) for some \( v' \in R_i(u') \), where \( f_{1u}(a) \in D_v \), i.e.,

\[
 u' = \tau'(f_1^+(a)) = f_0(\tau(a)) = f_0(u).
\]

Since \( f_0 \) has the lift property, we obtain \( v \in R_i(u) \) such that \( f_0(v) = v' \). Then by definition, \( a \rho_i^+(a|v) \). On the other hand, we obtain

\[
 f_{1v}^+(a|v) = f_{1u}(a|v) = f_{1u}(a)|f_0(v) = b',
\]

by 3.7.1. Therefore \( f_1^+ \) is a morphism. It is surjective, since every \( f_{1u} \) is surjective, by 3.3.1.

(2) Given \( g \), we define \( f_1 := (f_{1u})_{u \in F} \), with \( f_{1u} = g \upharpoonright D_u \). Then for any \( a \in D_u, u R_i v \)

\[
 f_{1v}(a|v) = g(a|v) \in D_{f_0(v)}.
\]

Since \( a \rho_i^+(a|v) \), it follows that \( f_{1u}(a) = g(a) \rho_i^+ g(a|v) \), i.e. \( g(a|v) = g(a)|f_0(v) \). Thus \((f_0, f_1)\) satisfies 3.7.1 (5), and so it is a morphism, \( g = f_1^+ \).

The definitions of a subframe, etc. from Section 3.3 can also be transferred to Kripke sheaves.

**Definition 3.7.15** Let \( \Phi = (F, D, \rho) \) be a Kripke sheaf over a propositional frame \( F = (W, R_1, \ldots, R_N) \), and let \( V \subseteq W \). Then we define the corresponding subsheaf as follows:

\[
 \Phi \upharpoonright V := (F \upharpoonright V, D \upharpoonright V, \rho \upharpoonright V),
\]

where \( F \upharpoonright V \) is the same as in Definition 1.3.13.

\[
 D \upharpoonright V := (D_u)_{u \in V}, \rho \upharpoonright V := (\rho_{uv})_{(u,v) \in R_i \upharpoonright V}.
\]

For \( u \in F \), the subsheaf generated by \( u \) or the cone \( \Phi \upharpoonright u \) is defined as \( \Phi \upharpoonright (W \upharpoonright u) \).
Exercise 3.7.16  (1) Prove an analogue to Generation lemma 3.3.18 for Kripke sheaves.

(2) Define disjoint sums of Kripke sheaves and prove an analogue to Proposition 3.3.36.

Lemma 3.7.17 For a Kripke sheaf $\Phi$ over a frame $F$

$$\text{ML}^{(=)}(\Phi) = \bigcap \{ \text{ML}^{(=)}(\Phi \uparrow u) \mid u \in F \},$$

and similarly for the intuitionistic case.

Proof  Cf. Lemma 3.3.21. ■

Note that if $u \approx_R v$ (i.e. if $u, v$ are in the same cluster) in a Kripke S4-frame $F$, then in every predicate Kripke frame $F = (F, D)$ we have $D_u = D_v$. So we can define the skeleton of $F$:

$$F^\sim := (F^\sim, D^\sim),$$

where $F^\sim$ is the skeleton of $F$ (Definition 1.3.40) and for a cluster $u^\sim$, $D^\sim_u := D_u$.

Analogously, one can easily see that for $u \approx_R v$ in an S4-based Kripke sheaf $\Phi = (F, D, \rho)$, the function $\rho_{uv}$ is a bijection between $D_u$ and $D_v$ with the converse $\rho_{vu}$. Thus for any model over $\Phi$,

$$u \models A(a_1, \ldots, a_n) \iff v \models A(a_1|v, \ldots, a_n|v).$$

So we can properly define the skeleton $\Phi^\sim$ of an S4-based Kripke sheaf $\Phi$. Then we obtain an analogue to 1.4.14:

Lemma 3.7.18 For an intuitionistic Kripke frame $F$, $\text{IL}(F) = \text{IL}(F^\sim)$; similarly, for intuitionistic Kripke sheaves.

Proof  Consider the case of Kripke sheaves. For an intuitionistic valuation $\xi$ in $\Phi$ there exists an intuitionistic valuation $\xi^\sim$ in $\Phi^\sim$ such that

$$\xi^\sim_u (P) = \xi_u (P)$$

for any $P, u$. Then the map $u \mapsto u^\sim$ is a p-morphism $(\Phi, \xi) \mapsto (\Phi^\sim, \xi^\sim)$. It remains to note that every intuitionistic valuation in $\Phi^\sim$ has the form $\xi^\sim$. ■

Therefore, in the semantics $K_{\mathcal{E}_{\text{int}}}$ is generated by Kripke sheaves over posets (and similarly, for the semantics $K_{\text{int}}$).

Finally note that Kripke sheaf morphisms are appropriate also for KFEs. Viz., we can define a morphism $F \longrightarrow F'$ of KFEs just as an arbitrary morphism of associated Kripke sheaves $\Theta(F) \longrightarrow \Theta(F')$.

Every strong morphism of KFEs corresponds to a morphism in this sense, but not the other way round. E.g. consider KFEs $F = (F, D, E)$ and $F' = (F, D', E')$, where $F$ is a reflexive singleton $u$, $|D_u| = 1$, $|D'_u| = 2$ and $E, E'$ are universal. Then obviously $\Theta(F) \cong \Theta(F')$, but there is no strong isomorphism from $F$ to $F'$. 

3.8 Transfer of completeness

In this section we show that in some simple cases Kripke and Kripke sheaf completeness transfer to extensions.

We begin with a lemma analogous to 1.3.45. Its proof is based on the existence of exact KFE-models for logics with equality, which will be proved in Chapter 6 (cf. Proposition 6.1.27).

Lemma 3.8.1 If an m.p.l.\((\equiv)\) L is conically expressive, then the rule \(\frac{A}{\Box^*A}\) is admissible in L.

Proof Since L has a characteristic model, it suffices to show that for any KFE-model, for any A, \(M \models A\) implies \(M \models \Box^*A\). We can argue similarly to 3.2.29, if we consider the Kripke model with the accessibility relation \(R^*\).

If \(M \models A(x_1,\ldots,x_n)\), then for any \(u \in M\) for any \(a_1,\ldots,a_n \in D_u\), \(M, u \models A(a_1,\ldots,a_n)\), by 3.5.7.

Hence, \(M, u \models \Box^*A(a_1,\ldots,a_n)\) by 3.3.39.

But obviously, \(\Box^*A(a_1,\ldots,a_n) = (\Box^*A)(a_1,\ldots,a_n)\). Thus \(M \models \Box^*A(x_1,\ldots,x_n)\), by 3.5.7.

Definition 3.8.2 A pure equality sentence is a predicate sentence with equality that does not contain predicate letters other than \(\equiv\).

In particular, every closed propositional formula is a pure equality sentence.

Theorem 3.8.3 If a conically expressive m.p.l.\(=\) or an s.p.l.\(=\) L is Kripke frame (respectively, Kripke sheaf) complete and C is a pure equality sentence in the language of L, then \(L + C\) is also Kripke frame (respectively, Kripke sheaf) complete.

Proof (I) Modal case. If \(L + C \not \models A\), then \(L \not \models \Box^*C \supset A\), since by Lemma 3.8.1, \(L + C \vdash \Box^*C\). By completeness, there exists a Kripke frame (or a KFE) \(F \models L\) and a Kripke (respectively, KFE) model \(M\) over \(F\) such that \(M, u \models \Box^*C \land \neg A\) for some \(u\). Hence \(M \uparrow u, u \models \Box^*C \land \neg A\) by Lemma 3.3.18 (1), and thus \(M \uparrow u \models C\), by Lemmas 3.3.39 and 1.3.19. But \(C\) is a simple equality sentence, so its truth value at every world does not depend on the valuation. Thus \(F \uparrow u \models C\). We also have \(F \uparrow u \models L\) by Lemma 3.3.18 (2), and therefore \(A\) is refuted in an \((L + C)\)-frame.

(II) Intuitionistic case. The proof is very similar. If \(L + C \not \models A\), then \(L \not \models C \supset A\). By completeness, there exists an intuitionistic Kripke frame \(F \models L\) and a Kripke model \(M\) over \(F\) such that \(M, u \not \models C \supset A\) for some \(u\). Hence \(M, v \not \models C\) and \(M, v \not \models A\) for some \(v\) accessible from \(u\). Then \(M \uparrow v, v \not \models C\) and \(M \uparrow v, v \not \models A\), by Lemma 3.3.18 (1), and thus \(M \uparrow v \not \models C\) (by 3.2.16). Since \(C\) is a pure equality sentence, it follows that \(F \uparrow v \models C\). \(F \uparrow v \models L\) by Lemma 3.3.18 (2), so \(A\) is refuted in an \((L + C)\)-frame.
An analogue of this theorem holds for strong completeness:

**Theorem 3.8.4** Let $L$ be an s.p.L.(=) or an m.p.L.(=), $\Gamma$ a set of pure equality sentences in the language of $L$. If $L$ is strongly Kripke (respectively, Kripke sheaf) complete, then $L + \Gamma$ is also strongly Kripke (Kripke sheaf) complete.

**Proof** (I) Modal case. If a theory $\Delta$ is $(L + \Gamma)$-consistent, then it is $(L + \Box^{\infty}\Gamma)$-consistent, since obviously $L + \Gamma \vdash \Box^{\infty}\Gamma$. So the theory $\Box^{\infty}\Gamma \cup \Delta$ is $L$-consistent. By strong completeness, there exists a Kripke frame (or a KFE) $F \models L$ such that $M, u \models \Box^{\infty}\Gamma \cup \Delta$ for some world $u$ in a Kripke model $M$ over $F$. Then $M, u \models \Box_\alpha B$ for any $B \in \Gamma$, $\alpha \in I_N^\infty$, so by 3.2.11 it follows that $M, v \models \Gamma$ for any $v \in R^*(u)$. Then as in the proof of 3.8.3, it follows that $F \uparrow u$ is an $(L + \Gamma)$-frame satisfying $\Delta$ at $u$.

(II) Intuitionistic case. If a theory $(\Delta, \Xi)$ is $L + \Gamma$-consistent, then the theory $(\Gamma \cup \Delta, \Xi)$ is $L$-consistent. By strong completeness, it is satisfiable in an $L$-frame $\Phi$ at some world $u$. Then similarly to the modal case, we obtain that $(\Delta, \Xi)$ is satisfiable in the $(L + \Gamma)$-frame $\Phi \uparrow u$. \[\square\]

Now let us consider equality expansions and prove some results generalising [Shimura and Suzuki, 1993].

**Lemma 3.8.5** Let $M = (\Phi, \xi)$ be an $L$-modal (respectively intuitionistic) Kripke model over $N$-modal (respectively, an S4-)KFE $F = (F, D^+, E)$ such that $M \models (\vdash) E^Q_A$ and $Q$ does not occur in $A$.

Let $\Phi' := (F, D^+, E')$, where

$$E'(a, b) := \{ u \mid M, u \models Q(a, b) \},$$

$M' := (\Phi', \xi')$, where for any $u \in F$, $P \in PL$

$$\xi'_u(P) := \begin{cases} 
\xi_u(P) & \text{if } P \text{ occurs in } A \text{ or } P = Q, \\
\emptyset & \text{otherwise}.
\end{cases}$$

Then

(1) $\Phi'$ is an $N$-modal (respectively, S4-) KFE;

(2) $M'$ is a modal (respectively, intuitionistic) KFE-model;

(3) for any $u \in M$, for any modal (respectively, intuitionistic) $D_u$-sentence $B$ in the same predicate letters as $A$

$$M, u \models (\vdash) B^Q \text{ iff } M', u \models (\vdash) B.$$

The same statements are true under the assumption $M \models (\vdash) E^Q$, with the following changes: $\xi' = \xi$; (3) holds for any $D_u$-sentence $B$ without occurrences of $Q$.  


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**Proof** (1) In fact,

\[ M \models \forall x \forall y (Q(x, y) \supset Q(y, x)) \]

implies

\[ E'(a, b) \subseteq E'(b, a) \]

for any \( a, b \in D^+ \); hence obviously \( E'(a, b) = E'(b, a) \).

Similarly the other members of \( E \) are responsible for the properties (2)–(4) from Lemma 3.5.2.

For example,

\[ M, u \models \forall x Q(x, x) \]

means that

\[ M, u \models Q(a, a) \]

for any \( a \in D_u \), hence

\[ u \in \bigcup_{a \in D^+} E'(a, a). \]

(2) Next, let us show that \( M' \) is a KFE-model. In fact, according to 3.5.2 and 3.5.3, we have to check

If \( P_t \) occurs in \( A \), this means

\[ (a_1, \ldots, a_n) \in \xi'_u(P^n_k) & u \in \bigcap_{i=1}^{n} E'(a_i, b_i) \Rightarrow (b_1, \ldots, b_n) \in \xi'_u(P^n_k). \]

If \( P^n_k \) occurs in \( A \), this means

\[ M, u \models P^n_k(a_1, \ldots, a_n) \land Q(a_1, b_1) \land \ldots \land Q(a_n, b_n) \supset P^n_k(b_1, \ldots, b_n), \]

which follows from \( M \models E^Q \).

If \( P^n_k = Q \), (\*) is equivalent to

\[ M, u \models Q(a_1, a_2) \land Q(a_1, b_1) \land Q(a_2, b_2) \supset Q(b_1, b_2), \]

which also follows from \( E^Q \).

If \( P^n_k \) does not occur in \( A \), \( P^n_k \neq Q \), (\*) holds trivially.

(3) Let \( MF^\neq_A(D_u) \) be the set of all \( D_u \) sentences constructed from \( MF^\neq_A \).

By definition of \( M' \),

\[ M', u \models B \text{ iff } M, u \models B \]

for any \( u \in M \), for any atomic \( B \in MF_A(D_u) \), and thus the equivalence

\[ M, u \models B^Q \text{ iff } M', u \models B \]

holds for any atomic \( B \in MF^\neq_A(D_u) \).

By induction we obtain that it holds for any \( B \in MF^\neq_A(D_u) \), since \( \Phi \), \( \Phi' \) are based on the same (\( F, D \)).

\[ \square \]
Theorem 3.8.6

(1) Let $C$ be a class of $N$-modal propositional Kripke frames such that $ML(C)$ is conically expressive. Then $ML^=(KEC) = ML(KEC)^\equiv$.

(2) If $C$ is a class of propositional Kripke $S4$-frames, then $IL^=(KEC) = IL(KEC)^\equiv$.

Proof (1) Since $ML(C)$ is conically expressive, $ML(KEC)$ is also conically expressive. So let us show that if a modal sentence $A$ does not contain $Q$, then

\[
\forall^* (\bigwedge_{A'} Q) \supset A^Q \in ML(KEC).
\]

(⇒) Suppose

\[
\forall^* (\bigwedge_{A'} Q) \supset A^Q \notin ML(KEC).
\]

Then

\[
M, u \vDash \forall^* (\bigwedge_{A'} Q) \land \neg A^Q
\]

for a KFE-model $M$ over $F \in C$ and for some $u \in M$. By the generation lemma

\[
M \upharpoonright u, u \vDash (\forall^* (\bigwedge_{A'} Q) \land \neg A^Q).
\]

Hence by a KFE-analogue of Lemma 3.3.39, $M \upharpoonright u \vDash Q^Q$ implies $M \upharpoonright u, u \vDash \neg A^Q$.

Then by Lemma 3.8.5, $M \upharpoonright u, u \vDash \neg A^Q$ implies

\[
(M \upharpoonright u)' \uparrow \neg A.
\]

Since $(M \upharpoonright u)'$ is a KFE-model over $F \upharpoonright u$, we have $A \notin ML^+(KEC \upharpoonright u))$, whence $A \notin ML^-(KEC)$ by Proposition 3.5.28.

(⇐) Suppose for some $F \in C$, $A \notin ML^+(KEF)$. Then there exists a KFE $\Phi = (F, D, \approx)$ and a model $N = (\Phi, \xi)$ such that $N, u \not\vDash A$ at some world $u$.

Consider a model $M$ over $\Phi$ with the trivial equality such that for any $v \in M, a, b \in D_v
\[
M, v \vDash Q(a, b) \iff a \approx_v b,
\]

and

\[
M, v \vDash B \iff N, v \vDash B
\]

for any $D_v$-sentence $B$ without $Q$ and equality. Then $M \vDash Q^Q$, as one can easily check, and $N = M'$, so we can apply Lemma 3.8.5. Thus $N, u \not\vDash A$ implies $M, u \not\vDash A^Q$, and therefore $\forall^* (\bigwedge_{A'} Q) \supset A^Q \notin ML(KEF) \subseteq ML(KEC)$.

The equivalence (♯) together with Proposition 2.14.4 show that sentences without $Q$ are the same in $ML^=(KEC)$ and $ML(KEF)^\equiv$. Now if $A$ contains $Q$, it can be replaced with another letter $Q'$ that does not occur in $A$. If $A' := [Q'(x, y)/Q(x, y)] A$, then $A = [Q(x, y)/Q'(x, y)] A'$; so for any logic $L$, $A \in L$ iff $A' \in L$. Therefore all sentences in these logics are the same. Since for any formula $A$ and a logic $L$, $A \in L$ iff $\neg A \in L$, the logics are really equal.
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(2) Similarly, in the intuitionistic case due to Proposition 2.14.5, it is sufficient to show that for a sentence $A$ and an extra predicate letter $Q$

$$A \in \text{IL}^= (K\mathcal{E}C) \iff \bigwedge \mathcal{E}_A^Q \supset A^Q \in \text{IL}(K\mathcal{E}C).$$

(##)
The proof is quite similar to the modal case.

$$(\Rightarrow)$$ Suppose $\mathcal{E}_A^Q \supset A^Q \notin \text{IL}(K\mathcal{E}C)$. Then for some intuitionistic KFE-model $M$ over $F \in \mathcal{C}$ and for some $u \in M$

$$M, u \models \mathcal{E}_A^Q \text{ and } M, u \not\models A^Q,$$

so by the generation lemma

$$M \upharpoonright u \models \mathcal{E}_A^Q \text{ and } M \upharpoonright u, u \not\models A^Q.$$

Then by Lemma 3.8.5,

$$(M \upharpoonright u)', u \not\models A,$$

which implies $A \not\in \text{IL}^= (K\mathcal{E}(C \uparrow))$, and so $A \not\in \text{IL}^= (K\mathcal{E}C)$ by Proposition 3.5.28.

$$(\Leftarrow)$$ Given $\Phi = (F, D, \lor)$ and an intuitionistic $N = (\Phi, \xi)$ such that $N, u \not\models A$, the same construction as in the modal case yields an intuitionistic $M$ such that $N = M'$. We leave the details to the reader. 

Theorem 3.8.7 Let $L$ be a strongly $K\mathcal{E}$-complete m.p.l. or s.p.l. Then $L^=$ is also strongly $K\mathcal{E}$-complete.

Proof

(I) Consider the modal case first. Let $\Gamma$ be an $L^=$-consistent $N$- modal theory. We may assume that $Q$ does not occur in $\Gamma$ — to avoid $Q$, we can appropriately rename all letters in $\Gamma$. Then the theory $\Box^\infty \mathcal{E}_N^Q \cup \Gamma^Q$ is $L$-consistent.

In fact, otherwise by 2.8.1

$$\Box^\infty \mathcal{E}_N^Q \vdash_L \neg(A_1^Q \land \ldots \land A_k^Q)$$

for some $A_1, \ldots, A_k \in \Gamma$. Hence $L^\equiv \vdash \neg(A_1 \land \ldots \land A_k)$ by 2.14.4, which means that $\Gamma$ is $L^=$-inconsistent.

Since $L$ is strongly $K\mathcal{E}$-complete, there exists a KFE $\Phi = (F, D^+, E)^{19}$ such that $\Phi \vdash L$ and for some model $M = (\Phi, \xi)$ and some world $u_0$ we have $M, u_0 \models \Gamma^Q \cup \Box^\infty \mathcal{E}_N^Q$.

We may also assume that $u_0$ is the root of $M$; in fact, otherwise $M$ and $\Phi$ can be replaced with $M \upharpoonright u_0$, and $\Phi \upharpoonright u_0$, respectively, since by Lemma 3.3.18,

$$M \upharpoonright u_0, u_0 \models \Gamma^Q \cup \Box^\infty \mathcal{E}_N^Q \text{ and } \Phi \upharpoonright u_0 \models L.$$ 

Thus $M, u_0 \models \Box^\alpha B^Q$ for any $B \in \mathcal{E}_N$, and so by 3.2.11, $M \models \mathcal{E}_N^Q$, since every $u \in M$ is covered by some $R_{\alpha}(u_0)$ (Lemma 1.3.19).

---

19 We use an equivalent definition of a KFE from Lemma 3.5.2.
Now consider $\Phi' = (F, D^+, E')$, where
\[
E'(a, b) := \{ u \mid M, u \models Q(a, b) \},
\]
and $M' = (\Phi', \xi')$ as in Lemma 3.8.5. Then $\Phi'$ is a KFE and $M', u_0 \models \Gamma$ by Lemma 3.8.5.

Since all equality axioms are valid in $\Phi'$, it remains to show that $\Phi' \models L$. It is again sufficient to check that $\Phi' \models A$ for any $A \in \mathcal{L}$ without $Q$, since otherwise $Q$ can be renamed. So for an arbitrary KFE-model $M_2 = (\Phi', \theta)$, let us show that $M_2 \models A$.

Since $Q$ does not occur in $A$ the truth values of $A$ do not depend on $\theta^+(Q)$. So we may further assume that for any $u$,
\[
M_1, u \models Q(a, b) \iff u \in E'(a, b).
\]

$M_2$ remains a KFE-model under this assumption, because
\[
E'(a_1, a_2) \cap E'(a_1, b_1) \cap E'(a_2, b_2) \subseteq E'(b_1, b_2).
\]

Now put $M_1 := (\Phi, \theta)$. Then $M_1$ is also a KFE-model.

In fact, the latter means

(1) $\left( a_1, \ldots, a_n \right) \in \theta_u(P^n_1) \land u \in \bigcap_{i=1}^{n} E(a_i, b_i) \Rightarrow \left( b_1, \ldots, b_n \right) \in \theta_u(P^n_1)$.

Since $M'_1$ is a KFE-model, we have

(2) $\left( a_1, \ldots, a_n \right) \in \theta_u(P^n_1) \land u \in \bigcap_{i=1}^{n} E'(a_i, b_i) \Rightarrow \left( b_1, \ldots, b_n \right) \in \theta_u(P^n_1)$.

Now (1) follows from (2), and the observation that

(3) $E(a, b) \subseteq E'(a, b)$ holds for any $a, b \in D^+$.

To check (3), note that it is equivalent to

(4) $M, u \models a = b \supset Q(a, b)$

provided $a, b \in D_u$.

But this holds, since
\[
M, u \models a = b \land Q(a, a) \supset Q(a, b)
\]
as soon as $M$ is a KFE-model, and $M, u \models Q(a, a)$ as we already know.

By definition, it follows that $M_2 = M'_1$ as in Lemma 3.8.5. Now $\Phi \models L$ implies $M_1 \models A$. $A = A^Q$, since $A$ is without equality; hence $M_2 \models A$ by Lemma 3.8.5.

(II) The intuitionistic case is considered in a very similar way. If $(\Gamma, \Delta)$ is an $L^\infty$-consistent intuitionistic theory, then $(\Gamma^Q \cup \mathcal{E}_0^Q, \Delta^Q)$ is also $L$-consistent. In fact, otherwise by 2.7.13,
\[
\Gamma^Q \cup \mathcal{E}_0^Q \not\models_L \Delta^Q
\]
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for some finite $\Delta_1 \subseteq \Delta$, which implies $\bigwedge \Gamma_1^Q \cup \varepsilon_0^Q \vdash \bigvee \Delta_1^Q$ for some finite $\Gamma_1 \subseteq \Gamma$ by 2.8.1. Hence $L^\equiv \vdash \bigwedge \Gamma_1 \supset \bigvee \Delta_1$ by 2.14.5, so $(\Gamma, \Delta)$ is $L^\equiv$-inconsistent.

By strong completeness, there exists a KFE $\Phi = (F, D^+, E)$ such that $\Phi \vDash L$ and for some intuitionistic model $M = (\Phi, \xi)$ and some world $u_0, M, u_0 \vDash (\Gamma^Q \cup \varepsilon_0^Q, \Delta^Q)$. As in the modal case, we may assume that $u_0$ is the root of $M$, and thus $M \vDash \varepsilon_0^Q$. So we can define a KFE $\Phi' := (F, D^+, E')$ such that

$E'(a, b) := \{u \mid M, u \vDash Q(a, b)\}$

and $M' = (\Phi', \xi')$ as in 3.8.5. Then by 3.8.5, $M'$ is a KFE-model such that

$M, u \vDash A^Q$ iff $M', u \vDash A$

for any $u \in M$ and $D_u$-sentence $A$. Hence $M', u_0 \vDash (\Gamma, \Delta)$.

$\Phi' \vDash L$ is proved as in the modal case — every intuitionistic model over $\Phi'$ corresponds to an intuitionistic model over $\Phi$ (for the basic language $\mathcal{L}_0$).

Theorem 3.8.8 Let $L$ be a KE-complete s.p.l. or a conically expressive m.p.l. Then $L^\equiv$ is also KE-complete.

Proof Essentially the same as for 3.8.7, but now we should take care of finiteness.

(I) Let us begin with the modal case. Consider an arbitrary formula $A$ in the language of $L$ such that $L^\equiv \not\vdash A$.

Then by 2.14.4

$\square^* \varepsilon_A^Q \nvdash L A^Q$,

i.e.

$L \nvdash (\bigwedge \square^* \varepsilon_A^Q) \supset A^Q$,

and thus by completeness, there exists a KFE $\Phi = (F, D^+, E)$ such that $\Phi \vDash L$ and for some world $u_0$ in some KFE-model $M = (\Phi, \xi)$ we have $M, u_0 \vDash \square^* \varepsilon_A^Q \cup \{\neg A^Q\}$.

As in 3.8.7, we may assume that $u_0$ is the root of $M$ and define

$\Phi' := (F, D^+, E')$,

where

$E'(a, b) := \{u \mid M, u \vDash Q(a, b)\}$.

and $M' = (\Phi', \xi')$ according to Lemma 3.8.5. Then $M'$ is a KFE-model.

By 3.8.5 we obtain

$M, u \vDash B^Q$ iff $M', u \vDash B$

for any $u \in M$, for any $D_u$-sentence $B$ using predicate letters only from $A$, and so $M', u_0 \nvdash A$. The proof of $\Phi' \vDash L$ is exactly the same as in 3.8.7.

(II) In the intuitionistic case suppose $L^\equiv \not\vdash A$. Then by 2.14.5, $\varepsilon_A^Q \nvdash L A^Q$, i.e. $L \nvdash \bigwedge \varepsilon_A^Q \supset A^Q$, and thus the latter formula is refuted in an intuitionistic
KFE-model $M = (\Phi, \xi)$ over a KFE $\Phi$ validating $L$. Again we may assume that $M$ is rooted and

$$M, u_0 \models (\xi^Q_A, \{A^Q\})$$

for the root $u_0$ of $M$. Now we define $\Phi'$, $M'$ as in 3.8.5. Then $M'$ is intuitionistic and

$$M, u \models B^Q \iff M', u \models B$$

for any $u$, whenever $B$ is a $D_u$-sentence in predicate letters from $A$. Eventually we obtain $\Phi' \models L$ and $M', u_0 \not\models A$. i.e. $\Phi'$ separates $A$ from $L^\equiv$.

Remark 3.8.9 The analogue of 3.8.8 does not hold for $\mathcal{K}$-completeness. In fact, $\mathcal{QH}$ and $\mathcal{QS4}$ are $\mathcal{K}$-complete, as we shall see in Chapter 7, but the equality-expansions $\mathcal{QH}^\equiv$, $\mathcal{QS4}^\equiv$ are $\mathcal{K}$-incomplete, by 3.10.6 below.

Corollary 3.8.10 If $L$ is an s.p.l. or a conically expressive m.p.l., then $L$ is $\mathcal{KE}$-complete iff $L^\equiv$ is $\mathcal{KE}$-complete.

Proof 'If' follows from 2.16.15, 'only if' - from 3.8.8.

Now let us prove an analogue of 2.16.14 for strong completeness.

Proposition 3.8.11 Let $L$ be an m.p.l. or an s.p.l., $S$ a semantics of Kripke frames or Kripke sheaves for the corresponding logics with equality. If $L^\equiv$ is conservative over $L$ and strongly $S$-complete, then $L$ is also strongly $S$-complete.

Proof Consider the modal case. Suppose $\Gamma$ is an $L$-consistent theory. Then $\Gamma$ is $L^\equiv$-consistent, due to the conservativity. So $\Gamma$ is satisfiable in an $L^\equiv$-frame (in the corresponding semantics), which is obviously an $L$-frame.

Corollary 3.8.12 If $L$ is an s.p.l. or a conically expressive m.p.l., then $L$ is strongly $\mathcal{KE}$-complete iff $L^\equiv$ is strongly $\mathcal{KE}$-complete.

Proof 'If' follows from 2.14.8 and 3.8.11, 'only if' - from 3.8.7.

3.9 Simulation of varying domains

Consider a formula $A$ in a language with or without equality, and let $U$ be a new unary predicate letter. Let $A_U$ be the formula obtained from $A$ by relativising all quantifiers under $U$; formally:

$$A_U := A \text{ for } A \text{ atomic;}$$

$$(\exists x B)_U = \exists x(U(x) \land B_U);$$

$$(\forall x B)_U = \forall x(U(x) \supset B_U);$$

$$(B * C)_U = B_U * C_U \quad \text{for } * \in \{\lor, \land, \land\}.\)$$

Next, put

$$A^*_U := \exists x U(x) \supset A_U.$$
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Let $M = (F, \theta)$ be a Kripke model over a rooted frame $F = (W, R, D)$ with root $v_0$, such that $M, v_0 \models \exists x U(x)$. Consider the frame $F^U := (W, R, D^U)$ with

$$D^U_v := \{ a \in D_v \mid M, v \models U(a) \}$$

and a Kripke model $M^U := (F^U, \theta_U)$ such that

$$(\theta_U)^+(P(a_1, \ldots, a_n)) := \{ v \mid a_1, \ldots, a_n \in D^U_v \} \cap \theta^+(P(a_1, \ldots, a_n)) = \{ v \mid M, v \models P(a_1, \ldots, a_n) \land \bigwedge_{i=1}^n U(a_i) \}$$

for $P \neq U$,

$$(\theta_U)^+(U(a)) := \{ v \mid a \in D_v \}.$$ 

$F^U$ is well defined. In fact, $D^U_{v_0} \neq \emptyset$ since $M, v_0 \models \exists x U(x)$; by truth preservation $D^U_w \subseteq D^U_v$ whenever $wRv$.

The model $M^U$ is intuitionistic, since $M$ is intuitionistic.

We can obviously extend the relativisation to formulas with constants. Then the following holds:

**Lemma 3.9.1** For any world $v$ in $M$, for any $D_v$-sentence $B$ that does not contain $U$

$$M, v \models B_U \iff M^U, v \models B.$$ 

**Proof** Easy by in induction. By definition,

$$M, v \models B \iff M^U, v \models B.$$ 

for any atomic $D_v$-sentence $B$. 

**Lemma 3.9.2** For every Kripke model $M' = (F, D', \theta')$ over a poset $F$ with root $v_0$ there exists a Kripke model $M = (F \odot V, \theta)$ over $F$ with a constant domain such that $M, v_0 \models \exists x U(x)$ and $M^U = M'$.

**Proof** In fact, put

$$V := (D')^+;$$

$$\theta^+(U(a)) := \{ v \mid a \in D'_v \} ;$$

$$\theta^+(P(a)) := \{ v \mid M', v \models P(a) \land \bigwedge_{i=1}^n U(a_i) \}$$

for $P \in PL^n$, $a = (a_1, \ldots, a_n)$.

Then obviously $M, u_0 \models \exists x U(x)$ and

$$a \in D^U_v \iff M, v \models U(a) \iff a \in D'_v,$$

and for any $v \in M$, $P \in PL^n$

$$a \in (D_v)^n \text{ iff } M, v \models P(a) \text{ iff } M', v \models P(a).$$

Therefore $M' = M^U$. 

Theorem 3.9.3

(1) Let $F$ be a rooted poset, $A$ a sentence without equality and without occurrences of $U$. Then

$$A \in \text{IL}(\mathcal{K}F) \iff A_U \in \text{IL}(\mathcal{C\mathcal{K}}F).$$

(2) Similarly, if $A \in IF^\approx$, then

$$A \in \text{IL}^\approx(\mathcal{K}\mathcal{E}F) \iff A_U \in \text{IL}^\approx(\mathcal{C\mathcal{K}\mathcal{E}}F).$$

Proof We prove only (1); the proof of (2) is similar. First, there exist a model $M_0$ over $F \cap V$ and a world $v_0 \in F$ such that $M_0, v_0 \models \exists x \bar{U}(x)$ and $M_0, v_0 \not\models A_U$. Put $M := M_0 \downarrow v_0$. Then by the generation lemma and Lemma 3.9.1, $M_U, v_0 \not\models A$. Thus $A \not\models \text{IL}(\mathcal{K}(\mathcal{F} \uparrow v_0))$, and so $A \not\in \text{IL}(\mathcal{K}\mathcal{F})$.

The other way round, suppose $A \not\in \text{IL}(\mathcal{K}\mathcal{F})$, i.e. there exists a frame $F' = (F, D')$ with a model $M'$ such that $M', v_0 \not\models A$; now $v_0$ is root of $F$. Consider a model $M$ according to Lemma 3.9.2. Then $M' = M_U$, so $M, v_0 \not\models A_U$ by 3.9.1. Hence $A_U \in \text{IL}(\mathcal{K}\mathcal{F})$.

Corollary 3.9.4

(1) $\text{IL}(\mathcal{K}F) \leq_m \text{IL}(\mathcal{C\mathcal{K}}F)$.

(2) $\text{IL}^\approx(\mathcal{K}\mathcal{E}F) \leq_m \text{IL}^\approx(\mathcal{C\mathcal{K}\mathcal{E}}F)$.

3.10 Examples of Kripke semantics

According to the general definition from Chapter 2, the Kripke semantics generated by a class $\mathcal{C}$ of Kripke sheaves (or KFEs) is

$$S(\mathcal{C}) = \{ML(\Phi) \mid \Phi \in \mathcal{C}\}.$$

Let us consider three examples of Kripke semantics.

Definition 3.10.1 A Kripke frame with equality $F = (F, D, \preceq)$ is called monic if

$$(\ast) \quad \forall u, v \in F \forall a, b \in (D_u \cap D_v) \quad (a \preceq_u b \iff a \preceq_v b);$$

$F$ is a KFE with closed equality (or CE-KFE) if for any $i$ for any $a, b \in D^+$ for any $u, v \in F$

$$(\ast\ast) \quad \forall i \forall u, v \in F \quad (uR_i v \Rightarrow \forall a, b \in D_u \quad (a \preceq_u b \iff a \preceq_v b));$$

or equivalently, the premise $uR_i v$ can be replaced with $uR^*v$. An intuitionistic KFE with closed equality is also called a KFE with decidable equality, or a DE-KFE.
3.10. EXAMPLES OF KRIPKE SEMANTICS

If \( F \) is presented in the form \((F, D^+, E)\) as in Lemma 3.3.2, then the above conditions are rewritten as follows:

\[
(*) \quad \forall a, b \in D^+ \left( E(a, b) = \emptyset \lor E(a, b) = E(a) \cap E(b) \right);
\]

\[
(**) \quad \forall a, b \in D^+ \left( \forall i \left( E(a) \cap E(b) \cap R_i^{-1} E(a, b) \subseteq E(a, b) \right) \right).
\]

Definition 3.10.2 A disjoint Kripke sheaf \( \Phi = (F, D, \rho) \) is called monic if for any \( a, b \in D^+, u, v \in F \)

\[
(1) \quad aR^*u \& bR^*u \& aR^*v \& bR^*v \& (a|u) = (b|u) \Rightarrow (a|v) = (b|v);
\]

\( \Phi \) is a CE-sheaf (or a DE-sheaf in intuitionistic case) if

\[
(2) \quad \forall u, v \in F \forall a, b \in D_u \forall i \left( uR_i v & (a|v) = (b|v) \Rightarrow a = b \right).
\]

The necessary modification for the non-disjoint case is quite obvious.

These three types of KFEs and Kripke sheaves fully correspond to each other and generate equal semantics:

Lemma 3.10.3

\[
(1) \quad A \text{ disjoint Kripke sheaf } \Phi = (F, D, \rho) \text{ is monic iff the KFE } G(\Phi) \text{ is monic. The other way round, a KFE } F \text{ is monic iff the Kripke sheaf } \Theta(F) \text{ is monic.}
\]

\[
(2) \quad F \text{ is a CE-KFE iff } \Theta(F) \text{ is a CE-sheaf.}
\]

Thus a Kripke sheaf \( \Phi \) is CE iff the KFE \( G(\Phi) \) is CE. Recall that \( \Theta(G(\Phi)) \) is isomorphic to \( \Phi \).

Proof

Consider the monic case (the CE-case is quite obvious).

Let \( F = (F, D, \succ) \) be a monic KFE,

\[
\Theta(F) = (F, D', \rho), \quad D'_u = (D_u / \succ_u), \quad \rho_{uv}(a_u) = a_v \text{ if } v \in F^u, a \in D_u \subseteq D_v.
\]

Assume \( wR^*y, wR^*v, w'R^*u, w'R^*v, a_w \in D'_w, b_w \in D_u \cap D_v \) and \( a \succ_w b \).

Then \( a \succ_v b \) since \( F \) is monic, i.e. \( \rho_{uv}(a_w) = \rho_{w'}(b_w) \).

Now, let \( \Phi = (F, D, \rho) \) be a monic disjoint Kripke sheaf, \( G(\Phi) = (F, D', \succ') \).

Let \( u, v \in F, a, b \in D'_u \cap D'_v, a \succ'_u b \), i.e. \( a \in D_w, b \in D_{w'} \) for some \( w, w' \in (R^*)^{-1}(u) \cap (R^*)^{-1}(v) \) and \( \rho_{uw}(a) = \rho_{w'v}(b) \). Then \( \rho_{uv}(a) = \rho_{w'v}(b) \) since \( \Phi \) is monic, i.e. \( a \succ_v b \).

Example 3.10.4 (Warning) In general if \( \Theta(F) \) is monic, then \( F \) is not necessarily monic (even in the intuitionistic case). In fact, let us consider the KFE \( F = (F, D, \succ) \) based on the poset \( F = \{u, v\} \), in which \( u \) and \( v \) are incomparable, \( D_u = D_v = \{a, b\}, \succ_u = (D_u)^2, \succ_v = id_{D_v} \). Then \( F \) is not monic, since \( a \succ_u b, \) but not \( a \succ_v b \). On the other hand,

\[
\Theta(F) = (F, D', \rho), D'_u = \{a_u\}, D'_v = \{a_v, b_v\}, \rho_{uu} = id_{D'_u}, \rho_{vv} = id_{D'_v}.
\]
This Kripke sheaf is monic, since
\[ aR^*w \& aR^*w' \Rightarrow w = w' \]
in \( \Theta(F) \).

Speaking informally, the individual \( a_u = b_u \) in \( \Theta(F) \) does not know that it corresponds to \( a_v \neq b_v \).

One can construct a similar example based on a rooted S4-frame.

**Lemma 3.10.5**

1. For any N-modal predicate Kripke frame \( F, i \leq N \)
\[ F \models \forall x \forall y (x \neq y \Rightarrow \Box_i(x \neq y)) \text{.}\]

2. For any intuitionistic Kripke frame \( F, F \models DE \).

**Proof**

1. If \( u \models a \neq b \) for \( a, b \in D_u \), then \( a \neq b \), and thus \( v \models a \neq b \) for any \( v \in R_i(u) \); hence \( u \models \Box_i(a \neq b) \).

2. We have to show that \( u \models a = b \lor a \neq b \) for any \( a, b \in D_u \). In fact, if \( a = b \), then \( u \models a = b \), by definition. If \( a \neq b \), then \( v \models a = b \) for any \( v \in R(u) \); thus \( u \models a \neq b \), by definition.

**Proposition 3.10.6** The logics \( QK^=, QH^= \) are \( K \)-incomplete.

Moreover, let \( \Lambda \) be a modal propositional logic such that \( K \subseteq \Lambda \subseteq S4 \) or \( S4 \subseteq \Lambda \subseteq S5 \), or an intermediate propositional logic \( \neq CL \). Then \( QA^= \) is \( K \)-incomplete.

**Proof** In this case a two-element reflexive chain \( Z_2 \) validates \( \Lambda \). Thus \( \Lambda \) (and \( QA^= \)) is valid in the Kripke sheaf \( \Phi = (Z_2, D, \rho) \), in which \( D = \{a, b\}, D_1 = \{c\}, a|1 = b|1 = c \). Obviously, \( \Phi \not\models CE \) and \( \Phi \not\models DE \). Therefore we obtain \( QA^= \not\models CE \) in the modal case and \( QA^= \not\models DE \) in the intuitionistic case. But by Lemma 3.10.5, \( QK^= \models_K CE, QH^= \models_K DE \), which yields incompleteness.

The semantics generated by monic KFEs (or Kripke sheaves) is denoted by \( MK^= \), \( MK^=_{int} \), and the semantics generated by CE-KFEs (or Kripke sheaves) is denoted by \( KCE^= \), \( KCE^=_{int} \). Obviously every monic KFE is CE (and similarly for Kripke sheaves). Lemma 3.10.5 shows that every simple KFE (or a simple Kripke sheaf — with inclusions \( \rho_{uv} \)) is also monic. So we have:
\[ K^= \subseteq MK^= \subseteq KCE^=, \]
and similarly for superintuitionistic logics. All these inclusions are actually strict, as we will see later on.

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20 Recall that this formula is denoted by \( CE \) (Section 2.6).
Lemma 3.10.7

(1) A KFE (or a Kripke sheaf) is CE iff it validates CE : ∀x∀y (◊(x = y) ⊃ (x = y)).

(2) An intuitionistic KFE (or a Kripke sheaf) is DE iff it validates DE : ∀x∀y ((x = y) ∨ (x ≠ y)).

Proof

(1) immediate

(2) (for KFEs). Let a, b ∈ D_u. Clearly u ∤ (a = b) ∨ (a ≠ b) iff ¬(a ≻_u b) & ∃v (uRv & a ≻_v b).

Example 3.10.4 shows that monic KFEs do not have an adequate logical characterisation. In fact, KFEs F and G(Θ(F)) have the same modal logic, but only one of them is monic. A similar example can be constructed for monic Kripke sheaves.

In some special cases the three semantics introduced are equivalent. Let us give two examples.

Recall that simple Kripke sheaves correspond to Kripke frames with trivial equality.

Lemma 3.10.8 Every CE-Kripke sheaf over an S4-tree F is isomorphic to a simple Kripke sheaf over F.

Proof Let Φ = (F, D, ρ) be the original Kripke sheaf (which we suppose disjoint) and let F = (W, R). Consider the following relation between individuals:

\[ a ∼ b : = \exists d (dρa & dρb). \]

Since F is a tree, ∼ is an equivalence relation. In fact, if a ∼ b, b ∼ c and dρa, dρb, eρc, eρb, d ∈ D_u, e ∈ D_v, then both u, v see the world of b, and thus they are comparable. We may assume that uRv. So dρd' for some d' ∈ D_v, and also d'ρb. Hence dρd', so by (CE) it follows that d' = e, and thus dρc, which implies a ∼ c.

Then we factorise the domains: D̃_u := D_u/ ∼. It follows that D̃_u ⊆ D̃_v whenever uRv, hence we obtain a PKF (F, D̃). One can easily show that Φ is isomorphic to Θ(F).

Definition 3.10.9 A propositional Kripke frame (W, R) is called

- directed iff ∀x, y ∈ W R(x) ∩ R(y) ≠ ∅;
- weakly directed iff ∀x, y ∈ W (R(x) ∩ R(y) is directed (if it is non-empty) as the frame with the relation R or R⁻¹ restricted to it).
Lemma 3.10.10 Every CE-sheaf over a weakly directed frame is monic.

Proof Assume that individuals \( a, b \) in a given CE-sheaf \( \Phi \) are both related to worlds \( u, v \). Also assume that \( (a|u) = (b|u), uRw, vRw \). Then

\[
(a|w) = (a|u)|w = (b|u)|w = (b|w),
\]

i.e.

\[
(a|v)|w = (b|v)|w,
\]

and thus \( (a|v) = (b|v) \) by (2). Similarly we obtain \( (a|v) = (b|v) \) in the case, when \( a, b \) are related to some world in \( R^{-1}(u) \cap R^{-1}(v) \).

Lemma 3.10.11 Let \( \Phi = (W, R, D, \rho) \) be a monic Kripke sheaf over a directed frame. Then \( \text{ML}(\Phi) = \text{ML}(F) \) for some PKF \( F \) over \( (W, R) \).

Proof Since \( (W, R) \) is directed and the inheritance relation \( \rho \) is transitive, we can construct the direct limit

\[
D_0^+ = \lim_{\rightarrow}(D_u : u \in W).
\]

Namely, we take \( D_0^+ = D^+/\sim \), where

\[
a \sim b \text{ iff } a, b \text{ have a common inheritor.}
\]

There exist natural embeddings \( i_u : D_u \rightarrow D_0^+ \) (the injectivity follows from (1)), such that the diagram

\[
\begin{array}{ccc}
D_v & \xrightarrow{\rho_{uv}} & D_v \\
\downarrow i_v & & \downarrow i_v \\
D_u & \xrightarrow{i_u} & D_0^+
\end{array}
\]

commutes. Consider the PKF \( F = (W, R, D_0) \) with \( (D_0)_u = i_u(D_u) \). It can be easily proved that \( \Phi \) and \( \Theta(F) \) (see Section 3.3) are isomorphic. Thus, \( \text{ML}(\Phi) = \text{ML}(F) \) by Lemma 3.3.4.

Corollary 3.10.12 \( \text{Kdir} = \text{MKdir} = \text{KCEdir} \), where \( \text{dir} \) means the restriction to directed frames.

Definition 3.10.13 A KFE is called a KFE with a constant domain if all its individual domains are equal.

We use the notation \( F = (F \odot V, \times) \) for KFEs with a constant domain (where \( V = D_u \) for all \( u \in F \)).
Definition 3.10.14 A KFE with a set of possible worlds $W$ is called flabby if the measure of existence of every its individual is $W$.

So, a KFE is flabby iff it has a constant domain.

Now let us describe flabby Kripke sheaves corresponding to flabby KFEs.

Definition 3.10.15 A Kripke sheaf $\Phi = (F, D, \rho)$ is called flabby if there exists a domain $D_0$ and a family of surjections $\rho_{0u} : D_0 \rightarrow D_u$ for $u \in F$ such that $\rho_{0v} = \rho_{0u} \circ \rho_{uv}$, whenever $u \asymp^* v$ in $F$.

Lemma 3.10.16

(1) If $F = (F, D, \asymp)$ is a flabby KFE, then $\Theta(F)$ is a flabby Kripke sheaf.

(2) For a flabby Kripke sheaf $\Phi = (F, D, \rho)$ there exists a flabby KFE $F$ such that $\Theta(F)$ is isomorphic to $\Phi$.

Proof

(1) Let $D_0$ be the domain of any world in $F$ and $\rho_0(a) = a_u(= a/ \asymp_u)$ for $u \in F$.

(2) Let $F = (F \odot D_0, \asymp)$, where $a \asymp_u b$ iff $\rho_{0u}(a) = \rho_{0u}(b)$.

Definition 3.10.17 A Kripke sheaf $(F, D, \rho)$ is called meek if all the maps $\rho_{uv}$ (for $u \asymp^* v$) are surjective. A KFE $(F, D, \asymp)$ is called meek if

$$\forall \bar{v} \forall u \forall v \in R_1(u) \\forall b \in D_v \exists a \in D_u \ a \asymp_u b,$$

i.e. if any individual in any accessible world is locally equal to some individual from the present world.

Obviously, every flabby Kripke sheaf (or KFE) is meek — if $\rho_{0u} \circ \rho_{uv}$ is surjective, then $\rho_{uv}$ is surjective. Every flabby KFE is meek, since it has a constant domain.

Lemma 3.10.18 Meek Kripke sheaves, meek KFEs, flabby Kripke sheaves, and flabby KFEs generate the same semantics.

Proof Note that every meek Kripke sheaf over a rooted frame is flabby.

Lemma 3.10.19 $CD^T \in (QS4 + Ba)$.

Proof This follows from Kripke completeness of $QS4 + Ba$, see Chapter 7. A syntactic proof is an exercise for the reader.
Remark 3.10.20 In Chapter 7 we will show that

\[ \text{IL}(\text{S4-PKF}s \text{ with constant domains}) = \text{QH} + \text{CD}, \]

\[ \text{ML}(N\text{-modal PKFs with constant domains}) = \text{QS4} + \{Ba_1, \ldots, Ba_N\}. \]

Remark 3.10.21 In a meek Kripke sheaf the following holds:

\[ u \models \forall x B(x, a_1, \ldots, a_n) \iff \forall c \in D_u \ u \models B(c, a_1, \ldots, a_n). \]

Lemma 3.10.22 (i) Let \( \Phi \) be an S4-based Kripke sheaf. Then:

\[ \text{CD} \in \text{IL}(\Phi) \iff Ba \in \text{ML}(\Phi) \iff \Phi \text{ is meek}. \]

(ii) For an S4-based KFE \( F \),

\[ F \models \text{CD} \iff F \text{ is meek.} \]

(iii) For an S4-based PKF \( F \),

\[ F \models \text{CD} \iff \text{every its cone } F^u \text{ has a constant domain}. \]

(1m) For a N-modal Kripke sheaf \( \Phi \),

\[ \Phi \models \bigwedge_{i=1}^{N} Ba_i \iff \Phi \text{ is meek.} \]

(2m) For an N-modal KFE \( F \)

\[ F \models \bigwedge_{i=1}^{N} Ba_i \text{ if } F \text{ is meek.} \]

(3m) For an N-modal PKF \( F \),

\[ F \models \bigwedge_{i=1}^{N} Ba_i \text{ iff every } F^u \text{ has a constant domain.} \]

Proof

(1) Let us consider only the 1-modal case; the intuitionistic case is similar.

(Only if.) Let \( v, w \in F, v R u \) and suppose there is \( b_0 \in D_w - \rho_{vu}(D_v) \). Take the valuation \( \xi \) in \( \Phi \) such that \( \xi, u \models P(a) \iff vu \) and \( a \in \rho_{vu}(D_v) \).

Then \( v \models \forall x \square P(x) \), since \( u \models P(\rho_{vu}(a)) \) for any \( a \) such that \( vu \) and for any \( a \in D_v \). On the other hand, \( w \not\models P(b_0) \). It follows that \( w \not\models \forall x P(x) \) and \( v \not\models \square \forall x P(x) \).

(If.) Suppose that \( \xi, u \models \forall x \square P(x) \) and \( \xi, u \not\models \square \forall x P(x) \) for some valuation \( \xi \) in \( \Phi \) and \( u \in F \). Then \( \xi, v \not\models P(b) \) for some \( v \) such that \( vu \) and some \( u \in D_v \). Since \( \Phi \) is meek, we can take \( a \in D_u \) such that \( b = \rho_{uv}(a) \). Then \( u \not\models \square P(a) \) and \( u \not\models \forall x \square P(x) \). This is a contradiction.
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(2) (Only if.) Let \( v, w \in F, vRw \), and suppose there is \( b_0 \in D_w \) such that \( \forall b \in D_v \ b \not\approx_w b_0 \). Take the valuation \( \xi \) in \( F \) such that:

\[
\xi, u \models Q \iff vRu \& u \neq v,
\]

\[
\xi, u \models P(a) \iff vRu \& \exists a' \in D_v \ a \approx_v a'.
\]

Then \( v \models \forall x(Q \lor P(x)) \), since \( v \models P(a) \) for any \( a \in D_v \) and \( u \models Q \) for any \( u \in R(v) \) with \( u \neq v \). On the other hand, \( v \not\models Q \) and \( v \not\models \forall xP(x) \) since \( w \not\models P(b_0) \).

(If.) Suppose that \( \xi, u \models \forall x(Q \lor P(x)), \xi, u \not\models Q, u \not\models \forall xP(x) \) for some valuation \( \xi \) in \( F \) and \( u \in F \). Then \( v \not\models P(b) \) for some \( v \in R(u), b \in D_v \). Since \( F \) is meek, we can take \( a \in D_u \) such that \( b \approx_v a \). Then \( u \not\models P(a) \) and \( u \not\models Q \lor P(a) \). This is a contradiction.

(3) This directly follows from (2) and from the obvious fact that a PKF \( F \) has a constant domain iff its corresponding KFE with trivial equality is meek.

As we noticed, every flabby Kripke sheaf is meek. On the other hand, a rooted meek Kripke sheaf is flabby.

Therefore, flabby Kripke sheaves generate the semantics equivalent to meek Kripke sheaves. Moreover, this semantics equals the semantics of flabby KFEs.

Let \( CK \) be the semantics generated by all PKFs with constant domains.

**Definition 3.10.23** A Kripke sheaf is called bijective if all its transition maps are bijections.

Let \( BK \) be the semantics generated by all bijective Kripke sheaves.

**Proposition 3.10.24** \( CK \subseteq BK \).

**Proof** \( CK \subseteq BK \) since every PKF with a constant domain \( F \) corresponds to the bijective Kripke sheaf \( \Theta(F) \). To prove that the converse inclusion is not true, consider the bijective Kripke sheaf \( \Phi_0 := (W, R, D, \rho) \), where

\[
W = \{u, v, w\}, \quad D = \{a, b, c, d\}, \quad D_u = \{a\}, \quad D_v = \{b\}, \quad D_w = \{c, d\},
\]

\[
R = id_W \cup \{(u, v)\}, \quad \rho = id_D \cup \{(a, b)\},
\]

see Fig. 3.4.

Let

\[
C' := \forall x \forall y (P(x) \supset P(y)),
\]

\[
K' := q \supset \Box q.
\]

Then obviously,

\[
\Phi_0 \models C' \lor K',
\]

but

\[
\Phi_0 \not\models C'.
\]
CHAPTER 3. KRIPKE SEMANTICS

However, for every PKF $F = F \odot V$, we have

$$|V| = 1 \Rightarrow F \models C,$$

and

$$|V| > 1 \& F \not\models K' \Rightarrow F \not\models C' \lor K';$$

so

$$F \models C' \lor K' \text{ only if } (F \models C' \text{ or } F \models K').$$

Thus $\text{ML}(\Phi_0) \notin \mathcal{C}K$.

The above example also shows:

**Lemma 3.10.25** $\mathcal{C}K$ lacks the collection property (CP) from Section 2.16.

**Proof** In fact,

$$\text{ML}(\Phi_0) = \text{ML}(\Phi_1) \cap \text{ML}(\Phi_2),$$

where $\Phi_1$ are $\Phi_2$ the restrictions of $\Phi_0$ respectively to $\{u,v\}$ and $\{w\}$; on the other hand,

$$\text{ML}(\Phi_1), \text{ML}(\Phi_2) \in \mathcal{C}K.$$  

Nevertheless, $\mathcal{C}K$ and $\mathcal{B}K$ are equivalent, as we shall see below.

**Lemma 3.10.26** Let $\Phi = (F, D, \rho)$ be a bijective Kripke sheaf over a rooted frame $F$. Then $\text{ML}(\Phi) \in \mathcal{C}K$.

**Proof** Let $u_0$ be the root of $F$, $D_0 = D_{u_0}$, and consider the PKF $F = F \odot D_0$. Since $\rho$ gives rise to a family of bijections between $D_0$ and each $D_u$, the Kripke sheaves $\Theta(F)$ and $\Phi$ are isomorphic. So, $\text{ML}(\Phi) = \text{ML}(F)$ by Lemma 3.3.4.

**Corollary 3.10.27** $\mathcal{C}K_r = \mathcal{B}K_r$ ($r$ means the restriction to rooted frames).
3.11. **ON LOGICS WITH CLOSED OR DECIDABLE EQUALITY**

**Lemma 3.10.28** Every logic from $\mathcal{B} \mathcal{K}$ is $\mathcal{C} \mathcal{K}$-complete.

**Proof** Let $\Phi$ be a bijective Kripke sheaf over $F$. Then

$$\text{ML}(\Phi) = \bigcap_{u \in F} \text{ML}(\Phi \uparrow u)$$

by Lemma 3.7.17. Thus $\text{ML}(\Phi)$ is $\mathcal{C} \mathcal{K}$-complete by Lemmas 3.10.26 and 2.16.3.

Lemmas 3.10.24, 3.10.28 and Corollary 2.12.4 yield

**Corollary 3.10.29** $\mathcal{C} \mathcal{K} \simeq \mathcal{B} \mathcal{K}$.

Here is the sequence of all Kripke semantics considered above:

$$\mathcal{B} \mathcal{K} \simeq \mathcal{C} \mathcal{K} \prec \mathcal{K} \mathcal{K} \prec \mathcal{K} \mathcal{C} \mathcal{E} \prec \mathcal{K} \mathcal{E}.$$  

3.11 On logics with closed or decidable equality

In this section we show that unlike equality-expansions, extensions with closed or decidable equality may be nonconservative.

3.11.1 Modal case

Consider the following modal formula:

$$\mathcal{B} \mathcal{A}^1 := \Diamond \exists x P(x) \supset \exists x \Diamond \exists y (P(y) \land \Diamond (x = y)).$$

Note that

$$\mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A}^1 \subseteq \mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A}.$$  

We will see that these inclusions are actually strict (Lemma 3.11.4). Nevertheless:

**Lemma 3.11.1** $\mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A} = \mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A}^1$.

**Proof** We shall use 'naive reasoning' in $\mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A}^1$. Assume that $\Diamond \exists x P(x)$. Then (from $\mathcal{B} \mathcal{A}^1$)

$$\exists x \Diamond \exists y (P(y) \land \Diamond (x = y)).$$

Using $\Diamond (x = y) \supset x = y$, we obtain

$$\exists x \Diamond \exists y (P(y) \land (x = y)),$$

i.e. $\exists x \Diamond P(x)$.

**Corollary 3.11.2** $\mathcal{Q} \mathcal{S} 4^= + \mathcal{B} \mathcal{A}^1 \models \mathcal{K} \mathcal{B} \mathcal{A}$.

**Lemma 3.11.3** Let $\Phi$ be an $\mathcal{S} 4$-based Kripke sheaf. Then $\Phi \models \mathcal{B} \mathcal{A}^1$ iff $\Phi$ satisfies:
(1) \( \forall u \forall v \ (uRv \Rightarrow \forall b \in D_v \ \exists a \in D_u \ \exists w \ (vRw \& \rho_{uw}(a) = \rho_{vw}(b))) \).

**Proof** (Only if.) Let \( u, v \in F, uRv \) and suppose there is \( b \in D_v \) such that \( \forall a \in D_u \ \forall w \ (uRw \Rightarrow \rho_{uw}(a) \neq \rho_{vw}(b)) \). Take the valuation \( \xi \) in \( \Phi \) such that

\[
\xi, s \vdash P(c) \text{ iff } s = v \text{ and } c = b.
\]

Then clearly \( \xi, u \vdash \Box \exists x P(x) \). On the other hand, suppose \( \xi, u \vdash \exists x \Box y (P(y) \land \Box (x = y)) \). Then there are \( a \in D_u, w \in F \) and \( c \in D_w \) such that \( \xi, w \vdash P(c) \land \Box (\rho_{uw}(a) = c) \). Since \( \xi, w \vdash P(a) \), we have \( w = v \) and \( c = b \). We have \( \xi, v \vdash \Box (\rho_{uw}(a) = b) \), which leads to a contradiction.

(If.) Suppose there is a valuation \( \xi \) in \( \Phi \) and \( u \in F \) such that \( \xi, u \vdash \Box \exists x P(x) \) and \( \xi, u \nvdash \exists x \Box (\exists y (P(y) \land \Box (x = y))) \). Then there are \( v \in F \) and \( b \in D_v \) such that \( uRv \) and \( \xi, v \vdash P(b) \). Since \( \Phi \) satisfies (1), there are \( a \in D_u \) and \( w \in F \) such that \( vRw \) and \( \rho_{uw}(a) = \rho_{vw}(b) \). It follows that \( \xi, v \vdash \Box (\rho_{uw}(a) = b) \) and thus \( \xi, v \vdash \exists y (P(y) \land \Box (\rho_{uw}(a) = y)) \). We have a contradiction.

**Lemma 3.11.4** \( \text{QS}_4^= + Ba_1 \not\vdash K \vDash Ba \).

**Proof** Consider the Kripke sheaf \( \Phi \) depicted at Figure 3.5. It is easily seen that \( \Phi \) satisfies (2), but it is not meek since \( b_1 \) does not have a predecessor in \( u_0 \); thus \( \Phi \vdash Ba^1 \), \( \Phi \not\vdash Ba \).

![Figure 3.5.](image-url)

**Corollary 3.11.5**

(1) \( LB_1 := \text{QS}_4^= + Ba^1 \) is \( K \)-incomplete.

(2) \( \mathcal{K} \prec \mathcal{K} \vDash \).
Remark 3.11.6 One can repeat the argument for logics without equality: $Ba^1$ should be replaced with

$$Ba^1 := \Box \forall x Q(x, x) \land \Diamond \exists x P(x) \supset \exists x \Diamond \exists y (P(y) \land \Diamond Q(x, y)).$$

3.11.2 Intuitionistic case

We shall use the following intuitionistic formulas:

$$E := \neg \exists x P(x) \supset \exists x \neg P(x);$$
$$E_1 := \exists x \forall y (\neg P(x) \supset \neg P(y));$$
$$E_2 := \exists x \forall y (\neg P(y) \supset \neg P(x));$$
$$F := \exists x \forall y (P(x) \supset P(y));$$
$$F_0 := \forall x \neg P(x) \supset \neg \forall x P(x);$$
$$G := \exists x \forall y (P(y) \supset P(x));$$
$$U := \forall x \forall y (P(x) \supset P(y));$$
$$U_0 := \forall x \forall y (\neg P(x) \supset \neg P(y));$$
$$C := \neg p \supset p;$$
$$J := \neg q \lor \neg q;$$
$$Z := (p \supset q) \lor (q \supset p).$$

The corresponding logics were considered in [Umezawa, 1956], where many inclusions and equalities between them were proved, in particular:

1. $QH + E = QH + E_1 = QH + E_2,$
2. $QH + F_0 = QH + \neg \neg F.$

One can easily see that

$$QH^\neg + U_0 \subset QH^\neg + \neg \neg U \subset QH^\neg + U = QH^\neg + \forall x \forall y (x = y).$$

We also have:

Lemma 3.11.7

1. $QH^{\neg s} + U = QH^{\neg s} + U_0;$
2. $QH^{\neg s} + U_0 \lor A = QH^{\neg s} + U \lor A$ for every sentence $A$ without occurrences of $P$.

Proof

1. From $R(x) \supset \forall y \neg R(y)$ by substitution we obtain

$$z = x \supset \forall y \neg z = y,$$

and

$$x = x \supset \forall y \neg x = y.$$

Now, $x = x$ and $\neg \neg x = y \supset x = y$ (in $QH^{\neg s}$) yield $\forall x \forall y (x = y)$.
2. Similar. 

\qed
Lemma 3.11.8 \((U \lor J) \in (\text{QH}^= + E)\).

Proof (‘Naive argument’). Due to 3.11.7 it is sufficient to show that
\[(U_0 \lor J) \in (\text{QH}^= + E)\].

\(E_1\) and \(E_2\) provide \(x_1, x_2\) such that

\[\forall y (\neg P(y) \supset \neg P(x_1))\]

and

\[\forall y (\neg P(x_2) \supset \neg P(y)).\]

If \(x_1 = x_2\) we have \(U_0\), so suppose \(x_1 \neq x_2\). Let

\[R(x) := (\neg q \land (x = x_1)) \lor (\neg q \land (x = x_2)).\]

It is easily seen that \(\neg \exists x R(x)\), so due to \(E\), there exists \(x_0\) such that \(\neg R(x_0)\). Since \((x_0 = x_1) \lor (x_0 \neq x_1)\), we obtain \(J\).

Corollary 3.11.9 \(\text{QH} + E \models_K U \lor J\).

Lemma 3.11.10 \(\text{QH} + E \not\models_K U \lor J\).

Proof The Kripke sheaf \(\Phi\) depicted at Figure 3.6, does not validate \(U \lor J\), as it is easily seen. If \(u \models \neg \exists x P(x)\) in some model on \(\Phi\) then \(v_1 \models P(b), V_2 \models P(c_i)\) for some \(i\) and so, \(u \not\models \neg P(a_i)\).

Corollary 3.11.11

(1) \(\text{QH} + E\) is \(K\)-incomplete.
3.12 Translations into classical logic

In Section 1.8 we showed that in many cases Kripke-complete modal or superintuitionistic propositional logics are recursively axiomatisable and complete w.r.t. countable frames. To prove these results, we used the standard translation of propositional modal formulas into classical first-order formulas. In this section we extend this translation to modal first-order formulas, cf. [van Benthem, 1983].

Let us consider the two-sorted classical predicate language $\mathcal{L}2_N$. The variables of the first sort (ranging over individuals) are taken from the same set $\text{Var}$ as in our basic language $\mathcal{L}_N$. The second sort has a countable set of variables (ranging over worlds) $\text{Wvar} := \{w_n \mid n \in \omega\}$. $\mathcal{L}2_N$ also contains equality\(^{21}\)

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\(^{21}\)Strictly speaking, there are two kinds of equality - for worlds and for individuals.
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and binary predicate letters \( R_1, \ldots, R_N, U \); \( R_i \) are of type \((\text{worlds} \times \text{worlds})\) and \( U \) is of type \((\text{worlds} \times \text{individuals})\); thus atomic formulas are of the form \( R_i(w_m, w_n) \) or \( U(w_m, v_n) \) or \( w_m = w_n \) or \( v_m = v_n \). \( \mathcal{L}_2N^\star \) is the expansion of \( \mathcal{L}_2N \) with \((n + 1)\)-ary predicate symbols \( P_j^n \) of type \((\text{worlds} \times \text{individuals}^n)\) corresponding to \( n \)-ary predicate symbols \( P_j^n(x) \) of our basic language \( \mathcal{L}_N \). Consider the following classical \( \mathcal{L}_2N^\star \)-theory:

\[
\Gamma_0 := \{ Exd \} \cup \{ A_j^n \mid n, j \geq 0 \},
\]

where

\[
Exd := \forall w_0 \exists v_0 U(w_0, v_0) \land \forall v_0 \exists w_0 U(w_0, v_0) \land \bigwedge_{i=1}^N \forall w_1 \forall w_2 \forall v_0 (R_i(w_1, w_2) \land U(w_1, v_0) \supset U(w_2, v_0)),
\]

\[
A_j^n := \forall w_0 \forall v_1 \ldots \forall v_n (P_j^n(w_0, v_1, \ldots, v_n) \supset \bigwedge_{i=1}^n U(w_0, v_i)).
\]

The intended interpretation of the formula \( U(w_0, x) \) is \( x \in D_{w_0} \). So the conjuncts of \( Exd \) respectively mean that every individual domain is non-empty, every individual belongs to some domain, and the domains are expanding with respect to \( R_i \). The formula \( P_j^n(w_0, x) \) asserts that \( w_0 \models P_j^n(x) \); thus \( A_j^n \) means that an atomic formula can be true at world \( w_0 \) only for individuals from \( D_{w_0} \).

Given a Kripke frame \( F = (F, D) \) based on a propositional frame \( F = (W, \rho_1, \ldots, \rho_N) \), we can construct the following (associated) \( \mathcal{L}_2N^\star \)-structure \( F^\star \) expanding \( F^\star \) and satisfying \( Exd \), in which the universes of sorts 1 and 2 are the sets \( W \) and \( D^+ \) respectively, and for any \( u_1, u_2, u, a \)

\[
F^\star \models R_i(u_1, u_2) \text{ iff } u_1 \rho_i u_2,
\]

\[
F^\star \models U(u, a) \text{ iff } a \in D_u.
\]

Next, a Kripke model \( M = (F, D, \xi) \) gives rise to an associated \( \mathcal{L}_2N^\star \)-structure \( M^\star \) satisfying \( \Gamma_0 \) and expanding \( F^\star \) such that for any \( u, a \)

\[
M^\star \models P_j^n(u, a) \text{ iff } a \in D_u^n, M, u \models P_j^n(a).
\]

Then we have

**Lemma 3.12.1** Every classical model of \( Exd \) is associated with some predicate Kripke frame. Every classical model of \( \Gamma_0 \) is associated with some predicate Kripke model.

**Proof** In fact, let \( \mu \) be a model of \( \Gamma_0 \). To define \( M \), let \( D^+ \) be the domain of the first sort, \( W \) the domain of the second sort. The accessibility relations \( \rho_i \) in \( M \) are taken from \( \mu \). Then put for every \( u \in W \),

\[
D_u := \{ a \in D^+ \mid \mu \models U(u, a) \},
\]

and

\[
\xi^+(P_k^n(a)) := \{ u \in W \mid \mu \models P_k^n(u, a) \},
\]

for \( a \in D_u^n \). From the above remarks it follows that \( \mu = M^\star \). In the same way a model of \( Exd \) can be presented as \( F^\star \) for some \( F \).
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Now by induction let us construct an $L^*_N$-formula $A^*(w_0, x)$, the standard translation of a first-order modal formula $A(x)$:

$P^n_j(x)^* := P^n_j(w_0, x),
(x_1 = x_2)^* := (x_1 = x_2),
\bot^* := \bot,
(B \supset C)^* := B^* \supset C^*,
(\exists y)B)^* := \exists y(U(w_0, y) \land B^*),
(\Box_iB)^*(w_0, x) := \forall w_1(R(w_0, w_1) \supset B^*(w_1, x)).$

Lemma 3.12.2 (1) $M, u \Vdash A(a)$ (modally) iff $M^* \Vdash A^*(u, a)$ (classically) for any $N$-modal Kripke model $M$, $N$-modal formula $A$, $u \in M$, $a \in D^*_u$.

(2) $F \models A$ iff $\forall \xi (F, \xi)^* \Vdash \forall w_0 A^*(w_0)$ for an $L_N$-sentence $A$ and an $N$-modal frame $F$.

Proof
(1) By induction. Let us consider the case $A(x) = \Box_k B(x)$.

$M, u \Vdash A(a) \iff \forall u_1 \in g_k(u) \quad M, u_1 \Vdash B(a)
\iff \forall u_1 \in g_k(u) \quad M^* \Vdash B^*(u_1, a)$ (by the induction hypothesis)
\iff $M^* \Vdash \forall w(R_k(u, w) \supset B^*(w, a)) \iff M^* \Vdash (\Box_k B)^*(a)$.

(2) $F \models A \iff \forall \xi \forall u \in F (F, \xi), u \Vdash A
\iff \forall \xi \forall u \in F (F, \xi)^* \Vdash A^*(u)$ (by (1)) \iff $\forall \xi (F, \xi)^* \Vdash \forall w_0 A^*(w_0)$.  

Lemma 3.12.3 (1) $M, u \Vdash A(a)$ (intuitionistically) iff $M^* \Vdash (A^T)^*(u, a)$ for any intuitionistic Kripke model $M$, an intuitionistic formula $A$, $u \in M$, $a \in D^*_u$.

(2) $F \Vdash A$ (intuitionistically) $\iff \forall \xi (F, \xi)^* \Vdash \forall w_0 (A^T)^*(w_0)$ for an intuitionistic sentence $A$ and an $S_4$-based frame $F$.

Proof
(1) $M, u \Vdash A(a)$ iff $M, u \Vdash A^T(a)$ (by Lemma 3.2.15)
iff $M^* \Vdash (A^T)^*(u, a)$ (by Lemma 3.12.2).

(2) Easily follows from (1).

The next two definitions are predicate analogues of Definition 1.8.2.

Definition 3.12.4 Let $C$ be a class of $N$-modal predicate Kripke frames. We say that $C$ is $\Delta$-elementary (respectively, $R$-elementary) if the class of associated $L^*_N$-structures $C^* := \{F^* \mid F \in C\}$ is $\Delta$-elementary (respectively, $R$-elementary).

Definition 3.12.5 A modal or superintuitionistic predicate logic $L$ is called $\Delta$-elementary (respectively, $R$-elementary) if the class $V(L)$ of all $L$-frames is $\Delta$-elementary (respectively, $R$-elementary).
Definition 3.12.6 A predicate Kripke frame $F = (F, D)$ is called countable if
the set of its worlds and the set of its individuals ($D^+$) are both countable. We
say that $F$ has a countable base if the set of worlds is countable and that $F$ has
a countable domain if $D^+$ is countable.

Definition 3.12.7 Let $C$ be a class of Kripke frames, $L = ML^{(=)}(C)$ its modal
logic. We say that $L$ has

- the countable frame property (c.f.p.) in $C$ if

$$L = ML^{(=)}\{(F \mid F \in C, \ F \ is \ countable )\},$$

- the countable domain property (c.d.p.) in $C$ if

$$L = ML^{(=)}\{(F \mid F \in C, \ F \ has \ a \ countable \ domain}\},$$

- the countable base property (c.b.p.) in $C$:

$$L = ML^{(=)}\{(F \mid F \in C, \ F \ has \ a \ countable \ base)\}).$$

and similarly for the intuitionistic case.

Obviously, the c.f.p. implies the c.d.p. and the c.b.p.

Proposition 3.12.8

(1) If a class of Kripke frames $C$ is $R$-elementary, then its modal predicate logic
$ML^{(=)}(C)$ is recursively axiomatisable (i.e. $RE$). Similarly, the intermedi-
ate logic $IL^{(=)}(C)$ is recursively axiomatisable for any $R$-elementary class
$C$ of intuitionistic Kripke frames.

(2) If $C$ is a $\Delta$-elementary class of Kripke frames, then $ML^{(=)}(C)$ has the
c.f.p. in $C$, and similarly for the intuitionistic case.

Proof Similar to 1.8.5. Let $C$ be the class of models of an $L_{2N}$-theory $\Sigma, C_1$
the class of all countable models of $\Sigma$. Then by Lemma 3.12.1, the $L_{2N}$-models
of $\Sigma \cup \Gamma_0$ are exactly the structures of the form $(F, \xi)^*$, where $F \models \Sigma$.
Now by Lemma 1.8.4 and Gödel's completeness theorem we obtain for any sentence $A$:

$$A \in ML(C) \Leftrightarrow \Sigma \cup \Gamma_0 \models \forall w_0 A^*(w_0) \ (\text{in the classical sense})$$

$$\Rightarrow \Gamma_0 \cup \Sigma \vdash \forall w_0 A^*(w_0) \ (\text{in classical predicate logic}).$$

Hence the set of all sentences in $ML(C)$ is $RE$. Since for any formula $A$, $A \in
ML(C)$ iff $\forall A \in ML(C)$, the whole logic is reducible to this set, and thus (1)
follows.

For the intuitionistic case, it suffices to note that $IL^{(=)}(C) = TML^{(=)}(C),$
by 3.2.29, thus $IL^{(=)}(C)$ is reducible to $ML^{(=)}(C)$. 
Similarly to the above, we obtain that for any sentence $A$: 

$A \in \mathbf{ML}(C_1)$ iff for any countable $L_2^{\star}$-structure $\mu$, $(\mu \models \Sigma \cup \Gamma_0 \Rightarrow \mu \vdash \forall w_0 A^\star(w_0))$.

By Lemma 3.12.1 and the Löwenheim-Skolem theorem, the latter is equivalent to $\Sigma \cup \Gamma_0 \models \forall w_0 A^\star(w_0)$, and thus (as we have shown) to $A \in \mathbf{ML}(C)$. Therefore $\mathbf{ML}(C) = \mathbf{ML}(C_1)$, which proves (2).

In the intuitionistic case note that

$$\mathbf{IL}(C) = T(\mathbf{ML}(C)) = T(\mathbf{ML}(C_1)) = \mathbf{IL}(C_1).$$

The following refinement of (2) is also useful.

**Proposition 3.12.9** Let $L$ be a $\Delta$-elementary modal or superintuitionistic predicate logic, $F$ an $L$-frame, $M$ a Kripke model over $F$, $S$ a countable set of worlds in $M$. Then there exists a countable reliable submodel of $M$ containing $S$, whose frame also validates $L$.

**Proof** Let $V(L)^\star$ be the class of models of an $L_2^N$-theory $\Sigma$. Then $M^\star \models \Sigma$. By the Löwenheim–Skolem–Tarski theorem, $M^\star$ has a countable elementary substructure $\mu$ containing $S$. Then $\mu \models \Gamma_0 \cup \Sigma$, so by Lemma 3.12.1 it follows that $\mu = M_1^\star$ for some Kripke model $M_1$. If $F_1$ is the frame of $M_1$, we obtain that $F_1^\star \models \Sigma$, i.e. $F_1 \in V(L)$.

Since $M_1^\star < M^\star$, we have for any $u \in M$, $A(a) \in L(u)$:

$$M^\star \models A(u, a) \text{ iff } M_1^\star \models A(u, a).$$

Hence by Lemma 3.12.3

$$M, u \models A(a) \text{ iff } M_1, u \models A(a),$$

which means that $M_1$ is reliable. 

**Remark 3.12.10** In the intuitionistic case we can define a slightly different translation. Namely, we can include the axioms of quasi-ordering (or partial ordering) for $R$ in $Exd$ and add the following truth-preservation clause to $A^n_j$:

$$\forall z_1 \forall z_2 \forall x (R(z_1, z_2) \land P^n_j(z_1, x) \supset P^n_j(z_2, x)).$$

For an intuitionistic predicate formula $A(x)$, the $L_2^\star$-formula $\tilde{A}(z_0, x)$ is constructed as follows:

- $P_j(x) = P_j^\star(z_0, x)$;
- $(x_1 = x_2) = (x_1 = x_2)$;
- $(A_1 \sigma A_2) = \tilde{A}_1 \sigma \tilde{A}_2$ for $\sigma = \land, \lor$;
- $(A_1 \supset A_2)(z_0, x) = \forall z_1 (R(z_0, z_1) \land \tilde{A}_1(z_1, x) \supset \tilde{A}_2(z_1, x))$;
- $\exists x_0 A_0(x_0, x) = \exists x_0 (U(x_0, x_0) \land A_0(x_0, x))$;
- $\forall x_0 A_0(x_0, x) = \forall z_1 \forall x_0 (R(z_0, z_1) \land U(z_1, x_0) \supset A_0(z_1, x_0, x))$. 


Recall (Section 3.2) that for a class of propositional Kripke frames $Z$, $\mathcal{K}(Z)$ denotes the class of all Kripke frames based on frames from $Z$. By $\mathcal{CK}(Z)$ we denote the class of all frames with constant domains from $\mathcal{K}(Z)$.

**Corollary 3.12.11** Let $L$ be a $\Delta$-elementary predicate logic, $M$ a Kripke model over an $L$-frame $F$, $u_0 \in M$. Then there exists a countable reliable submodel $M_0 \subseteq M$ over an $L$-frame containing $u_0$ such that $MT(M) = MT(M_0)$.

**Proof** For any sentence $A \not\in MT(M)$ (in the language of $L$) there exists a world $u_A \in M$ such that $M,u_A \not\models A$. Put

$$S := \{u_0\} \cup \{u_A \mid A \not\in MT(M)\}$$

and apply the previous proposition. The resulting submodel $M_0$ is reliable, so $MT(M) \subseteq MT(M_0)$.

On the other hand, if $A \not\in MT(M)$, then $M,u_A \not\models A$, so $M_0,u_A \not\models A$ by reliability of $M_0$. Therefore $MT(M_0) \subseteq MT(M)$.

**Corollary 3.12.12** Let $Z$ be a class of propositional Kripke frames.

1. If $Z$ is R-elementary, then the logics determined over $Z$ $ML^{=}((K(Z))$ and $ML^{=}((\mathcal{CK}(Z))$, are recursively axiomatisable.

2. If $Z$ is $\Delta$-elementary, then $ML^{=}((K(Z))$ has the c.f.p. in $\mathcal{K}(Z)$, and $ML^{=}((\mathcal{CK}(Z))$ has the c.f.p. in $\mathcal{CK}(Z)$.

Analogous properties hold for the intuitionistic case.

**Proof** For the case of constant domains we have to add the axiom

$$\forall z_1 \forall z_2 \forall x \ (U(z_1,x) \equiv U(z_2,x)),$$

or, if one prefers,

$$\forall z \forall x \ U(z,x).$$

Equivalently, we can drop the conjuncts containing $U$ in the definition of $\bar{A}$. ■

**Examples**

Consider the following $R$-elementary classes:

- $\mathcal{W}_n$, the class of all posets of width $\leq n$;
- $\mathcal{H}_n$, the class of all posets of height $\leq n$;
- $\mathcal{B}_n$, the class of all posets of branching $\leq n$.

So the superintuitionistic predicate logics of the following classes of posets are recursively axiomatisable (and satisfy the countable frame property):

- (a) $IL^{=}((\mathcal{W}_n))$, 
- (b) $IL^{=}((\mathcal{H}_n))$, 
- (c) $IL^{=}((\mathcal{B}_n))$. 


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(b) $\text{IL}^{(=)}(\mathcal{K}(P_n \cap W_m))$, etc.

If $m \geq 2$, explicit axiom systems for these intermediate logics are unknown. On the other hand, explicit recursive axiomatisations are known for the logics $\text{IL}^{(=)}(\mathcal{K}(P_n))$, $\text{IL}^{(=)}(\mathcal{K}(P_n \cap B_m))$, $\text{IL}^{(=)}(\mathcal{K}(B_m))$, see Chapter 7.

In particular, we have:

$$\text{IL}^{(=)}(\mathcal{K}(B_1)) = \mathcal{QH}^{(=)} + Z;$$

$$\text{IL}^{(=)}(\mathcal{K}(B_m)) = \mathcal{QH}^{(=)} \text{ for } m \geq 2.$$

In this connection note that the logic

$$\text{IL} \left( \mathcal{K} \left( B_m \cap \bigcup_{n<\omega} P_n \right) \right)$$

defines the class of frames of finite heights and of finite branching bounded by $m$ in all finite heights is not RE if $m \geq 2$, see Chapter 11; recall that the corresponding propositional logic is $\mathcal{H} + B_m$ (Chapter 1). The same holds for the logics $\text{IL}(\mathcal{W}_m \cap \bigcup_{n<\omega} P_n)$, $\text{IL}(\mathcal{P}_m \cap \bigcup_{n<\omega} W_n)$, etc.

Also note that explicit finite axiomatisations of the classes of frames with constant domains in the cases (a), (b) are known, see Chapter 8.

**Corollary 3.12.13** Every predicate logic determined over a finite propositional Kripke frame $F$ is recursively axiomatisable, and satisfies the c.d.p. (in $\mathcal{K}(F)$).

**Proof** (cf. [Skvortsov, 1995], [Skvortsov, 1991]). It is well known that every finite classical structure is finitely axiomatisable. So the class $\{F\}$ is RE-elementary for a finite propositional Kripke frame $F$; if $F = (\{a_1, \ldots, a_m\}, R)$, then up to isomorphism, $F$ is defined by the following $C_1$-sentence:

$$\exists z_1, \ldots, z_m \left( \bigwedge_{a \in R_j} R(z_i, z_j) \land \bigwedge_{a \notin R_j} \neg R(z_i, z_j) \land \forall z_0 \left( \bigvee_{i=1}^m (z_0 = z_i) \land \bigwedge_{i \neq j} (z_i \neq z_j) \right) \right)$$

However in many cases the predicate logics of RE-elementary classes of frames are not recursively axiomatisable. For example, for a class $\mathcal{Z}$ of rooted posets, $\text{IL}(\mathcal{Z})$ is RE iff $\mathcal{Z}$ is finite, see Chapter 11.

**Remark 3.12.14** We have defined a translation from a modal logic language into a classical language based on Kripke frame semantics. Similar translations can be defined for Kripke sheaves and for other Kripke-style semantics considered in further chapters. Recursive axiomatisability and countable frame (or countable domain) property also can be established for logics complete in these semantics. The constructions are rather straightforward, and we do not discuss them in detail.
We have established the countable domain property for any finite propositional Kripke frame. It turns out that the c.d.p. can be extended to any countable propositional Kripke frame as well. H. Ono proved this result for the intuitionistic case by a straightforward argument (see Theorem 1.1 in [Ono, 1972/73]). Here we give a model-theoretic proof for the modal case which is based again on the translation into a classical language.

**Proposition 3.12.15** Let $F$ be a propositional Kripke frame of infinite cardinality $\kappa$. Then

1. $\mathbf{ML}(=)(\mathcal{K}(F)) = \mathbf{ML}(=)\{(F,D) \mid \forall u \in F \ |D_u| \leq \kappa\}$, and similarly for the logic $\mathbf{IL}(=)(F)$ of any intuitionistic propositional Kripke frame $F$ of cardinality $\kappa$.

2. $\mathbf{ML}(=)(\mathcal{C}(F)) = \mathbf{ML}(=)\{(F \cup V) \mid |V| \leq \kappa\}$.

**Corollary 3.12.16** Let $\mathcal{Z}$ be a class of countable propositional frames. Then

$\mathbf{ML}(=)(\mathcal{K}(\mathcal{Z})) = \mathbf{ML}(=)\{(F,D) \mid F \in \mathcal{Z} \land \forall u \in F \ |D_u| \leq \aleph_0\}$,

and similarly for $\mathcal{C}(\mathcal{Z})$.

**Proof** We prove Proposition 3.12.15 for the modal case. For the intuitionistic case one can repeat the proof or just apply Gödel–Tarski translation.

(i) We add a set of individual constants of sort 1 of cardinality $\kappa$

$$T = \{c_\alpha \mid \alpha < \kappa\}$$

to $\mathcal{L}^{\kappa}_N$ and fix a bijection

$$\nu : T \rightarrow F.$$ 

We denote the resulting language by $\mathcal{L}^{\kappa}_N$.

Given a Kripke model $((F,D), \models)$, we can construct a model $\mathcal{M}_\kappa^+$ of $\mathcal{L}^{\kappa}_N$ by taking an $\mathcal{L}^{\kappa}_N$-model $\mathcal{M}^+$ and putting

$$\mathcal{M}_\kappa^+ \models u = c_\alpha \text{ iff } u = \nu(c_\alpha).$$

Due to the downward Löwenheim–Skolem–Tarski theorem, it has an elementary substructure of cardinality $\kappa$, which we denote by $\mathcal{M}_\kappa^{+'}$. Obviously, its universe of sort 1 must coincide with $F$, and so its $\mathcal{L}^{\kappa}_N$-reduct corresponds to some Kripke model $((F,D), \models')$ based on $F$. Since $D_2^{+'}$ is the universe of sort 2 in $\mathcal{M}_\kappa^{+'}$, we have $|D_2^{+'}| \leq \kappa$, and thus $\forall u \ |D_u| \leq \kappa$. For any modal predicate formula $A(x)$ we have

$$(F,D), u \models A(x) \text{ iff } \mathcal{M}_\kappa^+ \models \bar{A}(u,x) \text{ iff } \mathcal{M}_\kappa^{+'} \models \bar{A}(u,x) \text{ iff } (F,D'), u \models' A(x).$$

Therefore, if $A(x) \notin \mathbf{ML}(=)(\mathcal{K}(F))$, it is refuted in a frame $(F,D')$ such that $|D_u'| \leq \kappa$ for any $u \in F$.

(ii) Note that if $(F,D)$ has a constant domain, then $\mathcal{M}_\kappa^+ \models \forall z \forall x U(z,x)$ and thus $\mathcal{M}_\kappa^{+'} \models \forall z \forall x U(z,x)$ and $(F,D')$ has a constant domain.
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On the other hand, let $F$ be a propositional frame which is well-ordered of the type $\kappa$, $\kappa$ being an infinite cardinal, cf. [Skvortsov, 1989]. Consider the formula

$$KF = \neg \forall x(P(x) \lor \neg P(x)),$$

see Section 2.3. Then $KF \notin \mathbf{IL}'(\mathbb{K}(F))$, since $KF \notin \mathbf{IL}(F \odot V)$ if $|V| \geq \kappa$. In fact, let $(a_\beta \beta < \kappa)$ be a sequence of distinct elements in $V$, and consider the valuation $\xi$ such that for every $\alpha < \kappa$

$$\xi_\alpha(P) = \{a_\beta \mid \beta \leq \alpha\}.$$

Then

$$\xi_0, 0 \vdash \neg KF.$$

Also we have

$$KF \in \mathbf{IL}(F, D) \text{ if } |D^+| < \kappa.$$

In fact, take an arbitrary valuation $\xi$ in $(F, D)$; for every $a \in D^+$ take an ordinal $\beta_a \leq \kappa$ such that

$$\xi, \alpha \vdash P(a) \text{ iff } \beta_a \leq \alpha.$$

Also take $\alpha_0 < \kappa$ such that

$$\forall a (\beta_a < \kappa \Rightarrow \beta_a \leq \alpha_0).$$

Then

$$\xi, \alpha_0 \vdash \forall x(P(x) \lor \neg P(x)).$$

Also note that

$$|D^+| \leq \aleph_1 \text{ if } \forall \alpha < \kappa |D_\alpha| \leq \aleph_0.$$

In fact,

$$|\{\alpha < \kappa \mid D_\alpha \not\subseteq \bigcup_{\beta < \alpha} D_\beta\}| \leq \aleph_1.$$

Therefore, we cannot replace $|D_u| \leq \kappa$ with $|D_u| \leq \aleph_0$ in Proposition 3.12.15 say for $F = \aleph_2$. Let us also note that for $F = \kappa$

$$KF \notin \mathbf{IL}(F, D) \text{ if } D_\alpha = \{a_\beta \mid \beta \leq \alpha\},$$

and

$$\forall \alpha < \kappa |D_\alpha| < \kappa;$$

thus

$$KF \notin \mathbf{IL}\{(F, D) \mid \forall \alpha < \kappa |D_\alpha| \leq \aleph_0\}$$

for $F = \aleph_1$.

Let us consider another similar example. Let $F$ be a denumerable tree (with the root $O_F$), in which $F^\omega$ is nonlinear (e.g. the tree $\omega^*$ of all finite sequences of natural numbers, or the binary tree $\{0, 1\}^*$ of all finite $(0, 1)$-sequences, etc.). Let $\bar{F}$ be the coatomic tree obtained by adding maximal points $w_\tau$ above every branch (maximal chain) $\tau$ of $F$. Let us consider the following formula:

$$C^* := \forall x((\neg P(x) \supset q) \supset q) \land \neg \exists x P(x) \supset q.$$

Then
Lemma 3.12.17  (i) $C^* \in \text{IL}(\overline{F} \circ V)$ for a denumerable $V$;
(ii) $C^* \notin \text{IL}(\overline{F} \circ V)$ if $|V| \geq |\overline{F} - F|$ (e.g. if $|V| \geq 2^{\aleph_0}$).

Proof

(i) Suppose that

$$u_0 \not\models q, u_0 \models \forall x[\neg P(x) \supset q] \supset q], u_0 \models \neg \exists x P(x), V = \{a_n : n > 0\}.$$ 

Then

$$\forall u \not\models q \forall n > 0 \exists v (u R v \& v \models \neg P(a_n) \& v \not\models q),$$

and there exists a chain $u_0 R u_1 R u_2 R \ldots$ such that $\forall n > 0 [u_n \models \neg P(a_n) \& , u_n \not\models q].$ Then $w_\tau \models \neg \exists x P(x)$ for a branch $\tau$ containing all $u_n.$ This is a contradiction.

(ii) Take different elements $a_\tau$ from $V$ for $w_\tau \in (\overline{F} - F),$ and the following valuation in $\overline{F} \circ V$:

$$u \models P(a) \text{ iff } \exists \tau (u = w_\tau \& a = a_\tau);$$

$$u \models q \text{ iff } \exists \tau (u = w_\tau).$$

Then $O_F \not\models q, O_F \models \neg \exists x P(x).$ We also have

$$\forall u \in F \forall a \in D_0 u \not\models \neg P(a) \supset q$$

since for $a = a_\tau,$ there exists $v \in F^u$ such that $v \not\in \tau,$ i.e. $v \models \neg P(a).$ Thus $O_F \models \forall x[(\neg P(x) \supset q) \supset q).$

Remark 3.12.18 This example is also related to the formula $KF.$ It is known [Gabbay, 1981] that the predicate logic $QH + KF$ is Kripke-complete (w.r.t. denumerable atomic trees). Moreover, $QH + KF = \text{IL}(\omega^*) = \text{IL}(\{D| D \in \omega^*, \forall u | D_u | \leq \aleph_0\})$ (and similarly for the binary tree $\{0, 1\}^*.$ The corresponding predicate logic with constant domains is also Kripke-complete:

$$QH + KF + CD = \text{IL}(\{F' \circ V | \{0, 1\}^* \subset F' \subset \omega^*, F' \text{ is coatomic}\}$$

for a denumerable $V.$ On the other hand, the above mentioned example shows that

$$QH + K + CD \subset \text{IL}(\omega^* \circ V)$$

for a denumerable $V.$

Remark 3.12.19 For the case of constant domains (without equality), Proposition 3.12.15 also shows that

$$\text{ML}_C(QF) = \text{ML}(F \circ V)$$

for any countable $F$ and any constant infinite domain $V.$ This follows from 3.4.11:

$$\text{ML}(F \circ V) \subseteq \text{ML}(F \circ V'),$$

provided $|V| \geq |V'|.$
3.12. TRANSLATIONS INTO CLASSICAL LOGIC

The translation into classical logic and the well-known generalised form of the Löwenheim–Skolem theorem shows that

$$ML^\equiv(F \odot V) = ML^\equiv(F \odot V')$$


The intuitionistic case is quite similar. On the other hand, if a poset $F$ contains an infinite cone then

$$C^* \in (IL^\equiv(F \odot V) - IL^\equiv(F \odot V')),$$


Thus we obtain the following claim (cf. Theorem 1 in [Skvortsov, 1995]):

**Theorem 3.12.20** $IL^\equiv(F \odot V) = IL^\equiv(F \odot V')$ for any infinite constant domains $V, V'$ iff all the cones in $F$ are finite.
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Chapter 4

Algebraic semantics

4.1 Modal and Heyting valued structures

As we know from Chapter 1, every propositional Kripke frame $F$ corresponds to a modal algebra $MA(F)$; so Kripke semantics can be treated as a particular case of algebraic semantics. Likewise, in the predicate case Kripke frames with equality admit a straightforward algebraic generalisation.

Recall that by Lemma 3.5.2, every KFE can be represented as a propositional Kripke frame $F$ with individuals such that for every two individuals $a, b$ their measure of equality $E(a, b)$ is defined. $E(a, b)$ is a set of possible worlds, and thus an element of the corresponding modal algebra $MA(F)$. This suggests for the following generalization of KFEs: replace $MA(F)$ by an arbitrary modal algebra (or a Heyting algebra for the intuitionistic case) and $E$ with a function taking values in this algebra. To make the corresponding semantics sound, $E$ should satisfy the properties cited in Lemma 3.5.2.

Thus we come to the following two definitions.

**Definition 4.1.1** Let $\Omega$ be a complete Heyting algebra (a 'locale'). An $\Omega$-valued set is a triple $(\Omega, D, E)$, where $D$ is a set, $E : D \times D \rightarrow \Omega$ is a map such that for any $a, b, c \in D$

\[(E1) \ E(a, b) = E(b, a);\]
\[(E2) \ E(a, b) \land E(b, c) \subseteq E(a, c).\]

If also

\[(E3) \ \bigvee_{a \in D} E(a, a) = 1,\]

the triple $(\Omega, D, E)$ is called an $\Omega$-valued structure or a Heyting valued structure (H.v.s) over $\Omega$. $D$ is called its individual domain, the elements of $D$ are called individuals. The elements of $\Omega$ are called truth values.

---

\[^1\text{Cf. [Borceaux, 1994; Fourman and Scott, 1979; Dragalin, 1988].}\]
Obviously, (E3) implies that \( D \) is non-empty.

**Definition 4.1.2** Let \( \Omega = (\Omega, \cup, \cap, -, 0, 1, \Box_1, \ldots, \Box_N) \) be a complete modal algebra. An \( \Omega \)-valued structure or a modal valued structure (m.v.s) over \( \Omega \) is a triple \((\Omega, D, E)\), where \( D \) is a set, \( E: D \times D \to \Omega \) is a map such that for any \( a, b, c \in D \), \( i \in \{1, \ldots, N\} \)

\[
\begin{align*}
(E1) & \ E(a, b) = E(b, a), \\
(E2) & \ E(a, b) \cap E(b, c) \leq E(a, c), \\
(E3) & \ \bigcup_{a \in D} E(a, a) = 1, \\
(E4) & \ E(a, b) \leq \Box_i E(a, b).
\end{align*}
\]

The condition (E3) again implies the non-emptiness of \( D \).

Since every Boolean algebra is a Heyting algebra (where \( \lor = \cup \), \( \land = \cap \) and \( a \rightarrow b = a \oplus b = -a \land b \)), an m.v.s \((\Omega, D, E)\) over a modal algebra \( \Omega \) corresponds to the H.v.s \((\Omega^\circ, D, E)\), where \( \Omega^\circ \) is the Boolean part of \( \Omega \).

The other way round, if \( \Omega \) is an \( S4 \)-algebra, then an m.v.s. \((\Omega, D, E)\), corresponds to the H.v.s. \((\Omega^\circ, D, E)\), where \( \Omega^\circ \) is the pattern of \( \Omega \).

Definitions and results for H.v.s. and m.v.s. are often quite similar. In these cases we talk about 'structures' and denote all operations in a Heyting-algebraic style.

As in Chapter 3, we call \( E(a, b) \) the measure of equality of individuals \( a, b \); they are 'fully equal' if \( E(a, b) = 1 \) and 'fully different' if \( E(a, b) = 0 \). The truth value \( E(a, a) \) (which in general may be not equal to 1) is called the measure of existence of \( a \).

We also introduce the measure of equality for tuples \( a = (a_1, \ldots, a_n), \ b = (b_1, \ldots, b_n) \in D^n: \)

\[
E(a, b) := E(a_1, b_1) \land \ldots \land E(a_n, b_n)
\]

and the following abbreviations:

\[
E_{ab} := E(a, b), \ E_a := E(a) := Eaa.
\]

We can also include the degenerate case \( n = 0 \). A 0-tuple is just \( \lambda \) (the void sequence), and we put

\[
E \lambda \lambda := 1.
\]

As we said above, Definitions 4.1.1, 4.1.2 generalise the situation in Kripke frames with equality. In that case \( D \) corresponds to the set of all individuals \( D^+ \), the conditions (E1)-(E3) in Definition 4.1.2 are exactly the same as in Lemma 3.5.2 (1), and the condition (E4) in 4.1.2 is obviously equivalent to (E4) in 3.5.2 (1). Therefore we have

\[\text{[Note: Information about ambiguous notation and inclusion of degenerate cases.]}\]
Lemma 4.1.3

(1) Let $F = (F, D, \bowtie)$ be a KFE, $F = (W, R_1, \ldots, R_m)$, and for every two individuals $a, b \in D^+$ put $E(a, b) = \{u \mid a \bowtie u b\}$. Then $\text{MV}(F) := (MA(F), D^+, E)$ is an m.v.s.

(2) Similarly, if $F$ is an S4-frame, the triple $\text{HV}(F) := (HA(F), D^+, E)$ is an H.v.s.

Proof In fact, (E1)-(E4) in 4.1.2 follow readily from 3.5.2(1). In particular, (E3) means that every individual domain $D_u$ is non-empty.

The conditions (E1), (E2) in the above Definitions 4.1.1, 4.1.2 correspond to the symmetry and the transitivity of equality; more exactly, they are intended to make the corresponding first-order formulas true (see below). To prove the reflexivity, the condition (E3) is necessary. The condition (E4) corresponds to a theorem in 2.6.18(3):

$$X = Y \vdash \Box_i(x = y);$$

it is necessary for verifying the substitution instances of the equality axiom (Ax17).

Lemma 4.1.4 For any structure $(\Omega, D, E)$, individuals $a, b \in D$ and tuples $a, b, c \in D^n$:

(1) $E(a, b) \leq E(a, a)$;

(2) $Eab \leq Eaa$;

(3) $Eab \land Ebc \leq Eac$;

(4) $E(a) \leq \Box_i E(a)$ (in the modal case);

(5) $\bigvee_{a \in D^n} E(a) = 1$.

Proof This easily follows from Definition 4.1.2. In fact,

$$E(a, b) = E(a, b) \land E(b, a) \leq E(a, a);$$

hence

$$Eab = \bigwedge_{i=1}^n E_a b_i \leq \bigwedge_{i=1}^n E_a a_i = Eaa;$$

$$Eab \land Ebc = \bigwedge_{i=1}^n (E_a b_i \land E_b c_i) \leq \bigwedge_{i=1}^n E_a c_i = Eac;$$

$$\bigvee_{a \in D^n} E(a) = \bigvee\{E_a a_1 \land \ldots \land E_a a_n \mid a_1, \ldots, a_n \in D\} \geq \bigvee_{a \in D} \bigwedge_{i=1}^n Eaa = 1.$$
Definition 4.1.5 A structure $F = (\Omega, D, E)$ has a constant domain (or briefly, $F$ is a CD-m.v.s. or a CD-H.v.s. or a CD-structure) if the equality is trivial, i.e. for any $a, b \in D$

$$E(a, b) = \begin{cases} 1 & \text{iff } a = b, \\ 0 & \text{iff } a \neq b. \end{cases}$$

Such a structure is denoted simply by $(\Omega, D)$.

Definition 4.1.6 A structure $(\Omega, D, E)$ is called flabby if $E(a, a) = 1$ for every $a \in D$ and keen if $E(a, b) = 0$ whenever $a \neq b$.

So a CD-structure is both flabby and keen.

Every structure $F = (\Omega, D, E)$ gives rise to a flabby structure (of the same type) $F_{fl} := (\Omega, D, E_{fl})$, where

$$E_{fl}(a, b) := \begin{cases} 1 & \text{if } a = b, \\ E(a, b) & \text{otherwise}, \end{cases}$$

to a keen structure $F_{ke} := (\Omega, D, E_{ke})$, where

$$E_{ke}(a, b) := \begin{cases} E(a, a) & \text{if } a = b, \\ 0 & \text{otherwise}, \end{cases}$$

and to a CD-structure

$$F_{cd} := (\Omega, D).$$

Definition 4.1.7 For a structure $F = (\Omega, D, E)$, an $n$-ary $F$-predicate (or an $\Omega$-valued predicate on $D$) is a map $A : D^n \rightarrow \Omega$. Such a predicate is called strict (or Scott) if $A(a) \leq E(a, a)$ for any $a \in D^n$, and congruential if for any $a, b \in D^n$, for any $i \in I_n$

$$\forall j \neq i \ a_j = b_j \Rightarrow E_{a_i b_i} \land A(a) \leq A(b).$$

For $a, b \in D^n$ we will use the notation

$$a =_i b := \forall j \neq i \ a_j = b_j.$$  

Lemma 4.1.8 For any congruential predicate $A : D^n \rightarrow \Omega$, for any $a, b \in D^n$

1. $E_{a b} \land A(a) \leq A(b)$,

2. $E_{a b} \leq A(a) \to A(b)$.

Proof We prove (1) by induction on $n$.

If $n = 1$, the condition $a =_1 b$ holds trivially, so the claim is obvious.

Suppose (1) holds for $n$ for any congruential $A$ and consider $a, b \in D^{n+1}$.
Let $a = a_1 a', \ b = b_1 b'$. Then

$$(*) \ A(a) \land E_{a b} = A(a) \land E_{a_1 b_1} \land E_{a'b'} \leq A(b_1 a') \land E_{a'b'}$$
since $\mathcal{A}$ is congruential. The $n$-ary predicate $\mathcal{B} : c \mapsto \mathcal{A}(b_1c)$ is also congruential. In fact, for $c, d \in D^n$ the condition $c =_{i+1} b_1d$, thus
\[
\mathcal{B}(c) \land Ec_id_i = \mathcal{A}(b_1c) \land Ec_id_i \leq \mathcal{A}(b_1d) = \mathcal{B}(d),
\]
since $\mathcal{A}$ is congruential.

By the induction hypothesis,
\[
\mathcal{A}(b_1a') \land Ea'b' = \mathcal{B}(a') \land Ea'b' \leq \mathcal{B}(b') = \mathcal{A}(b),
\]
therefore by $(*),$ we obtain (1):
\[
\mathcal{A}(a) \land Eab \leq \mathcal{A}(b).
\]
Now (1) implies
\[
Eab \leq \mathcal{A}(a) \rightarrow \mathcal{A}(b)
\]
hence by symmetry
\[
Eba \leq \mathcal{A}(b) \rightarrow \mathcal{A}(a),
\]
which eventually implies (2).

A 0-ary predicate $\mathcal{A} : \{\lambda\} \rightarrow \Omega$ can be treated just as an element $\mathcal{A}(\lambda)$ of $\Omega$. This predicate is always congruential and strict, since $E(\lambda, \lambda) = 1$.

Lemma 4.1.9

(1) A structure $F$ is flabby iff all $F$-predicates are strict.

(2) A structure $F$ is keen iff all $F$-predicates are congruential.

Proof

(1) (Only if.) $E(a_i, a_i) = 1$ implies $\mathcal{A}(a) \leq E(a, a)$.

(If). Suppose $Ea_0 \neq 1$ for some $a_0 \in D$. Then the predicate $\mathcal{A}$ sending every $a$ to 1 is not strict, since $\mathcal{A}(a_0) \not\leq Ea_0$.

(2) (Only if.) If $a \neq b$, then $Eab = 0$, and obviously $Ea_ib_i \land \mathcal{A}(a) \leq \mathcal{A}(b)$. If $a = b$, then $Eab \land \mathcal{A}(a) \leq \mathcal{A}(a) = \mathcal{A}(b)$.

(If). Suppose $E(a_0, b_0) = \alpha \neq 0$ for some $a_0 \neq b_0$ and consider the unary $F$-predicate $\mathcal{A}$ sending $a_0$ to $\alpha$ and every $a \neq a_0$ to 0. Then
\[
Ea_0b_0 \land \mathcal{A}(a_0) = \alpha \not\leq 0 = \mathcal{A}(b_0),
\]
thus $\mathcal{A}$ is not congruential.

Definition 4.1.10 For $n$-ary $F$-predicates we introduce the ordering
\[
\mathcal{A} \preceq \mathcal{B} := \forall a \in D^n B(a) \leq \mathcal{A}(a).
\]
Lemma 4.1.11 For an n-ary F-predicate $A$ put
\[ A^s(a) := A(a) \land Ea. \]

Then

1. $A^s \preceq A$;
2. $A^s$ is strict;
3. for any n-ary strict F-predicate $B$
   \[ B \preceq A \Rightarrow B \preceq A^s; \]
4. if $A$ is congruential, then $A^s$ is congruential.

Thus $A^s$ is the greatest strict predicate "below" $A$; we call it the strict version of $A$.

Proof (1), (2), (3) are obvious.
(4) Suppose $a, b \in D^n$ and $a =_i b$. Then

(##) $A^s(a) = A(a) \land Ea \land Ea_ib_i \leq A(b)$,

since $A$ is congruential. Also
\[ Ea_ib_i \leq Eb_i \]

by 4.1.4(4), hence

(###) $Ea \land Ea_ib_i \leq Eb$,

and thus
\[ A^s(a) \land Ea_ib_i \leq A(b) \land Eb = A^s(b). \]

by (##), (###). Therefore $A^s$ is congruential.

Note that trivially $A^s = A$ for 0-ary $A$.

Let us now prove a dual to Lemma 4.1.11.

Lemma 4.1.12 For an n-ary F-predicate $A$ put
\[ A^c(a) := A(a) \lor \bigvee_{i=1}^{n} \bigvee_{d=_{ia}} (A(d) \land Ea_id_i). \]

Then

1. $A \preceq A^c$;
2. $A^c$ is congruential;
(3) for any \( n \)-ary congruential \( F \)-predicate \( B \)
\[
A \preceq B \Rightarrow A^c \preceq B;
\]

(4) \( A \) is strict \( \iff A^c \) is strict.

So \( A^c \) is the least congruential predicate 'above' \( A \); we call it the congruential version of \( A \).

**Proof** \( (1), \ (3) \) are obvious.

(2) Suppose \( a =_k b \). Then

\[
A^c(a) \land E a_k b_k = A(a) \land E a_k b_k \lor \bigvee_{i=1}^{n} \bigvee_{d=i, a} (A(d) \land E a_i d_i \land E a_k b_k).
\]

by well-distributivity (Section 1.2).

Now if \( i = k \), then

\[
E a_i d_i \land E a_k b_k = E a_k d_k \land E a_k b_k \leq E b_k d_k
\]

by 4.1.1(1), (2). If \( i \neq k \), then \( a_i = b_i \), so

\[
E a_i d_i \land E a_k b_k = E b_i d_i \land E a_k b_k \leq E b_i d_i.
\]

Thus

\[
\bigvee_{i=1}^{n} \bigvee_{d=i, a} (A(d) \land E a_i d_i \land E a_k b_k) \leq \bigvee_{i=1}^{n} \bigvee_{d=i, b} (A(d) \land E b_i d_i).
\]

Also note that

\[
A(a) \land E a_k b_k = A(a) \land E b_i d_i
\]

for \( d = a, \ i = k \). Hence

\[
A^c(a) \land E a_k b_k \leq \bigvee_{i=1}^{n} \bigvee_{d=i, b} (A(d) \land E b_i d_i) \leq A^c(b).
\]

(4) Since \( A \preceq A^c \), it follows that \( A^c(a) \preceq E a \) implies \( A(a) \preceq E a \).

The other way round, assume that \( A \) is strict. Then for any \( i \) and \( d = i, a \),

\[
A(d) \land E a_i d_i \leq E d \land E a_i d_i = \bigwedge_{j \neq i} E a_j \land E a_i d_i \leq E a,
\]

since \( E a_i d_i \leq E a_i \) by 4.1.4(1). By assumption \( A(a) \leq E a \), so all the disjuncts in \( A^c(a) \) are bounded by \( E a \), and thus \( A^c(a) \leq E a \).

\[\blacksquare\]
Remark 4.1.13 Note that the disjuncts $A(a) \land Ea_i a_i$ corresponding to $d = a$ are obviously redundant in the definition of $A^c$. On the other hand, for a strict $A$ we have

$$A^c(a) = \bigvee_i \bigvee_{d=i} a_i (A(d) \land Ea_i d_i),$$

since $A(a) = A(a) \land Ea_i a_i$.

**Remark 4.1.14** For an arbitrary predicate $A$, we can construct a strict congruential version

$$(\ast) \quad (A^c)^c(a) = (A^c)^c(a) = \bigvee_i \bigvee_{d=i} a_i (A(d) \land Ea_i d_i).$$

In fact, $A^c$ is strict, so by Remark 4.1.13

$$\begin{align*}
(A^c)^c(a) &= \bigvee_i \bigvee_{d=i} a_i (A^c(d) \land Ea_i d_i) = \bigvee_i \bigvee_{d=i} a_i (A(d) \land Ed \land Ea_i d_i) \\
&= \bigvee_i \bigvee_{d=i} a_i (A(d) \land Ea \land Ea_i d_i).
\end{align*}$$

The latter equality holds since $Ea_i \land Ea_i d_i = Ed_i \land Ea_i d_i$ (which easily follows from 4.1.4 (1)), and thus for $d = a$

$$Ed \land Ea_i d_i = \bigwedge_{j \neq i} Ea_j \land Ed_i \land Ea_i d_i = \bigwedge_{j \neq i} Ea_j \land Ea_i \land Ea_i d_i = Ea \land Ea_i d_i.$$  

On the other hand,

$$\begin{align*}
(A^c)^c(a) &= A^c(a) \land Ea = A(a) \land Ea \lor \bigvee_i \bigvee_{d=i} a_i (A(d) \land Ea \land Ea_i d_i)
\end{align*}$$

by well-distributivity. The first disjunct is redundant as it equals $A(a) \land Ea \land Ea_i a_i$, hence ($\ast$) follows.

Note that

$$A^c \preceq (A^c)^c \preceq A^c,$$

but in general $A$ and $(A^c)^c$ are $\preceq$-incomparable. E.g. consider a unary $A$ such that $A(a) \preceq Eaa$ for some $a$. Then $A(a) \preceq (A^c)^c(a)$, since $(A^c)^c(a) \preceq Eaa$. And if $A(d) \land Ebd \preceq A(b)$ for some $b, d$, then $(A^c)^c(b) \preceq A(b)$, since $A(d) \land Ebd \preceq (A^c)^c(b)$.

**Exercise 4.1.15** Show that $(A^c)^c = A$ iff $A$ is strict and congruential.

**Remark 4.1.16** As we are interested in algebraic models satisfying the axioms of equality, atomic $D^+$-sentences should be evaluated by congruential predicates. This requirement also makes sense for logics without equality. A similar situation is in Kripke semantics — as we know from Chapter 3, including equality in semantics is crucial for completeness.
Remark 4.1.17 The strictness property is quite natural for evaluation of formulas — the underlying idea is that a $D^+$-sentence may be true only within the "life-zone" of all occurring individuals. But as we shall see, this does not matter for semantics, because replacing every $A$ with $A^s$ does not change the notion of validity.

However evaluating formulas with arbitrary predicates simplifies the inductive truth definition, see Definition 4.2.4. This happens because strict predicates are not closed under all Boolean operations (for example, under negation, since $A(a) \leq Ea$ does not imply $-A(a) \leq Ea$ if $Ea \neq 1$).

### 4.2 Algebraic models

**Definition 4.2.1** A valuation is an m.v.s. (H.v.s.) $F = (\Omega, D, E)$ is a map $\varphi : AFD \rightarrow \Omega$ sending every $D$-sentence to $\Omega$ such that

$$\varphi(P(a)) \land Ea, b \leq \varphi(P(b))$$

whenever $P \in PL^a$, $a, b \in D^n$, $a = b$. Then the $n$-ary $F$-predicate $\varphi_P : a \mapsto \varphi(P(a))$ is called associated to $\varphi$ and $P$. The pair $(F, \varphi)$ is called an algebraic model over $F$.

From the definitions it follows that all the predicates $\varphi_P$ are congruential.

**Definition 4.2.2** A valuation $\varphi$ is called strict if every predicate associated to $\varphi$ is strict, i.e. if

$$\varphi(P(a)) \leq Ea$$

for any $P \in PL^n$, $a \in D^n$.

**Definition 4.2.3** For an arbitrary valuation $\varphi$ we define its strict version $\varphi^s$ such that

$$\varphi^s(P(a)) := \varphi(P(a)) \land Ea.$$

for any $a \in D^n$, $P \in PL^n$.

Thus $(\varphi^s)_P = (\varphi_P)^s$, and so the definition is sound, since $(\varphi_P)^s$ is congruential by Lemma 4.1.11.

Certainly $\varphi^s = \varphi$ if $\varphi$ is strict.

**Definition 4.2.4** For a valuation $\varphi$ in an m.v.s. (respectively, H.v.s.) $F = (\Omega, D, E)$ we define its (unique) 'large' extension to all modal (respectively, intuitionistic) $D$-sentences in the natural way:

1. $\varphi(\bot) := 0$;
2. $\varphi(a = b) := E(a, b)$;
3. $\varphi(A \lor B) := \varphi(A) \lor \varphi(B)$;
(4) \( \varphi(A \land B) := \varphi(A) \land \varphi(B) \);

(5) \( \varphi(A \supset B) := \varphi(A) \rightarrow \varphi(B) \);

(6) \( \varphi(\Box_i A) := \Box_i \varphi(A) \);

(7) \( \varphi(\exists x A) := \bigvee_{d \in D} (Ed \land \varphi([d/x]A)) \).

(8) \( \varphi(\forall x A) := \bigwedge_{d \in D} (Ed \rightarrow \varphi([d/x]A)) \).

Then for any \( D \)-formula \( A \) and a distinct list of variables \( x \supseteq FV(A) \) of length \( n \), we may define the associated \( F \)-predicate

\[ \varphi_{A,X} : D^n \rightarrow \Omega \]

such that

\[ \varphi_{A,X}(a) := \varphi([a/x]A) \]

for any \( a \in D^n \). In particular, \( \varphi_{P(x),x} = \varphi_P \).

To simplify notation, we sometimes write \( \varphi_A \) rather than \( \varphi_{A,X} \).

Let us check that every predicate \( \varphi_{A,X} \) is congruential. We begin with a simple observation.

**Lemma 4.2.5**

(1) \( \varphi_P \) remains congruential after fixing some parameters. In precise terms: let \( P \in PL^n, c \in D^n \), let \( x, y \) be disjoint distinct list of variables of length \( n \) and \( m \) respectively, let \( \sigma : I_m \rightarrow I_n \) be an injection, and let \( B = [c/x][y/x \cdot \sigma]P(x) \). Then \( \varphi_{B,Y} \) is congruential.

(2) Let \( A \) be an atomic \( D \)-formula without equality with \( FV(A) \subseteq \{x\} \). Then for any algebraic model \( (F, \varphi) \) with the individual domain \( D \), for any \( a, b \in D \)

\[ \varphi([a/x]A) \land Eab \leq \varphi([b/x]A) \]

**Proof** (1) First note that

\[ \varphi([a/y]B) = \varphi(P(a')) \]

where

\[ a'_i := \begin{cases} c_i & \text{if } i \notin r(\sigma), \\ a_{\sigma^{-1}(i)} & \text{otherwise}. \end{cases} \]

This is clear, since

\[ B[y \mapsto a] = P(x)[x \cdot \sigma \mapsto a][x \mapsto c]. \]

Similarly we have

\[ \varphi([b/y]B) = \varphi(P(b')). \]
where \( b' \) is constructed from \( b \) in the same way as \( a' \). Now \( a =_i b \) implies \( a' =_i b' \), and thus, since \( \varphi_P \) is congruential,

\[
\varphi([a/y]B) \land Eab = \varphi(P(a')) \land Eab \leq \varphi(P(b')) = \varphi([b/y]B),
\]
as required.

(2) If \( A \) is a \( D \)-sentence, the claim is trivial. Otherwise we have

\[
A = [x^m/y]B,
\]
for some \( B \) as in (1), if \( A \) contains \( P \) and \( x \) occurs \( m \) times in \( A \).

By (1), \( \varphi_{B,y} \) is congruent, hence by Lemma 4.1.8,

\[
E(a^m, b^m) \land \varphi(B(a^m)) \leq \varphi(B(b^m)).
\]

Now since \( E(a^m, b^m) = Eab, B(a^m) = A(a) \) and \( B(b^m) = A(b) \), we obtain (2).

Lemma 4.2.6 The predicate \( \varphi_{A,x} \) is congruent for any \( D \)-formula \( A \) with \( FV(A) \subseteq \{x\} \).

**Proof** By induction on the complexity of \( A \) we prove

\[
\varphi([a/x]A) \land Eab \leq \varphi([b/x]A).
\]

(1) If \( A \) is \( \bot \), there is nothing to prove.

(2) If \( A \) is \( x = c \) or \( c = x \), then by 4.1.1 we have:

\[
\varphi(A(a)) \land Eab = Eac \land Eab \leq Ebc = \varphi(A(b)).
\]

(3) If \( A \) is \( x = x \), then

\[
\varphi(A(a)) \land Eab = Eaa \land Eab \leq Ebb = \varphi(A(b)).
\]

(4) If \( A \) is atomic without equality, the claim follows from Lemma 4.2.5.

(5) In the modal case, if \( A \) is \( \square_i B \), then by Definition 4.1.2(4) and the induction hypothesis

\[
\varphi(A(a)) \cap Eab \leq \square_i \varphi(B(a)) \cap \square_i Eab = \square_i (\varphi(B(a)) \cap Eab) \leq \square_i \varphi(B(b)) = \varphi(A(b)).
\]

(6) If \( A \) is \( B \supset C \) and \( \varphi_B, \varphi_C \) are congruent, then

\[
\varphi(A(a)) \land Eab = (\varphi(B(a)) \rightarrow \varphi(C(a))) \land Eab;
\]

thus

\[
\varphi(A(a)) \land Eab \land \varphi(B(b)) =
\]

\[
(\varphi(B(a)) \rightarrow \varphi(C(a))) \land \varphi(B(b)) \land Eab \leq \varphi(B(a)) \rightarrow \varphi(C(a)) \land \varphi(B(b)) \land Eab \quad \text{(since \( \varphi_B \) is congruent)}
\]

\[
\leq \varphi(C(a)) \land Eab \quad \text{(by properties of Heyting algebras)}
\]

\[
\leq \varphi(C(b)) \quad \text{(since \( \varphi_C \) is congruential)}.
\]

Therefore

\[
\varphi(A(a)) \land Eab \leq \varphi(B(b)) \rightarrow \varphi(C(b)) = \varphi(A(b)).
\]
(7) If $A(x)$ is $\exists y B(y, x)$, then we may assume that $y \neq x$ (otherwise $A$ is a sentence), and thus

$$\varphi(A(a)) = \bigvee_{d \in D} \varphi(B(d, a)).$$

So by well-distributivity, and the induction hypothesis applied to $B(d, x)$, we obtain

$$\varphi(A(a)) \land Eab = \bigvee_{d \in D} (\varphi(B(d, a)) \land Eab) \leq \bigvee_{d \in D} \varphi(B(d, b)) = \varphi(A(b)).$$

(8) If $A = B \lor C$, and the claim is proved for $B$, $C$, we have

$$\varphi(A(a)) \land Eab \leq (\varphi(B(a)) \lor \varphi(C(a))) \land Eab = (\varphi(B(a)) \land Eab) \lor (\varphi(C(a)) \land Eab) \leq \varphi(B(b)) \lor \varphi(C(b)) = \varphi(A(b)).$$

(9) The simple case $A = B \land C$ is left to the reader.

(10) Let $A = \forall y B(y, x)$, and assume that $y \in \text{FV}(B)$, and that the claim holds for $B$. Then

$$\varphi(A(a)) \land Eab \leq \varphi(A(b))$$

is equivalent to

$$Eab \land \bigwedge_{c \in D} (Ec \to \varphi(B(c, a))) \leq \bigwedge_{c \in D} (Ec \to \varphi(B(c, b))).$$

It suffices to show that for any $c \in D$,

$$Eab \land (Ec \to \varphi(B(c, a))) \leq Ec \to \varphi(B(c, b)),$$

i.e.

$$Eab \land (Ec \to \varphi(B(c, a))) \land Ec \leq \varphi(B(c, b)).$$

The latter follows by properties of Heyting algebras and the induction hypothesis:

$$Eab \land (Ec \to \varphi(B(c, a))) \land Ec \leq Eab \land \varphi(B(c, a)) \leq \varphi(B(c, b)).$$

Lemma 4.2.7 The predicate $\varphi_{A,x}$ is congruential for any D-formula $A$ and $r(x) \supseteq \text{FV}(A)$.

Proof Assume $a \neq b$ for $a, b$ of the same length as $x$. Fix all the parameters of $A$ but $x_i$, so put

$$B(x_i) := A(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n).$$

Then

$$\varphi(A(a)) \land Ea_ib_i = \varphi(B(a_i)) \land Ea_ib_i \leq \varphi(B(b_i)) = \varphi(A(b))$$

by Lemma 4.2.6.
4.2. ALGEBRAIC MODELS

However, as we mentioned in Remark 4.1.17, the predicate \( \varphi_A \) is not necessarily strict, even for a strict \( \varphi \). So we introduce another extension of \( \varphi \) to \( D \)-sentences.

Let \( F = (\Omega, D, E) \) be a structure, \( A \) a \( D \)-sentence of the corresponding type. Then we put

\[
E(A) := \bigwedge \{ Ea \mid a \in D, \ a \text{ occurs in } A \}.
\]

In particular, \( E(A) = 1 \) if \( A \) is a usual sentence (without constants from \( D \)).

**Definition 4.2.8** For a valuation \( \varphi \) in a structure \( F = (\Omega, D, E) \) we define an \( \Omega \)-valued function \( \varphi^b \) on corresponding \( D \)-sentences as follows.

1. \( \varphi^b(\bot) := 0 \);
2. \( \varphi^b(a = b) := E(a, b) \);
3. \( \varphi^b(A) := \varphi(A) \wedge E(A) \) for all other atomic \( A \);
4. \( \varphi^b(A \lor B) := E(A \lor B) \wedge (\varphi^b(A) \lor \varphi^b(B)) \);
5. \( \varphi^b(A \land B) := \varphi^b(A) \land \varphi^b(B) \);
6. \( \varphi^b(A \supset B) := E(A \supset B) \land (\varphi^b(A) \rightarrow \varphi^b(B)) \);
7. \( \varphi^b(\Box_i A) := E(\Box_i A) \cap \Box_i \varphi^b(A) \) (in the modal case);
8. \( \varphi^b(\exists x A) := \bigvee_{d \in D} \varphi^b([d/x]A) \).
9. \( \varphi^b(\forall x A) := E(\forall x A) \land \bigwedge_{d \in D} (Ed \rightarrow \varphi^b([d/x]A)) \).

We also define another associated predicate

\[
\varphi^b_{A, x} : D^n \rightarrow \Omega
\]

such that for any \( a \in D^n \)

\[
\varphi^b_{A, x}(a) := \varphi^b([a/x]A).
\]

Again we often abbreviate \( \varphi^b_{A, x} \) to \( \varphi^b_A \).

**Lemma 4.2.9** Let \( (F, \varphi) \) be an algebraic model, with the domain \( D \), \( A \) a corresponding \( D \)-sentence. Then

\[
\varphi^b_A(A) = \varphi(A) \wedge E(A).
\]

**Proof** By induction.

1. If \( A \) is \( \bot \), the claim is trivial.
2. If \( A \) is \( a = b \), then by 4.1.4 (1) and symmetry,

\[
\varphi(A) = Eab \leq Ea \wedge Eb = E(A).
\]
(3) If $A$ is $a = a$, then $\varphi(A) = Eaa = E(A)$.

(4) If $A$ is atomic without equality, the claim holds by definition.

(5) If $A$ is a modal formula $\Box_i B$, then by the induction hypothesis,

$$\varphi^\sharp(A) = \Box_i \varphi^\sharp(B) \cap E(A) = \Box_i \varphi(B) \cap \Box_i E(B) \cap E(A).$$

Next,

$$\Box_i E(B) \cap E(A) = \Box_i E(A) \cap E(A) = E(A)$$

by 4.1.4 (4), and thus

$$\varphi^\sharp(A) = \Box_i \varphi(B) \cap E(A) = \varphi(A) \cap E(A).$$

(6) If $A$ is $B \lor C$, then by the induction hypothesis,

$$\varphi^\sharp(A) = (\varphi^\sharp(B) \lor \varphi^\sharp(C)) \land E(A) =$$

$$(\varphi(B) \land E(B) \land E(A)) \lor (\varphi(C) \land E(C) \land E(A)).$$

Since $E(A) \leq E(B), E(C)$, we obtain

$$\varphi^\sharp(A) = (\varphi(B) \land E(A)) \lor (\varphi(C) \land E(A)) = \varphi(A) \land E(A).$$

(7) If $A$ is $B \land C$ and the claim holds for $B, C$, we have

$$\varphi^\sharp(A) = \varphi(B) \land \varphi(C) = \varphi(B) \land E(B) \land \varphi(C) \land E(C) =$$

$$(\varphi(B) \land \varphi(C)) \land (E(B) \land E(C)) = \varphi(A) \land E(A).$$

(8) If $A$ is $B \supset C$, then by the induction hypothesis

$$\varphi^\sharp(A) = E(A) \land (\varphi^\sharp(B) \rightarrow \varphi^\sharp(C)) =$$

$$E(A) \land (\varphi(B) \land E(B) \rightarrow \varphi(C) \land E(C)).$$

Let us show that this equals

$$E(A) \land \varphi(A) = E(A) \land (\varphi(B) \rightarrow \varphi(C)).$$

In fact,

$$\varphi(B) \land E(A) \leq \varphi(B) \land E(B),$$

hence

$$\varphi(B) \land E(A) \land (\varphi(B) \land E(B) \rightarrow \varphi(C) \land E(C)) \leq$$

$$\varphi(B) \land E(B) \land (\varphi(B) \land E(B) \rightarrow \varphi(C) \land E(C)) \leq$$

$$\varphi(C) \land E(C) \leq \varphi(C).$$

So

$$E(A) \land (\varphi(B) \land E(B) \rightarrow \varphi(C) \land E(C)) \leq \varphi(B) \rightarrow \varphi(C).$$

The other way round

$$((\varphi(B) \rightarrow \varphi(C)) \land \varphi(B) \leq \varphi(C),$$
hence,

\[ E(A) \land (\varphi(B) \rightarrow \varphi(C)) \land \varphi(B) \land E(B) \leq E(A) \land \varphi(C) \leq \varphi(C) \land E(C) \]

thus

\[ E(A) \land (\varphi(B) \rightarrow \varphi(C)) \leq \varphi(B) \land E(B) \rightarrow \varphi(C) \land E(C). \]

(9) If \( A = \exists x B(x) \) and \( x \) is a parameter of \( B(x) \), then by the induction hypothesis and Definition 4.2.4 we have

\[
\varphi^\sharp(A) = \bigvee_{d \in D} \varphi^\sharp(B(d)) = \bigvee_{d \in D} (\varphi(B(d)) \land E(B(d))) = \bigvee_{d \in D} (\varphi(B(d)) \land Ed \land E(A)) = E(A) \land \bigvee_{d \in D} (\varphi(B(d)) \land Ed) = E(A) \land \varphi(A).
\]

If \( A = \exists x B(x) \) and \( B(x) \) is a \( D \)-sentence, then \( B(x) = B(d) \) for any \( d \in D, E(B(d)) = E(A) \). So

\[
\varphi^\sharp(A) = \bigvee_{d \in D} \varphi^\sharp(B(d)) = \bigvee_{d \in D} (\varphi(B(d)) \land E(B(d))) = E(A) \land \bigvee_{d \in D} \varphi(B(d)) = E(A) \land \varphi(A).
\]

(10) Finally let \( A = \forall x B(x) \) and first assume that \( x \in FV(B(x)) \). Then

\[
\varphi^\sharp(A) = E(A) \land \bigwedge_{d \in D} (Ed \rightarrow \varphi^\sharp(B(d))) = E(A) \land \bigwedge_{d \in D} (Ed \rightarrow E(A) \land Ed \land \varphi(B(d)))
\]

by the induction hypothesis; thus

\[
\varphi^\sharp(A) \leq E(A) \land \bigwedge_{d \in D} (Ed \rightarrow \varphi(B(d))) = E(A) \land \varphi(A).
\]

The other way round,

\[
E(A) \land (Ed \rightarrow \varphi(B(d))) \land Ed \leq E(A) \land Ed \land \varphi(B(d)),
\]

\[
E(A) \land (Ed \rightarrow \varphi(B(d))) \land Ed \leq E(A) \land Ed \land \varphi(B(d)),
\]

hence

\[
(*) \quad E(A) \land (Ed \rightarrow \varphi(B(d))) \leq Ed \rightarrow E(A) \land Ed \land \varphi(B(d)),
\]

which implies

\[
E(A) \land \varphi(A) \leq \varphi^\sharp(A).
\]

If \( B(x) \) is closed, the argument is slightly different, because now \( B(d) = B(x) \) and \( E(B(d)) = E(A) \). By the induction hypothesis we have

\[
\varphi^\sharp(A) = E(A) \land \bigwedge_{d \in D} (Ed \rightarrow E(A) \land \varphi(B(d))) \leq E(A) \land \bigwedge_{d \in D} (Ed \rightarrow \varphi(B(d))) = E(A) \land \varphi(A).
\]
The other way round, (*) implies

\[ E(A) \wedge (Ed \rightarrow \varphi(B(d))) \leq Ed \rightarrow E(A) \wedge \varphi(B(d)) , \]

so

\[
E(A) \wedge \varphi(A) = E(A) \wedge \bigwedge_{d \in D} (Ed \rightarrow \varphi(B(d))) \leq E(A) \wedge \bigwedge_{d \in D} (Ed \rightarrow E(A) \wedge \varphi(B(d))) = \varphi^s(A).
\]

**Lemma 4.2.10** Let \((F, \varphi)\) be an algebraic model with a domain \(D\). Then for any \(D\)-formula \(A\) with \(FV(A) \subseteq r(x)\) the predicate \(\varphi^s_{A,x}\) is congruential; it is also strict if \(FV(A) = r(x)\).

**Proof** Strictness readily follows from the previous lemma, since

\[ \varphi^s_{A,x}(a) = \varphi^s(A(a)) \leq E(A(a)) = Ea. \]

To prove the congruentiality, assume that \(|x| = n\) and consider \(a =_i b\) in \(D^n\).

If \(x_i \notin FV(A)\), then \([a/x]A = [b/x]A\), so the inequality

\[ \varphi^s([a/x]A) \wedge Ea_ib_i \leq \varphi^s([b/x]A) \]

is trivial.

If \(x_i \in FV(A)\), then by the previous lemma and Lemma 4.2.7,

\[ \varphi^s([a/x]A) \wedge Ea_ib_i = \varphi([a/x]A) \wedge E([a/x]A) \wedge Ea_ib_i \leq \varphi([b/x]A) \wedge E([a/x]A) \wedge Ea_ib_i. \]

Since \(a =_i b\) and \(Ea_ib_i \leq Eb_i\), we have

\[ E([a/x]A) \wedge Ea_ib_i \leq E([b/x]A); \]

thus

\[ \varphi^s([a/x]A) \wedge Ea_ib_i \leq \varphi^s([b/x]A) \]

by 4.2.7.

**Lemma 4.2.11** Let \((F, \varphi)\) be an algebraic model with a domain \(D\), \(\varphi^s\) the corresponding strict valuation such that for any \(P \in PL^n, a \in D^n,\)

\[ \varphi^s(P(a)) = \varphi(P(a)) \wedge Ea, \]

i.e. (Lemma 4.1.11)

\[ \varphi^s_P = (\varphi_P)^s. \]

Then

\[ (\varphi^s)^s(A) = \varphi^s(A) \]

for any \(D\)-sentence \(A\).
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Proof By the choice of $\varphi^s$, for any $A \in AF_D$

$$\varphi^s(A) = \varphi(A) \land E(A) = \varphi^h(A),$$

so

$$(\varphi^s)^h(A) = \varphi^s(A) \land E(A) = \varphi^h(A).$$

Since the maps $(\varphi^s)^h$, $\varphi^h$ coincide on $AF_D$ and they are uniquely prolonged according to 4.2.8, they also coincide on all $D$-sentences.

Lemma 4.2.12 Let $(F, \varphi)$ be an algebraic model, $A$ a sentence of the corresponding type. Then

$$\varphi(A) = \varphi^s(A) = \varphi^h(A) = (\varphi^s)^h(A).$$

Proof By 4.2.11,

$$\varphi^h(A) = (\varphi^s)^h(A).$$

Since $E(A) = 1$, by Lemma 4.2.9 we also have

$$\varphi^h(A) = \varphi(A), \ (\varphi^s)^h(A) = \varphi^s(A).$$

Lemma 4.2.13 Let $F = (\Omega, D, E)$ be a structure, $(F, \varphi)$ an algebraic model, $A(x)$ a $D$-formula of the corresponding type with $r(x) = FV(A(x))$, $|x| = n$, $x$ distinct. Then

(1) $\varphi(\forall x A(x)) = \bigwedge_{a \in D^n} (Ea \rightarrow \varphi(A(a))).$

(2) $\varphi(\forall x A(x)) = 1$ iff $\forall a \in D^n \ Ea \leq \varphi(A(a)).$

Thus $\varphi(\forall x A(x))$ does not depend on the ordering of $x$, so we may use the notation $\varphi(\forall A)$.

Proof (1) By induction on $n$. The base is trivial. Consider the step from $n$ to $n + 1$. If $x = yz$, $|y| = n$, then $\forall x A(x) = \forall y \forall z A(y, z)$, so by the induction hypothesis and Lemma 1.2.3,

$$\varphi(\forall x A(x)) = \bigwedge_{a \in D^n} (Ea \rightarrow \varphi(\forall z A(a, z)))$$

$$= \bigwedge_{a \in D^n} (Ea \rightarrow \bigwedge_{c \in D} (Ec \rightarrow \varphi(A(a, c)))) = \bigwedge_{a \in D^n} \bigwedge_{c \in D} (Ea \rightarrow (Ec \rightarrow \varphi(A(a, c))))$$

$$= \bigwedge_{(a, c) \in D^{n+1}} (Ea \wedge Ec \rightarrow \varphi(A(a, c))) = \bigwedge_{b \in D^{n+1}} (Eb \rightarrow \varphi(A(b)))$$

as required.

(2) Readily follows from (1).
Definition 4.2.14 A closed formula \( A \) is called true in an algebraic model \((F, \varphi)\) (notation: \((F, \varphi) \models A\) or \((F, \varphi) \Vdash A\) in the intuitionistic case) if \( \varphi(A) = 1\); an arbitrary formula \( A \) is called true in \((F, \varphi)\) (with the same notation) if \( \forall A \) is true.

Definition 4.2.15 A predicate \( A : D^n \rightarrow \Omega \) is called true in a structure \( F = (\Omega, D, E) \) (notation: \( F \models A \) or \( F \Vdash A \)) if \( E \alpha \leq A(\alpha) \) for any \( \alpha \in D^n \).

Thus
\[
F \models A \text{ iff } \forall \alpha \in D^n \ E \alpha = A(\alpha).
\]
for a strict \( A \),
\[
F \models A \text{ iff } A(\lambda) = 1.
\]
for a 0-ary \( A \), and
\[
F \models A \text{ iff } F \models A^e.
\]

Lemma 4.2.16 The following conditions are equivalent (where \( \models \) stands for \( \Vdash \) in the intuitionistic case)

(1) \((F, \varphi) \models A\),
(2) \((F, \varphi^e) \models A\),
(3) \( F \models \varphi_A \), i.e. \( \forall \alpha \in D^n \ E \alpha \leq \varphi(A(\alpha)) \),
(4) \( F \models \varphi_A^e \), i.e. \( \forall \alpha \in D^n \ \varphi^e(A(\alpha)) = E \alpha \),
(5) \( F \models \varphi_A^b \), i.e. \( \forall \alpha \in D^n \ E \alpha \leq \varphi^b(A(\alpha)) \),
(6) \( F \models (\varphi^s)^b_A \), i.e. \( \forall \alpha \in D^n \ (\varphi^s)^b_A(\alpha) = E \alpha \).

Proof By 4.2.12 and 4.2.13.

Definition 4.2.17 A formula \( A \) is valid in a structure \( F \) (of the corresponding kind) if \( A \) is true in every model over \( F \) (or equivalently, in every strict model over \( F \)).

Validity is denoted again by \( \models \) in the modal case, \( \Vdash \) in the intuitionistic case.

Remark 4.2.18 If in the definition of validity we also allow for 'valuations' \( \varphi \) in \( F = (\Omega, D, E) \), for which \( \varphi_P \) is not congruential (they are just valuations in the corresponding \( D \)-structure \( (\Omega, D) \)), then some \( \mathbf{QK}^- \) (or \( \mathbf{QH}^- \)) theorems become nonvalid. For instance, if \( P \in \mathbf{PL}^1 \) and \( \varphi_P \) is not congruential, then the formula
\[
A := \forall x \forall y (x = y \land P(x) \supset P(y))
\]
is not true in \( (\Omega, \varphi) \) (in the sense of Definition 4.2.14).
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In fact, suppose

\[ Eab \land \varphi(P(a)) \not\leq \varphi(P(b)) \]

for some \( a, b \in D \). Then by Lemma 4.2.13 (the proof of which does not use congruentiality),

\[
\begin{align*}
\varphi(A) &\leq Ea \land Eb \rightarrow (Eab \land \varphi(P(a)) \rightarrow \varphi(P(b))) \\
&= Ea \land Eb \land Eab \land \varphi(P(a)) \rightarrow \varphi(P(b)) \\
&= Eab \land \varphi(P(a)) \rightarrow \varphi(P(b)) \neq 1,
\end{align*}
\]

by our assumption.

**Lemma 4.2.19** For any valuation \( \varphi \) in an m.v.s.,

\[
\begin{align*}
(1) &\quad \varphi^k(\neg A) = E(A) - \varphi^k(A); \\
(2) &\quad \varphi^k(\Diamond_i A) = E(A) \cap \Diamond_i \varphi^k(A).
\end{align*}
\]

**Proof** The equality (1) is checked easily, so let us check (2). In fact, by (1), we have:

\[
\begin{align*}
\varphi^k(\Diamond_i A) &= \varphi^k(\neg \Box_i - A) = E(A) \cap \neg \Box_i (E(A) \cap \neg \varphi^k(A)) = \\
&= E(A) \cap (-\Box_i E(A) \cup -\Box_i - \varphi^k(A)) = E(A) \cap \Diamond_i \varphi^k(A)
\end{align*}
\]

since \( E(A) \leq \Box_i E(A) \).

**4.3 Soundness**

We begin with an analogue of Lemma 3.2.23.

**Lemma 4.3.1** Let \( F \) be a modal or Heyting-valued structure with a set of individuals \( D \), and let \( A(x) \), \( B(x) \) be congruent formulas of the corresponding type, \( |x| = n \). Then for any \( a \in D^n \), for any valuation \( \phi \) in \( F \);

(I) \( \varphi(A(a)) = \varphi(B(a)) \), \( \varphi^d(A(a)) = \varphi^d(B(a)) \)

(II) for \( A, B \) without constants, \( F \vdash (\neg)A \iff F \vdash (\neg)B \).

**Proof**

(I) Along the same lines as 3.2.22. Consider the equivalence relation on modal (or intuitionistic) formulas:

\( A \sim B \) iff \( \text{FV}(A) = \text{FV}(B) \) and for any distinct list \( x \) such that \( r(x) = FV(A) \), for any \( a \in D^{\|x\|} \)

\[ \varphi([a/x]A) = \varphi([a/x]B). \]

It is sufficient to show that \( \sim \) has the properties 2.3.17(1)-(4).

(1) \( QyA \sim Qz(A[y \mapsto z]) \) for \( y \not\in BV(A), \ z \not\in V(A) \).

We only consider the case \( Q = \forall \). If \( FV(\forall yA) = r(x) \) for a distinct \( x \), we have two options:
(i) If \( y \not\in FV(A) \), then \( y \not\in V(A) \), \( A[y \mapsto z] = A \),
\[
\forall z (A[y \mapsto z]) = \forall z A, \quad [a/x] \forall y A = \forall y[a/x]A, \quad [a/x] \forall z A = \forall z [a/x]A.
\]
By Definition 4.2.4,
\[
\varphi(\forall y[a/x]A) = \bigwedge_{d \in D} (Ed \rightarrow \varphi([a/x]A)),
\]
since \( y \) does not occur in \([a/x]A\). Hence by Lemma 1.2.4 and 4.1.1(E3),
\[
\varphi(\forall y[a/x]A) = \bigvee_{d \in D} Ed \rightarrow \varphi([a/x]A) = \varphi([a/x]A).
\]
By the same reason
\[
\varphi(\forall z[a/x]A) = \varphi([a/x]A),
\]
so (1) holds in this case.

(ii) If \( y \in FV(A) \), then
\[
[a/x] \forall y A = \forall y[a/x]A, \quad [a/x] \forall z (A[y \mapsto z]) = \forall z [a/x] (A[y \mapsto z]),
\]
and so
\[
\varphi([a/x] \forall y A) = \bigwedge_{d \in D} (Ed \rightarrow \varphi([d/y][a/x]A)),
\]
\[
\varphi([a/x] \forall z (A[y \mapsto z])) = \bigwedge_{d \in D} (Ed \rightarrow \varphi([d/z][a/x] (A[y \mapsto z]))).
\]
It remains to note that
\[
[d/y][a/x]A = [d/z][a/x] (A[y \mapsto z]),
\]
since \( y \not\in BV(A) \), \( z \not\in V(A) \).

(2) Supposing \( A \sim B \), let us show that
\[
\forall y A \sim \forall y B.
\]
We have \( FV(A) = FV(B) \), \( FV(\forall y A) = FV(\forall y B) \); let \( r(x) = FV(\forall y A) \).
Then
\[
\varphi([a/x] \forall y A) = \bigwedge_{d \in D} (Ed \rightarrow \varphi([ad/xy]A)),
\]
\[
\varphi([a/x] \forall y B) = \bigwedge_{d \in D} (Ed \rightarrow \varphi([ad/xy]B)).
\]
Now \( A \sim B \) implies
\[
\varphi([ad/xy]A) = \varphi([ad/xy]B),
\]
and the claim follows.
The argument for \( \exists \) is quite similar.
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(3) $A \sim A' & B \sim B' \Rightarrow (A \ast B) \sim (A' \ast B')$.

Consider the case $* = \supset$. We argue similarly to 3.2.22. If $r(x) = FV(A \supset B)$, then $FV(A) = x \cdot \sigma$, $FV(B) = x \cdot \tau$ for injections $\sigma, \tau$. Now $A \sim A' & B \sim B'$ implies $FV(A \supset B) = FV(A' \supset B')$ and

$$
\varphi([a/x](A \supset B)) = \varphi([a \cdot \sigma/x \cdot \sigma]A \supset [a \cdot \tau/x \cdot \tau]B) = \\
= \varphi([a \cdot \sigma/x \cdot \sigma]A \rightarrow [a \cdot \tau/x \cdot \tau]B) \\
= \varphi([a \cdot \sigma/x \cdot \sigma]A') \rightarrow \varphi([a \cdot \tau/x \cdot \tau]B') \\
= \varphi([a/x](A' \supset B')).
$$

The proof of (4) is trivial.

Since $E(A(a)) = E(B(a))$, by Lemma 4.2.9 it follows that $\varphi^\sharp(A(a)) = \varphi^\sharp(B(a))$.

(II) is an obvious consequence of (I).

\[ \square \]

Theorem 4.3.2 (Soundness theorem)

(1) For an $N$-m.v.s. $F$, the set

$$ML(=)(F) := \{ A \in MF_N(=) \mid F \vdash A \}$$

is an m.p.l. (\(=\)).

(2) For an $H$-v.s. $F$, the set

$$IL(=)(F) := \{ A \in IF(=) \mid F \vDash A \}$$

is an s.p.l. (\(=\)).

Proof First let us show that the set of valid formulas is substitution closed. The argument resembles the proof of Lemma 3.2.16, but the detail is slightly different. Assume that $F = (\Omega, D, E) \vdash A$. Let $S = [C(x, y)/P(x)]$ be a simple formula substitution, where $x, y$ are distinct lists such that $r(y) \subseteq FV(C(x, y)) \subseteq r(xy)$. We may also assume that $P$ occurs in $A$ (otherwise $SA = A$) and $y \cap FV(A) = \emptyset$, due to Lemma 4.3.2 (cf. the proof of Lemma 3.2.16). Since $SA$ is defined up to congruence and validity respects congruence (Lemma 4.3.1), we may further assume that $A$ is clean and $BV(A) \cap y = \emptyset$; then $SA$ is obtained by replacing every subformula of the form $P(x')$ with $C(x', y)$. By Lemma 2.5.23 we also have $FV(SA) = y \cup FVe(S,A)$, where $FVe(S,A) \subseteq FV(A)$ is the set of essential parameters, see Definition 2.5.22. Let

$$FV(A) = r(z), \ FVe(S,A) = r(z'), \ z = z''.''
$$

For an algebraic model $(F, \varphi)$ let us show that $(F, \varphi) \models (\vDash) SA$, i.e. (Lemma 4.2.16) for any $b \in D^m$, $c \in D^j$ (where $m = |y|$, $j = |z'|$)

$$\varphi([c, b/z', y]SA) \geq Eb \wedge Ec.$$
Given tuples \( b, c \), we construct a valuation \( \eta \) in \( F \) such that for any \( a \in D^n \) (where \( n = |x| \))

- \( \eta(P(a)) := \varphi(C(a, b)) \);
- \( \eta(B) := \varphi(B) \) for any other \( B \in AF_D \).

Let us check that \( \eta \) is a valuation. In fact, let \( a = d \); then by Lemma 4.2.7

\[
Ea_i d_i \land \eta(P(a)) = Ea_i d_i \land \varphi(C(d, b)) \leq \varphi(C(a, b)) = \eta(P(d)).
\]

Then we claim that for any formula \( B(u) \), with a distinct \( u \) such that

\[ FV(B(u)) \subseteq r(u), \quad r(u) \cap r(y) = r(u) \cap BV(B(u)) = r(y) \cap V(B(u)) = \emptyset, \]

and for any \( d \in D^l \) (where \( l = |u| \))

\[ (1) \quad \eta(B(d)) = \varphi([d, b/u, y]SB), \]

or in a simpler notation,

\[ \eta(B(d)) = \varphi(SB(d, b)). \]

Note that \( SB(d, b) \) is a \( D \)-sentence, since \( FV(SB) \subseteq uy \), by Lemma 2.5.23.

The claim (1) is proved by induction.

The case when \( B \) is atomic and does not contain \( P \), is trivial.

Assume that \( B \) is atomic, \( B = P(x') \), \( x' = u \cdot \sigma \), where \( \sigma : I_n \rightarrow I_l \). Then we have:

\[ B(d) = [d/u]u \cdot \sigma/x\)P(x) = [d \cdot \sigma/x]P(x) = P(a), \]

where \( a = d \cdot \sigma \). On the other hand, \( SB = C(u \cdot \sigma, y) \), and so

\[ [d, b/u, y]SB = [d, b/u, y]C(u \cdot \sigma, y) = C(d \cdot \sigma, b) = C(a, b). \]

Thus by definition of \( \eta \),

\[ \eta(B(d)) = \varphi(SB(d, b)). \]

Let \( B = B_1 \ast B_2 \), \( SB = SB_1 \ast SB_2 \), where \( \ast \) is \( \lor \), \( \land \) or \( \supset \). Then by Definition 4.2.4 and the induction hypothesis

\[
\eta(B(d)) = (\eta(B_1(d)) \ast \eta(B_2(d))) = \varphi(SB_1(d, b)) \ast \varphi(SB_2(d, b))
\]

\[
= \varphi(SB_1(d, b)) \ast SB_2(d, b) = \varphi(SB(d, b)),
\]

where \( \ast \) is the corresponding operation in \( \Omega \).

If \( B = \forall x B_1 \), then by our assumption \( x \notin uy \), so \( SB = \forall x SB_1 \), and thus by Definition 4.2.4 and the induction hypothesis

\[
\eta(B(d)) = \bigwedge_{a \in D} (Ea \to \eta(B_1(a, d))) = \bigwedge_{a \in D} (Ea \to \varphi(SB_1(b, a, d)))
\]

\[
= \varphi(\forall x SB_1(b, d)) = \varphi(SB(b, d)).
\]
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The case \( B = \exists x B_1 \) is similar and is left to the reader.

Now let us verify (\( \# \)). Take an arbitrary \( e \in D^k-j \) and the corresponding \( d = c e \in D^k \). Then by (1) we have:

\[
(2) \quad \varphi([d, b/z, y]SA) = \eta([d/z]A).
\]

But

\[
(3) \quad [d, b/z, y]SA = [c, b/z', y]SA, \quad [d/z]A = [c/z']A,
\]

since \( FV(A) = r(z) \), \( FV(SA) = r(yz') \).

By our assumption, \( F \models A \), and so by (2), (3), and Lemma 4.2.16, we obtain

\[
\varphi([c, b/z', y]SA) = \eta([d/z]A) = \eta([c/z']A) \geq Ec \geq Eb \land Ec,
\]

i.e. (\( \# \)) holds.

The remaining properties (m0)-(m3), (m5)=, (s1), (s3) from Definitions 2.6.1, 2.6.2, 2.6.3 are verified in a standard way. We check only some of them.

Note that the value \( \varphi(A) \) of a propositional formula \( A \) in a structure \( F = (\Omega, D, E) \) is exactly the same as in the algebra \( \Omega \). So, since all propositional axioms are valid in \( \Omega \) (Lemma 1.2.6), they are also valid in \( F \).

Let us check the validity of (Ax12): \( \forall x P(x) \supset P(y). \) In fact, consider an algebraic model \( (F, \varphi) \). We have

\[
Ea \land \varphi(\forall x P(x)) = Ea \land \bigwedge_{d \in D} (Ed \rightarrow \varphi(P(d)))
\leq Ea \land (Ea \rightarrow \varphi(P(a))) \leq \varphi(P(a)),
\]

hence

\[
Ea \leq \varphi(\forall x P(x)) \rightarrow \varphi(P(a)) = \varphi(\forall x P(x)) \rightarrow \varphi(P(a)) = \varphi(\forall x P(x) \supset P(a)),
\]

i.e. \( (F, \varphi) \models \forall x P(x) \supset P(y), \) by Lemma 4.2.16.

Let us also show the validity of (Ax14): \( \forall x (P(x) \supset q) \supset (\exists x P(x) \supset q) \). In fact, \( \varphi(\forall x (P(x) \supset q) \supset (\exists x P(x) \supset q)) \) if \( \varphi(\forall x (P(x) \supset q)) \leq \varphi(\exists x P(x) \supset q) = \varphi(\exists x P(x)) \rightarrow \varphi(q) \)
if \( \varphi(\forall x (P(x) \supset q)) \land \varphi(\exists x P(x)) \leq \varphi(q) \).

Transforming the left part of the latter inequality by well-distributivity, we obtain

\[
\bigwedge_{d \in D} (\varphi(P(d)) \rightarrow \varphi(q)) \land \bigvee_{a \in D} \varphi(P(a))
\leq \bigvee_{a \in D} [\varphi(P(a)) \land \bigwedge_{d \in D} (\varphi(P(d)) \rightarrow \varphi(q))] \leq \varphi(q)
\]

as required.

And finally let us consider modus ponens. Suppose \( F \models A(x, y), A(x, y) \supset B(y, z), \) where \( FV(A(x, y)) = r(x, y), FV(B(y, z)) = r(yz), r(x) \cap r(z) = \emptyset \) and the lists xy, yz are distinct (and some of them may be empty). Let l, m, n be
the lengths of $x, y, z$ respectively. By 4.2.16, for any $a \in D^l$, $b \in D^m$, $c \in D^n$ and valuation $\varphi$

$$E(ab) \leq \varphi(A(a, b)), \quad E(abc) \leq \varphi(A(a, b) \supset B(b, c)).$$

Hence

$$E(abc) = E(ab) \land E(abc) \leq \varphi(A(a, b)) \land \varphi(A(a, b) \supset B(b, c)) \leq \varphi(B(b, c)).$$

As this happens for any $a \in D^l$, we obtain

$$\bigvee_{a \in D^l} E(abc) \leq \varphi(B(b, c)).$$

But

$$\bigvee_{a \in D^l} E(abc) = \bigvee_{a \in D^l} (E(a) \land E(bc)) \leq (\bigvee_{a \in D^l} E(a)) \land E(bc) = E(bc),$$

by 4.1.4(5). Thus

$$E(bc) \leq \varphi(B(b, c)),$$

which implies $F \models B(y, z)$, by 4.2.16.

Let us now consider a particular kind of H.v.s. — those arising from S4-m.v.s.

Definition 4.3.3 The pattern of an S4-m.v.s. $F = (\Omega, D, E)$ is the H.v.s. $F^\circ = (\Omega^\circ, D, E)$.

An H.v.s. of this form is called 'basic'. More generally:

Definition 4.3.4 A locale is said to be basic if it is isomorphic to the pattern of a complete S4-algebra. An H.v.s. over a basic locale is also called basic.

For basic H.v.s. the intuitionistic truth definition matches with the modal definition. In precise terms, this means the following.

Lemma 4.3.5 Let $\varphi$ be a valuation in an S4-m.v.s $F = (\Omega, D, E)$, and put

$$\psi(A) := \Box \varphi(A)$$

for every $A \in AF_D$.

Then $\psi$ is a valuation in $F^\circ$ and $\psi(A) = \varphi(A^T)$ for every $A \in IF_D^\circ$.

Proof By Lemma 4.2.7, $\psi$ is congruential. The required equality is easily proved by induction. Let us consider the induction step for $A = \forall x B(x)$, where $B(x)$ is a $D$-formula, $FV(B(x)) \subseteq \{x\}$. Then according to Definitions 4.2.4, 2.11.1 and Proposition 1.2.7, we have:

$$\psi(A) = \bigwedge_{d \in D} (Ed \rightarrow \psi(B(d))) = \Box \bigcap_{d \in D} (Ed \supset \varphi(B^T(d))) \leq \Box \bigcap_{d \in D} (Ed \supset \varphi(B^T(d))) = \varphi(A^T).$$
On the other hand,
\[ \bigcap_{d \in D} (Ed \supset \varphi(B^T(d))) \leq Ed \supset \varphi(B^T(d)), \]

hence
\[ \varphi(A^T) = \Box \bigcap_{d \in D} (Ed \supset \varphi(B^T(d))) \leq \Box (Ed \supset \varphi(B^T(d))), \]

and thus
\[ \varphi(A^T) = \bigcap_{d \in D} \Box (Ed \supset \varphi(B^T(d))), \]

which implies
\[ \varphi(A^T) = \Box \varphi(A^T) \leq \Box \bigcap_{d \in D} (Ed \supset \varphi(B^T(d))) = \psi(A), \]

and eventually
\[ \varphi(A^T) = \psi(A). \]

The case \( A = \exists x B(x) \) is left to the reader.

Lemma 4.3.6 Let \( F = (\Omega, D, E) \) be an \( S_4 \)-m.v.s., \( \psi \) a valuation in \( F^\circ \), \( \psi^\sim \) the same valuation in \( F \). Then \( \psi(A) = \psi^\sim(A^T) \) for every \( A \in IF^\circ_D \).

Proof Apply Lemma 4.3.5 to the case when \( \varphi = \psi^\sim \).

Proposition 4.3.7 For any \( S_4 \)-m.v.s. \( F \) and intuitionistic formula \( A(x) \), \( F^\circ \models A(x) \) iff \( F \models A^T(x) \), i.e. \( IL(=)(F^\circ) = s(ML(=)(F)) \).

Proof We may assume that \( x = FV(A) \).

\( \Leftarrow \) Suppose \( F \models A^T \) and consider a valuation \( \psi \) in \( F^\circ \). Let \( \psi^\sim \) be the same valuation in \( F \); then \( \psi^\sim(\forall x A^T(x)) = 1 \), and thus \( \psi^\sim(\Box \forall x A^T(x)) = 1 \). By Lemma 2.11.7,
\[ \text{QS4} \vdash \Box \forall x A^T(x) \equiv (\forall x A(x))^T; \]

hence by Lemmas 4.3.2, 4.3.6
\[ \psi^\sim(\Box \forall x A^T(x)) = \psi^\sim((\forall x A(x))^T) = \psi(\forall x A(x)) = 1. \]

Since \( \psi \) is arbitrary, it follows that \( F^\circ \models A \).

\( \Rightarrow \) Suppose \( F^\circ \models A \). For an arbitrary valuation \( \varphi \) in \( F \), let us show that \( \varphi(\forall x A^T(x)) = 1 \). Let \( \psi \) be a valuation in \( F^\circ \) described in Lemma 4.3.5. Then by Lemmas 2.11.7, 4.3.2 and since \( F^\circ \models A \), we obtain
\[ \varphi(\Box \forall x A^T(x)) = \varphi((\forall x A(x))^T) = \psi(\forall x A(x)) = 1, \]

and therefore \( \varphi(\forall x A^T(x)) = 1 \).
Definition 4.3.8 We introduce general algebraic semantics for our four types of logics \((\mathcal{M}, \mathcal{M}^=, \mathcal{S}, \mathcal{S}^=)\):

\[
\begin{align*}
\mathcal{A}_m &= \{ \mathcal{ML}^=(F) \mid F \text{ is an m.v.s.} \}, \\
\mathcal{A}_m^e &= \{ \mathcal{ML}(F) \mid F \text{ is an m.v.s.} \}, \\
\mathcal{A}_s &= \{ \mathcal{IL}^=(F) \mid F \text{ is an H.v.s.} \}, \\
\mathcal{A}_s^e &= \{ \mathcal{IL}(F) \mid F \text{ is an H.v.s.} \}.
\end{align*}
\]

An algebraic semantics is a semantics generated by a class of m.v.s (or H.v.s.).

Let us also introduce some particular cases of algebraic semantics.

Definition 4.3.9 For superintuitionistic logics the semantics

\[
\mathcal{A}_s^e = \{ \mathcal{IL}^=(F) \mid F \text{ is a basic H.v.s.} \}.
\]

is called basic general algebraic.

The question, whether every locale is basic, seems open, and so we do not know if \(\mathcal{A}_s^e\) and \(\mathcal{A}_s\) are equivalent.

Neighbourhood frames generate a special kind of modal algebras, so we can define the associated algebraic semantics.

Definition 4.3.10 A neighbourhood frame with equality is a triple \(\Phi = (F, D, E)\) such that \(F\) is a neighbourhood frame and \((\mathcal{M}A(F), D, E)\) is an m.v.s. The latter m.v.s. is denoted by \(\mathcal{MV}(\Phi)\). If \(F\) is a topological space, we call \(\Phi\) a topological frame with equality and define

\[
HV(\Phi) := (HA(F), D, E) (= MV(\Phi)^o).
\]

The corresponding logics are defined in an obvious way:

\[
\begin{align*}
\mathcal{ML}^=(\Phi) &= \mathcal{ML}^e(\mathcal{MV}(\Phi)); \\
\mathcal{IL}^=(\Phi) &= \mathcal{IL}^e(HV(\Phi)).
\end{align*}
\]

Definition 4.3.11 We define general neighbourhood/topological semantics as follows:

\[
\begin{align*}
\mathcal{T}_m &= \{ \mathcal{ML}^=(\Phi) \mid \Phi \text{ is a neighbourhood frame with equality} \}; \\
\mathcal{T}_s &= \{ \mathcal{IL}^=(\Phi) \mid \Phi \text{ is a topological frame with equality} \}.
\end{align*}
\]

From the definitions and since Kripke frames correspond to a special kind of neighbourhood frames, we have:

Lemma 4.3.12

\[
\begin{align*}
\mathcal{KE}_m &= \mathcal{T}_m \leq \mathcal{A}_m; \\
\mathcal{KE}_s &= \mathcal{T}_s \leq \mathcal{A}_s.
\end{align*}
\]
4.4 Morphisms of algebraic structures

In this section we consider maps between algebraic structures preserving validity; they are analogues of p-morphisms used in propositional logic. Let us begin with the intuitionistic case.

**Definition 4.4.1** Let \( F_1 = (\Omega, D_1, E_1) \) and \( F_2 = (\Omega, D_2, E_2) \) be H.v.s. A p-morphism \( \gamma : F_1 \to F_2 \) is a map \( \alpha : D_1 \times D_2 \to \Omega \) (an \( \Omega \)-valued graph) satisfying the following conditions for any \( a \in D_1, b \in D_2 \):

- (E1) \( \alpha(a, b) \land E_1(a) \leq E_2(b) \);
- (E2) \( \alpha(a, b) \land E_2(b) \leq E_1(a) \);
- (Q1) \( E_1(a) \leq \bigvee_{b \in D_2} \alpha(a, b) \) (‘totality’);
- (Q2) \( E_2(b) \leq \bigvee_{a \in D_1} \alpha(a, b) \) (‘surjectivity’).

If instead of (E1), (E2), \( \alpha \) satisfies the conditions

- (I1) \( \alpha(a_1, b_1) \land \alpha(a_2, b_2) \land E_1(a_1, a_2) \leq E_2(b_1, b_2) \) (‘functionality’);
- (I2) \( \alpha(a_1, b_1) \land \alpha(a_2, b_2) \land E_2(b_1, b_2) \leq E_1(a_1, a_2) \) (‘injectivity’);

it is called a p\(^{=}\)-morphism.

A p\(^{=}\)-morphism \( \alpha \) is called a p\(^{=}\)-embedding if for any \( a, a' \in D_1, b, b' \in D_2 \):

- (\( \varepsilon \)) \( E_2(b', b) \land \alpha(a, b) \leq \alpha(a, b') \).

It is obvious that (I1) implies (E1), and (I2) implies (E2), so every p\(^{=}\)-morphism is a p-morphism. It is also clear that every p\(^{=}\)-morphism \( \alpha : F_1 \to F_2 \) gives rise to the converse p\(^{=}\)-morphism \( \alpha^{-1} : F_2 \to F_1 \), where \( \alpha^{-1}(a, b) = \alpha(b, a) \). This is because the conditions (E1), (E2), (Q1), (Q2), (I1), (I2) are ‘symmetrical’.

**Definition 4.4.2** A p\(^{=}\)-morphism \( \alpha \) is called a p\(^{=}\)-equivalence if both \( \alpha, \alpha^{-1} \) are p\(^{=}\)-embeddings, i.e. \( \alpha \) satisfies (\( \varepsilon \)) and

- (\( \varepsilon' \)) \( E_1(a', a) \land \alpha(a, b) \leq \alpha(a', b) \).

The diagram below shows the correlation between different kinds of morphisms:

\[
\begin{array}{ccc}
p^{=} - \text{morphism} & \Rightarrow & p - \text{morphism} \\
\uparrow & & \uparrow \\
p^{=} - \text{embedding} & \Rightarrow & p - \text{embedding} \\
\uparrow & & \uparrow \\
p^{=} - \text{equivalence} & \Rightarrow & p - \text{equivalence} \\
\end{array}
\]

For \( a = (a_1, \ldots, a_n) \in D_1^n, b = (b_1, \ldots, b_n) \in D_2^n \), let \( \alpha(a, b) := \bigwedge_{i=1}^n \alpha(a_i, b_i) \).
Lemma 4.4.3 If \( \alpha : F_1 \rightarrow F_2 \) is a \( p \)-morphism of H.v.s., then for any \( a, c \in D_1^n; b \in D_2^n \),
\[
\alpha(a, b) \land \alpha(c, b) \land E_1(c) \leq E_1(a, c).
\]

Proof In fact, the condition (E1) yields:
\[
\alpha(c_i, b_i) \land E_1(c_i) \leq E_2(b_i).
\]
On the other hand, from (I2) we have:
\[
\alpha(a_i, b_i) \land \alpha(c_i, b_i) \land E_2(b_i) \leq E_1(a_i, c_i).
\]
These two inequalities imply
\[
\alpha(a_i, b_i) \land \alpha(c_i, b_i) \land E_1(c_i) \leq E_1(a_i, c_i),
\]
whence the statement follows easily.

Lemma 4.4.4 Let \( \alpha : F_1 \rightarrow F_2 \) be a \( p(=) \)-morphism, \( x = (x_1, \ldots, x_n), A(x) \in IF(=), FV(A) \subseteq x \), \( a \in D_1^n, b \in D_2^n \). Then \( \alpha(a, b) \land E_1(A(a)) \leq E_2(A(b)) \).

Proof Easy by (E1).

We will use the following abbreviations:

\[
\varphi A := \varphi(A), \quad \psi \sim A := \psi \sim(A).
\]

Definition 4.4.5 Let \( \alpha : F_1 \rightarrow F_2 \) be a \( p \)-morphism of H.v.s. Valuations \( \varphi_1 \) in \( F_1 \) and \( \varphi_2 \) in \( F_2 \) are said to be matching if
\[
\alpha(a, b) \leq \varphi_1 P(a) \leftrightarrow \varphi_2 P(b),
\]
for any \( P \in PL^n, a \in D_1^n, b \in D_2^n \).

The above condition is equivalent to
\[
\alpha(a, b) \land \varphi_1 P(a) = \alpha(a, b) \land \varphi_2 P(b)
\]
and can be replaced with the following two:
\[
\alpha(a, b) \land \varphi_1 P(a) \leq \varphi_2 P(b);
\]
\[
\alpha(a, b) \land \varphi_2 P(b) \leq \varphi_1 P(a).
\]

Lemma 4.4.6 Let \( \alpha : F_1 \rightarrow F_2 \) be a \( p(=) \)-morphism, \( x = (x_1, \ldots, x_n), A(x) \in IF(=), FV(A) \subseteq x \), \( a \in D_1^n, b \in D_2^n \).

Then
\[
\alpha(a, b) \leq \varphi_1 \sim A(a) \leftrightarrow \varphi_2 \sim A(b),
\]
whenever valuations \( \varphi_1 \) in \( F_1 \) and \( \varphi_2 \) in \( F_2 \) are matching.
\section*{4.4. MORPHISMS OF ALGEBRAIC STRUCTURES}

\textbf{Proof}  By induction on $A$ we check the following two properties:

(\*) $\alpha(a, b) \land \varphi_1^- A(a) \leq \varphi_2^- A(b)$;

(\**) $\alpha(a, b) \land \varphi_2^- A(b) \leq \varphi_1^- A(a)$.

- The atomic case is obvious; if $A$ is $(x_1 = x_2)$, apply the conditions (I1), (I2).
- The case $A = B \land C$ is trivial.
- Let $A = B \lor C$. Then

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land E_1(A(a)) \land (\varphi_1^- B(a) \lor \varphi_1^- C(a)) \leq E_2(A(b)) \land (\varphi_2^- B(b) \lor \varphi_2^- C(b)) = \varphi_2^- A(b)$$

by the inductive hypothesis and Lemma 4.4.4; similarly we obtain (\**).

- Let $A = B \supset C$. Then we have:

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land E_1(A(a)) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a))$$

$$\leq \alpha(a, b) \land E_2(A(b)) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a))$$

by Lemma 4.4.4.

Next,

$$\alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \land \varphi_2^- B(b)$$

$$\leq \alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \land \varphi_1^- B(a)$$

by the induction hypothesis for $B$

$$\leq \alpha(a, b) \land \varphi_1^- C(a) \text{ by the property of Heyting algebras}$$

$$\leq \varphi_2^- C(b)$$

by the inductive hypothesis for $C$, so we obtain:

$$\alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \leq \varphi_2^- B(b) \rightarrow \varphi_2^- C(b).$$

Thus

$$\alpha(a, b) \land \varphi_1^- A(a) \leq E_2(A(b)) \land (\varphi_2^- B(b) \rightarrow \varphi_2^- C(b)) = \varphi_2^- A(b).$$

Similarly one can check (\**).

- Let $A(x) = \exists y B(y, x)$, $y \in FV(B)$. Then by Definition 4.2.1, we have:

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land \bigvee_{c \in D_1} \varphi_1^- B(c, a)$$

$$= \bigvee_{c \in D_1} (\alpha(a, b) \land E_1(c) \land \varphi_1^- B(c, a)).$$

Now, since by (Q1),

$$E_1(c) \leq \bigvee_{d \in D_2} \alpha(c, d)$$

\section*{4.4.3. THEOREM}

\textbf{Proof}  By induction on $A$ we check the following two properties:

(\*) $\alpha(a, b) \land \varphi_1^- A(a) \leq \varphi_2^- A(b)$;

(\**) $\alpha(a, b) \land \varphi_2^- A(b) \leq \varphi_1^- A(a)$.

- The atomic case is obvious; if $A$ is $(x_1 = x_2)$, apply the conditions (I1), (I2).
- The case $A = B \land C$ is trivial.
- Let $A = B \lor C$. Then

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land E_1(A(a)) \land (\varphi_1^- B(a) \lor \varphi_1^- C(a)) \leq E_2(A(b)) \land (\varphi_2^- B(b) \lor \varphi_2^- C(b)) = \varphi_2^- A(b)$$

by the inductive hypothesis and Lemma 4.4.4; similarly we obtain (\**).

- Let $A = B \supset C$. Then we have:

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land E_1(A(a)) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a))$$

$$\leq \alpha(a, b) \land E_2(A(b)) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a))$$

by Lemma 4.4.4.

Next,

$$\alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \land \varphi_2^- B(b)$$

$$\leq \alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \land \varphi_1^- B(a)$$

by the induction hypothesis for $B$

$$\leq \alpha(a, b) \land \varphi_1^- C(a) \text{ by the property of Heyting algebras}$$

$$\leq \varphi_2^- C(b)$$

by the inductive hypothesis for $C$, so we obtain:

$$\alpha(a, b) \land (\varphi_1^- B(a) \rightarrow \varphi_1^- C(a)) \leq \varphi_2^- B(b) \rightarrow \varphi_2^- C(b).$$

Thus

$$\alpha(a, b) \land \varphi_1^- A(a) \leq E_2(A(b)) \land (\varphi_2^- B(b) \rightarrow \varphi_2^- C(b)) = \varphi_2^- A(b).$$

Similarly one can check (\**).

- Let $A(x) = \exists y B(y, x)$, $y \in FV(B)$. Then by Definition 4.2.1, we have:

$$\alpha(a, b) \land \varphi_1^- A(a) = \alpha(a, b) \land \bigvee_{c \in D_1} \varphi_1^- B(c, a)$$

$$= \bigvee_{c \in D_1} (\alpha(a, b) \land E_1(c) \land \varphi_1^- B(c, a)).$$

Now, since by (Q1),

$$E_1(c) \leq \bigvee_{d \in D_2} \alpha(c, d)$$
and by the inductive hypothesis,
\[ \alpha(a, b) \land \alpha(c, d) \land \varphi_1^\sim B(c, a) \leq \varphi_2^\sim B(d, b), \]

we obtain:
\[
\bigvee_{c \in D_1} (\alpha(a, b) \land E_1(c) \land \varphi_1^\sim B(c, a)) \leq \bigvee_{d \in D_2} \varphi_2^\sim B(d, b) = \varphi_2^\sim A(b).
\]

So \( A \) satisfies \((*)\). To check \((**)*\), note:

\[
\begin{align*}
\alpha(a, b) \land \varphi_2^\sim A(b) &= \alpha(a, b) \land \bigvee_{d \in D_2} \varphi_2^\sim B(d, b) \\
&= \bigvee_{d \in D_2} (\alpha(a, b) \land E_2(d) \land \varphi_2^\sim B(d, b)) \leq \bigvee_{c \in D_1} \varphi_1^\sim B(c, a) = \varphi_1^\sim A(a),
\end{align*}
\]

by the inductive hypothesis and since \( E_2(d) \leq \bigvee_{c \in D_1} \alpha(c, d) \) by \((Q2)\).

- Let \( A(x) = \exists y B(x, y) \), \( y \not\in \text{FV}(B) \); then

\[
\begin{align*}
\alpha(a, b) \land \varphi_1^\sim A(a) &= \alpha(a, b) \land \bigvee_{c \in D_1} \varphi_1^\sim B(a) \\
&= \alpha(a, b) \land \varphi_1^\sim B(a) \leq \varphi_2^\sim B(b) = \bigvee_{d \in D_2} \varphi_2^\sim B(b) = \varphi_2^\sim B(b).
\end{align*}
\]

- Let \( A(x) = \forall y B(y, x) \). For any \( d \in D_2 \) we have:

\[
\begin{align*}
\alpha(a, b) \land \bigwedge_{c \in D_1} (E_1(c) \rightarrow \varphi_1^\sim B(c, a)) \land E_2(d) \\
&= \alpha(a, b) \land \bigwedge_{c \in D_1} (E_1(c) \rightarrow \varphi_1^\sim B(c, a)) \land E_2(d) \land \bigvee_{c \in D_1} \alpha(c, d) \quad \text{(by \((Q2)\))} \\
&\leq \bigvee_{c \in D_1} (\alpha(a, b) \land E_2(d) \land \alpha(c, d) \land (E_1(c) \rightarrow \varphi_1^\sim B(c, a))) \\
&\quad \text{(by distributivity)} \\
&\leq \bigvee_{c \in D_1} (\alpha(a, b) \land \alpha(c, d) \land E_1(c) \land (E_1(c) \rightarrow \varphi_1^\sim B(c, a))) \\
&\quad \text{(since in Heyting algebras \( x \land (x \rightarrow y) \leq y \))} \\
&= \bigvee_{c \in D_1} (\alpha(ca, db) \land \varphi_1^\sim B(c, a)) \leq \varphi_2^\sim B(d, b) \quad \text{(by IH)}.
\end{align*}
\]

Hence we obtain \((*)\):

\[
\begin{align*}
\alpha(a, b) \land \varphi_1^\sim A(a) &= \alpha(a, b) \land E_1(A(a)) \land \bigwedge_{c \in D_1} (E_1(c) \rightarrow \varphi_1^\sim B(c, a)) \leq \\
&\leq E_2(A(b)) \land \bigwedge_{d \in D_2} (E_2(d) \rightarrow \varphi_2^\sim B(d, b)) = \varphi_2^\sim A(b).
\end{align*}
\]

Similarly, for any \( c \in D_1 \) we have:

\[
\alpha(a, b) \land \bigwedge_{d \in D_2} (E_2(d) \rightarrow \varphi_2^\sim B(d, b)) \land E_1(c) \leq \varphi_1^\sim B(c, a),
\]
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and thus we obtain (**):

\[ \alpha(a, b) \land \varphi_2^\sim A(b) = \alpha(a, b) \land E_2(A(b)) \land \bigwedge_{d \in D_2} (E_2(d) \to \varphi_2^\sim B(d, b)) \leq \]
\[ \leq E_1(A(a)) \land \bigwedge_{c \in D_1} (E_1(c) \to \varphi_1^\sim B(c, a)) = \varphi_1^\sim A(a). \]

Lemma 4.4.7 If \( \alpha : F_1 \longrightarrow F_2 \) is a \( p(=) \)-embedding, then

1. every valuation \( \varphi_1 \) in \( F_1 \) matches with some valuation \( \varphi_2 \) in \( F_1 \);

2. \( \text{IL}^{(=)}(F_2) \subseteq \text{IL}^{(=)}(F_1) \).

Proof

(1) Given \( \varphi_1 \), we define

\[ \varphi_2 P(b) := \bigvee_{c \in D_1^n} (\alpha(c, b) \land \varphi_1 P(c)) \]

for every \( b \in D_2^n \). Then

\[ \alpha(a, b) \land \varphi_2 P(b) = \bigvee_{c \in D_1^n} (\alpha(a, b) \land \alpha(c, b) \land \varphi_1 P(c)) \]
\[ = \bigvee_{c \in D_1^n} (\alpha(a, b) \land \alpha(c, b) \land E_1(c) \land \varphi_1 P(c)) \]
\[ (\text{since } \varphi_1 P(c) \leq E_1(c) \text{ for the valuation } \varphi_1) \]
\[ \leq \bigvee_{c \in D_1^n} (E_1(a, c) \land \varphi_1 P(c)) \leq \varphi_1 P(a), \]

by Lemma 4.4.3 and Definition 4.2.1.

This \( \varphi_2 \) is a valuation in \( F_2 \). Indeed,

\[ E_2(d, b) \land \varphi_2 P(b) = \bigvee_{c \in D_1^n} (\alpha(c, b) \land E_2(d, b) \land E_1(c) \land \varphi_1 P(c)) \leq \]
\[ \bigvee_{c \in D_1^n} (\alpha(c, d) \land \varphi_1 P(c)) = \varphi_2 P(d), \]

by (e). Also \( \varphi_2 P(b) \leq E_2(b) \), since by (E1), \( \alpha(c, b) \land E_1(c) \leq E_2(b) \) for any \( c \in D_1^n \).

(2) Let us show that \( A \not\in \text{IL}^{(=)}(F_1) \) implies \( A \not\in \text{IL}^{(=)}(F_2) \). We may assume that \( A \) is a sentence. Let \( \varphi_1 \) be a valuation in \( F_1 \) such that \( \varphi_1^\sim A \neq 1 \), and let \( \varphi_2 \) be a valuation in \( F_2 \) matching with \( \varphi_1 \). By Lemma 4.4.6 (for \( n = 0 \)), \( \varphi_1^\sim A \leftrightarrow \varphi_2^\sim A = 1 \), and thus \( \varphi_2^\sim A \neq 1 \).

Corollary 4.4.8 If \( \alpha : F_1 \longrightarrow F_2 \) is a \( p(=) \)-equivalence, then \( \text{IL}^{(=)}(F_1) = \text{IL}^{(=)}(F_2) \).
Definition 4.4.9 Let $F_1 = (\Omega, D_1, E_1)$ and $F_2 = (\Omega, D_2, E_2)$ be two H.v.s. A map $g : D_1 \rightarrow D_2$ is called a strong $p$-morphism if for any $a \in D_1$, $b \in D_2$

1. $E_1(a) = E_2(g(a))$;
2. $E_2(b) \leq \bigvee_{a \in D_1} E_2(b, g(a))$.

A strong $p^\approx$-morphism is a map $g$ satisfying (ii) and

3. $E_1(a_1, a_2) = E_2(g(a_1), g(a_2))$.

for any $a_1, a_2 \in D_1$. An isomorphism is a bijection $g : D_1 \rightarrow D_2$ satisfying (iii).

Note that (ii) obviously holds if $g$ is surjective and that (iii) implies (i). So a surjective $g$ is a strong $p^\approx$-morphism iff (iii) holds. It is also obvious that every isomorphism is a strong $p^\approx$-morphism and that every strong $p^\approx$-morphism is a strong $p$-morphism.

Lemma 4.4.10 For H.v.s. $F_1 = (\Omega, D_1, E_1)$, $F_2 = (\Omega, D_2, E_2)$ and a map $g : D_1 \rightarrow D_2$, consider $\alpha : D_1 \times D_2 \rightarrow \Omega_2$ such that

$\alpha(a, b) := E_2(b, g(a))$

for $a \in D_1, b \in D_2$. Then

- $g$ is a strong $p$-morphism iff $\alpha$ is a $p$-morphism;
- $g$ is a strong $p^\approx$-morphism iff $\alpha$ is a $p^\approx$-morphism (or a $p$-embedding, $p$-equivalence, $p^\approx$-embedding, $p^\approx$-equivalence).

Proof First note that $\alpha$ satisfies

$$(E1) \quad \alpha(a, b) \land E_1(a) \leq E_2(b).$$

In fact, this means

$$E_2(b, g(a)) \land E_1(a) \leq E_2(b),$$

which obviously holds, by Lemma 4.1.4(1).

Let us show that $(E2)$ for $\alpha$ and (i) for $g$ are equivalent. The condition

$$(E2) \quad \alpha(a, b) \land E_2(b) \leq E_1(a)$$

is equivalent to

$$E_2(b, g(a)) \leq E_1(a);$$

this follows from (i) and implies $E_2(g(a)) \leq E_1(a)$ if we take $b = g(a)$.

The condition

$$(Q1) \quad E_1(a) \leq \bigvee_{b \in D_2} \alpha(a, b)$$

implies $E_1(a) \leq E_2(b, g(a))$ for all $b \in D_2$ and

$$E_1(a) \leq E_2(g(a)).$$
is equivalent to
\[ E_1(a) \leq \bigvee_{b \in D_2} E_2(b, g(a)); \]
this follows from (i), since \( E_2(g(a)) = E_2(g(a), g(a)) \leq \bigvee_{b \in D_2} E_2(b, g(a)). \) On the other hand, this condition implies \( E_1(a) \leq E_2(g(a)), \) since \( E_2(b, g(a)) \leq E_2(g(a)) \) for any \( b \in D_2. \)

Next, the condition
\[(Q2) \quad E_2(b) \leq \bigvee_{a \in D_1} \alpha(a, b) \]
is equivalent to
\[ E_2(b) \leq \bigvee_{a \in D_1} E_2(b, g(a)), \]
which is the same as the condition (ii).

Therefore \( g \) is a strong p-morphism iff \( \alpha \) is a p-morphism.

Next, let us show that \( g \) satisfies (iii) if \( \alpha \) satisfies (I1) and (I2). In fact, the condition
\[(I1) \quad \alpha(a_1, b_1) \land \alpha(a_2, b_2) \land E_1(a_1, a_2) \leq E_2(b_1, b_2) \]
means
\[ E_2(b_1, g(a_1)) \land E_2(b_2, g(a_2)) \land E_1(a_1, a_2) \leq E_2(b_1, b_2), \]
and
\[(I2) \quad \alpha(a_1, b_1) \land \alpha(a_2, b_2) \land E_2(b_1, b_2) \leq E_1(a_1, a_2) \]
means
\[ E_2(b_1, g(a_1)) \land E_2(b_2, g(a_2)) \land E_2(b_1, b_2) \leq E_1(a_1, a_2). \]
These two conditions follow from (iii) and taken together, imply (iii) if we put \( b_1 = g(a_1), \quad b_2 = g(a_2). \)

The condition
\[(\varepsilon) \quad E_2(b', b) \land \alpha(a, b) \land E_1(a) \leq \alpha(a, b') \]
means
\[ E_2(b', b) \land E_2(b, g(a)) \land E_1(a) \leq E_2(b', g(a)) \]
and holds by Definition 4.1.2.

The condition
\[(\varepsilon') \quad E_1(a', a) \land \alpha(a, b) \land E_2(b) \leq \alpha(a', b) \]
means
\[ E_1(a', a) \land E_2(b, g(a)) \land E_2(b) \leq E_2(b, g(a')) \]
and thus follows from (iii).

This proves the second part of the Lemma.
Remark 4.4.11 One can consider a more natural, but slightly more restrictive
definition of \( p^{(=)} \)-morphism. Viz., the conditions (E1), (E2) can be replaced
with a stronger one:

\[
(E') \quad \alpha(a, b) \leq E_2(b) \land E_1(a).
\]

In this case the conjunct \( E_1(a) \) in (\( \epsilon \)), etc. should be omitted. Note that a strong
\( p^{(=)} \)-morphism \( g \) gives rise to the \( p^{(=)} \)-morphism \( \alpha \) in this stronger sense.

Problem 4.4.12 Does the existence of a \( p^{(=)} \)-morphism (\( p^{(=)} \)-embedding in
our form) imply the existence of a \( p^{(=)} \)-embedding satisfying (\( E' \))?

Now let us consider the modal case. It is clear that if \( F = (\Omega, D_1, E) \) is an
m.v.s., then its Boolean part \( F^b = (\Omega^b, D_1, E) \) is an H.v.s. So we can give

Definition 4.4.13 A \( p^{(=)} \)-morphism (respectively, \( p^{(=)} \)-embedding, \( p^{(=)} \)-equivalence) \( \alpha : F_1 \rightarrow F_2 \) from a \( \mu \)-m.v.s. \( F_1 = (\Omega, D_1, E_1) \) to an m.v.s. \( F_2 = (\Omega, D_2, E_2) \) is a \( p^{(=)} \)-morphism (respectively, \( p^{(=)} \)-embedding, \( p^{(=)} \)-equivalence)
of the Boolean parts \( F_1^b \rightarrow F_2^b \) satisfying the condition

\[
(E0) \quad \alpha(a, b) \leq \square_i \alpha(a, b)
\]

for all \( i \in I_\mu \).

Thus a \( p \)-morphism is a map \( \alpha : D_1 \times D_2 \rightarrow \Omega \) satisfying (E0) and the
conditions (for any \( a \in D_1, b \in D_2 \)):

\[
(E1) \quad \alpha(a, b) \cap E_1(a) \leq E_2(b),
\]

and so on.

The following is rather trivial.

Lemma 4.4.14 If \( F_1, F_2 \) are m.v.s. over the same \( S4 \)-algebra, \( F_1^b, F_2^b \) are
the corresponding H.v.s., \( \alpha : F_1 \rightarrow F_2 \) is a \( p^{(=)} \)-morphism (respectively, \( p^{(=)} \)-
embedding, \( p^{(=)} \)-equivalence), then the same map \( \alpha \) is a \( p^{(=)} \)-morphism (respec-
tively, \( p^{(=)} \)-embedding, \( p^{(=)} \)-equivalence) \( F_1^b \rightarrow F_2^b \) of the corresponding H.v.s.

The definition of matching valuations in the modal case is the same as Def-
nition 4.4.5:

Definition 4.4.15 Let \( \alpha : F_1 \rightarrow F_2 \) be a \( p \)-morphism of m.v.s. Valuations \( \varphi_1 \)
in \( F_1 \) and \( \varphi_2 \) in \( F_2 \) are matching if

\[
\alpha(a, b) \leq (\varphi_1 P(a) \equiv \varphi_2 P(b)),
\]

for any \( P \in PL^n, a \in D_1^n, b \in D_2^n \).

Lemma 4.4.16 Let \( \alpha : F_1 \rightarrow F_2 \) be a \( p^{(=)} \)-morphism, \( A(x) \in MF^{(=)},
FV(A) \subseteq x = (x_1, \ldots, x_n), a \in D_1^n, b \in D_2^n \). Then

\[
\alpha(a, b) \leq (\varphi_1 A(a) \equiv \varphi_2 A(b))
\]

whenever valuations \( \varphi_1 \) in \( F_1 \) and \( \varphi_2 \) in \( F_2 \) are matching.
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Proof By induction on $A$.

The atomic case and the cases of $\land$, $\lor$, $\subset$, $\exists$, $\forall$ were considered in the proof of Lemma 4.4.6.

Consider the case $A = \Box_i B$. By induction hypothesis, we have:

$$\alpha(a, b) \cap \varphi_1 B(a) \leq \varphi_2 B(b),$$

and thus

$$\Box_i \alpha(a, b) \cap \Box_i \varphi_1 B(a) \leq \Box_i (\alpha(a, b) \cap \varphi_1 B(a)) \leq \Box_i \varphi_2 B(b).$$

Hence, by (E0),

$$(1) \quad \alpha(a, b) \cap \Box_i \varphi_1 B(a) \leq \Box_i \varphi_2 B(b).$$

By Lemma 4.4.4 (which obviously holds in the modal case),

$$(2) \quad \alpha(a, b) \cap E_1(A(a)) \leq E_2(A(b)).$$

So

$$\alpha(a, b) \cap \varphi_1 A(a) = \alpha(a, b) \cap E_1(A(a)) \cap \Box_i \varphi_1 B(a) \leq E_2(A(b)) \cap \Box_i \varphi_2 B(b)$$

by (1) and (2)

$$= \varphi_2 A(b).$$

Let us also define some auxiliary kinds of morphism.

Definition 4.4.17 Let $F_i = (\Omega, D_i, E_i)$, $i = 1, 2$, be two H.v.s (respectively, m.v.s) over the same algebra $\Omega$. A regular morphism $\gamma : F_1 \to F_2$ is a mapping $\gamma : D_1 \to D_2$ such that for any $a, b \in D_1$

$$(1) \quad E_1(a, b) \leq E_2(\gamma(a), \gamma(b)),$$

$$(2) \quad E_1(a, a) = E_2(\gamma(a), \gamma(a)).$$

$\Omega$-Hvs (respectively, $\Omega$-mvs) denotes the category of H.v.s. (respectively, m.v.s.) over $\Omega$ and regular morphisms.

In particular, every strong $p^m$-morphism in the sense of Definition 4.4.9 is a regular morphism.

Remark 4.4.18 For H.v.s. there also exists another kind of morphisms, which we call weak morphisms [Goldblatt, 1984; Borceaux, 1994]. A weak morphism is defined as a mapping $\alpha : D_1 \times D_2 \to \Omega$ with the following properties:

- $E_2(b, b') \land \alpha(a, b') \leq \alpha(a, b)$;

- $\alpha(a, b) \land E_1(a, a') \leq \alpha(a', b)$;
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- $\alpha(a, b) \land \alpha(a, b') \leq E_2(b, b')$;
- $E_1(a, a) = \bigvee \{\alpha(a, b) \mid b \in D_2\}$.

It is clear that every regular morphism $\gamma$ corresponds to a weak morphism $\alpha$, such that

$$\alpha(a, b) = E(b, \gamma(a)).$$

Moreover, the category of $\Omega$-sets and weak morphisms is equivalent to its full subcategory of complete $\Omega$-sets, and in the latter subcategory all weak morphisms correspond to regular morphisms [Borceaux, 1994]. However, weak morphisms are not logically faithful (see the discussion below). Thus the above equivalence is not sufficient for showing the equivalence between the semantics of H.v.s. and the semantics of complete H.v.s.

4.5 Presheaves and $\Omega$-sets

In this section we consider an equivalent representation of algebraic semantics using presheaves. The connection between presheaves and $\Omega$-sets is well known in topos theory. Every presheaf corresponds to an $\Omega$-set in a standard way [Fourman and Scott, 1979]; this construction can be used to define logics of presheaves. On the other hand, for the case of sheaves there exist a connection in the opposite direction; Higgs’s theorem [Higgs, 1984], also cf. [Borceaux, 1994; Goldblatt, 1984; Makkai and Reyes, 1977; Fourman and Scott, 1979], states the equivalence between the category of sheaves and sheaf morphisms and some full subcategory of $\Omega$-sets (so called ‘complete’ $\Omega$-sets). It also turns out that the latter subcategory is ‘representative’, i.e. equivalent to the whole category of $\Omega$-sets [Goldblatt, 1984]. Unfortunately, this result does not help for our purpose, because the $\Omega$-set isomorphisms used in this theorem are not logically faithful. So in this section we describe another construction showing that the logic of any m.v.s. can be presented as the logic of some presheaf.

First, let us recall (Chapter 3) that every poset $(F, R)$ corresponds to a category $\text{Cat}(F, R)$, in which $(u, v)$ is the unique morphism from $u$ to $v$ whenever $uRv$. For every locale $\Omega$ let $\text{Cat}(\Omega) := \text{Cat}(\Omega, \geq)$.

**Definition 4.5.1** A presheaf (of sets) over a locale $\Omega$ is defined as a functor

$$\mathcal{F} : \text{Cat}(\Omega) \to \text{SET}.$$ 

This means that we have a family of sets $(\mathcal{F}(u) : \Omega)$ and a family of functions (restriction maps)

$$\mathcal{F}(u, v) : \mathcal{F}(u) \to \mathcal{F}(v),$$

where $u, v \in \Omega$, $u \geq v$, and the following conditions hold:

1. $\mathcal{F}(u, u) = \text{id}_{\mathcal{F}(u)}$ (the identity function);
A presheaf $\mathcal{F}$ is called inhabited if $\sqrt{\{u \mid \mathcal{F}(u) \neq \emptyset\}} = 1$.

The set $\mathcal{F}(u)$ is called the (domain) of $\mathcal{F}$ at $u$; its elements are called individuals (or sections) over $u$. The set of all the individuals

$$\mathcal{F}^* = \bigcup\{\mathcal{F}(u) \mid u \in \Omega\}$$

is called the total domain of $\mathcal{F}$.

**Definition 4.5.2** A morphism $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ of presheaves over $\Omega$ is a functor morphism (natural transformation) i.e. a family of mappings

$$f_u : \mathcal{F}_1(u) \rightarrow \mathcal{F}_2(u),$$

such that the following diagram commutes (provided $u \geq v$).

![Diagram](attachment:diagram.png)

An isomorphism is an invertible morphism, as usual; this is equivalent to bijectivity of every $f_u$.

Presheaves over $\Omega$ and morphisms constitute a category $\text{Psh}(\Omega)$.

**Definition 4.5.3** A presheaf $\mathcal{F}$ over $\Omega$ is disjoint if

$$\forall u, v \in \Omega \ (u \neq v \Rightarrow \mathcal{F}(u) \cap \mathcal{F}(v) = \emptyset).$$

**Lemma 4.5.4** Every presheaf is isomorphic to some disjoint presheaf.
Proof Almost obvious. Given a presheaf \( \mathcal{F} \), let

\[
\mathcal{F}'(u) := \mathcal{F}(u) \times \{u\},
\]

\[
\mathcal{F}'(u,v)(a,u) := (\mathcal{F}(u,v)(a),v).
\]

Then \( \mathcal{F}' \) is a disjoint presheaf, and there exists an isomorphism \( f : \mathcal{F} \rightarrow \mathcal{F}' \) such that

\[
f_u(a) = (a, u).
\]

In fact, \( f \) is obviously bijective, and the corresponding diagram commutes, since

\[
\mathcal{F}'(u,v) \circ f_u : a \mapsto (a, u) \mapsto (\mathcal{F}(u,v)(a),v)
\]

and also

\[
f_v \circ \mathcal{F}(u,v) : a \mapsto (\mathcal{F}(u,v)(a),v).
\]

Thanks to the above lemma, we can assume that all presheaves are disjoint.

In a disjoint presheaf \( \mathcal{F} \) every individual belongs to a unique \( \mathcal{F}(u) \); the corresponding \( u \) is called the extent of \( a \) and denoted by \( |a| \).

Definition 4.5.5 If \( v \leq |a| \), the restriction of \( a \) to \( v \) is

\[
a|v := \mathcal{F}(|a|, v)(a).
\]

Definition 4.5.1 for disjoint presheaves can be reformulated as follows.

Lemma 4.5.6 Let \( \mathcal{F} \) be a disjoint presheaf, \( a \in \mathcal{F}^* \), \( w \leq v \leq |a| \). Then

1. \( a | (|a|) = a; \)
2. \( (a|v)|w = a|w. \)

The following simple lemma yields an equivalent definition of a morphism of disjoint presheaves as a map of domains preserving extents and commuting with restrictions:

Lemma 4.5.7 Let \( f : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) be a morphism of disjoint presheaves. Consider a map \( g : \mathcal{F}_1^* \rightarrow \mathcal{F}_2^* \) such that

\[
g(a) := f|_a|(a).
\]

Then \( g \) has the following properties.

1. \( |g(a)| = |a|; \)
2. \( g(a|v) = g(a) | v. \)

The other way round, given a map \( g \) satisfying (1), (2), we can define a morphism \( f : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) by putting

\[
f|_a|(a) := g(a).
\]
4.5. PRESHEAVES AND Ω-SETS

Proof Straightforward.

Proposition 4.5.8 Let \( \mathcal{F} \) be an inhabited disjoint presheaf over a locale \( \Omega \), and for \( a, b \in \mathcal{F}^* \) let

\[
X(a, b) = \{ u \in \Omega \mid u \leq |a| \wedge |b|, \ a|u = b|u \},
\]

\[
E(a, b) = \bigvee X(a, b).
\]

Then the triple

\[
HV(\mathcal{F}) = (\Omega, \mathcal{F}^*, E)
\]

is an H.v.s.

Proof The equality \( E(a, b) = E(b, a) \) is trivial, since \( X(a, b) = X(b, a) \).

\[
\bigvee \{E(a, a) \mid a \in \mathcal{F}^*\} = 1
\]

holds, because \( E(a, a) = |a| \) and \( \mathcal{F} \) is inhabited.

Since by Lemma 4.5.7,

\[
w \leq v \leq |a| \Rightarrow (a \mid v) \mid w = a \mid w,
\]

\( X(a, b) \) is \( \geq \)-stable, that is

\[
v \in X(a, b) \& v \geq w \Rightarrow w \in X(a, b).
\]

So for any \( u \in X(a, b), v \in X(b, c) \) we have

\[
(u \wedge v) \in X(a, b) \cap X(b, c) \subseteq X(a, c).
\]

Hence it follows that

\[
E(a, b) \wedge E(b, c) = \bigvee \{u \wedge v \mid u \in X(a, b), v \in X(b, c)\} \leq \bigvee X(a, c) = E(a, c).
\]

Obviously, if the locale \( \Omega \) is basic, isomorphic to \( \Xi^0 \), for some \( \text{S}_4 \)-algebra \( \Xi \), and \( E \) is the same as in Proposition 4.5.8, then

\[
MV(\mathcal{F}) = (\Xi, \mathcal{F}^*, E)
\]

is an \( \text{S}_4 \)-m.v.s.

Lemma 4.5.9 If \( \mathcal{F}_1, \mathcal{F}_2 \) are isomorphic disjoint presheaves over \( \Omega \), then the corresponding H.v.s. \( HV(\mathcal{F}_1), HV(\mathcal{F}_2) \) are isomorphic; similarly for m.v.s. (and basic \( \Omega \)).

Proof Let \( X_1, X_2 \) be the functions corresponding to \( \mathcal{F}_1, \mathcal{F}_2 \) described in Proposition 4.5.8. If \( g : \mathcal{F}_1^* \longrightarrow \mathcal{F}_2^* \) is a bijective map satisfying Lemma 4.5.7 (i), (ii), then one can easily check that \( X_1(a, b) = X_2(g(a), g(b)) \). Thus \( E_1(a, b) = E_2(g(a), g(b)) \), which means that \( g \) is an isomorphism of H.v.s.
**Definition 4.5.10** If $\mathcal{F}$ is an inhabited presheaf over a locale $\Omega$, we define its superintuitionistic logic (with or without equality) via the corresponding H.v.s.:

$$\text{IL}^{(=)}(\mathcal{F}) = \text{IL}^{(=)}(HV(\mathcal{F}_1)),$$

where $\mathcal{F}_1$ is a disjoint presheaf isomorphic to $\mathcal{F}$.

If $\Omega$ is basic, we also define the modal logic

$$\text{ML}^{(=)}(\mathcal{F}) = \text{ML}^{(=)}(MV(\mathcal{F}_1)).$$

These logics are well-defined. In fact, by Lemma 4.5.9, an isomorphism of disjoint presheaves $\mathcal{F}_1, \mathcal{F}_2$ gives rise to an isomorphism of the associated H.v.s. (m.v.s.), and thus the corresponding logics are equal, by Corollary 4.4.8.

**Proposition 4.5.11**

1. For any H.v.s. $G$ over a Heyting algebra $\Omega$ there exists a disjoint presheaf $\mathcal{G}$ over $\Omega$ such that $\text{IL}^{(=)}(\mathcal{G}) = \text{IL}^{(=)}(G)$.

2. For any m.v.s. $G$ over an $\mathbf{S}_4$-algebra $\Omega$ there exists a disjoint presheaf $\mathcal{G}$ over $\Omega^0$ such that $\text{ML}^{(=)}(\mathcal{G}) = \text{ML}^{(=)}(G)$.

**Proof** We prove only (1), leaving (2) for the reader.

For every $u \in \Omega$ let

$$\mathcal{G}_0(u) = \{(u, a) \mid a \in D, u \leq E a a\},$$

and let $\approx_u$ be the following equivalence relation in $\mathcal{G}_0(u)$:

$$(u, a) \approx_u (u, b) \iff u \leq E a b$$

We denote the equivalence class $(u, a)/\approx_u$ by $[u, a]$.

Now we define the presheaf $\mathcal{G}$ as follows:

(*) $\mathcal{G}(u) := \mathcal{G}_0(u)/\approx_u$;

(**) $\mathcal{G}(u, v)([u, a]) := [v, a]$ (provided $v \leq u$).

The map $\mathcal{G}(u, v)$ is well-defined, since $[u, a] = [u, b]$ means that $u \leq E a b$, which implies $v \leq E a b$, i.e. $[v, a] = [v, b]$.

Now let us show that $\text{IL}^{(=)}(\mathcal{G}) = \text{IL}^{(=)}(G)$.

It is clear that (*), (**), really define a presheaf, and we may assume that $\mathcal{G}$ is disjoint. $\mathcal{G}$ is inhabited, since

$$\{u \mid \mathcal{G}(u) \neq \emptyset\} = \bigvee\{u \mid \mathcal{G}_0(u) \neq \emptyset\} = \bigvee\{u \mid \exists a \in D u \leq E a a\} = \bigvee\{E a a \mid a \in D\} = 1.$$

Recall that $\text{IL}^{(=)}(\mathcal{G}) = \text{IL}^{(=)}(HV(\mathcal{G}))$, where $HV(\mathcal{G}) = (\Omega, \mathcal{G}^*, E^*)$ is the H.v.s., in which

$\mathcal{G}^* = \bigcup_{u \in \Omega} \mathcal{G}(u) = \{[u, a] \mid a \in D, u \leq E a a\},$

$E^*([u, a], [v, b]) = \bigvee\{w \mid w \leq u \land v, \ [w, a] = [w, b]\}$

$= \bigvee\{w \mid w \leq u \land v, w \leq E a b\} = u \land v \land E a b.$
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In particular, we have:

\[ E^*[u, a] = u \land E(a) = u. \]

Consider the map \( \alpha : G^* \times D \rightarrow \Omega \) such that

\[ \alpha([u, a], b) := u \land Eab. \]

This map is well-defined, since if \([u, a] = [u, a']\), i.e. if \( u \leq Eaa'\), then \( u \land Eab = u \land Ea'b \). Let us show that

\[ \alpha : G \rightarrow HV(G) \]

is a \( p^=\)-equivalence (Definitions 4.4.1, 4.4.2).

**(Q1)** \( E(b) \leq \bigvee_{c \in G^*} \alpha(c, b) \), since \( E(b) = \alpha([E(b), b], b) \).

**(Q2)** \( EE^*[u, a] \leq \bigvee_{b \in D} \alpha([u, a], b) \), since \( E^*[u, a] = u = u \land Eaa = \alpha([u, a], a) \).

(I) \( \alpha([u_1, a_1], b_1) \land \alpha([u_2, a_2], b_2) \land Eb_1b_2 = u_1 \land Ea_1b_2 \land u_2 \land Ea_2b_2 \land Eb_1b_2 \leq u_1 \land u_2 \land Eaa \leq E^*[u_1, a_1, [u_2, a_2]]. \)

(ii) \( \alpha([u_1, a_1], b_1) \land \alpha([u_2, a_2], b_2) \land E^*[u_1, a_1, [u_2, a_2]] = u_1 \land Ea_1b_1 \land u_2 \land Ea_2b_2 \land u_1 \land u_2 \land Eaa \leq Eb_1b_2. \)

(\varepsilon) \( E^*[u', a'], [u, a]] \land \alpha([u, a], b) = u' \land u \land Eaa \land u \land Eab \leq u' \land Eab = \alpha([u', a'], b). \)

(\varepsilon') \( Ebb \land \alpha([u, a], b) = Ebb \land u \land Eab \leq u \land Eab = \alpha([u, a], b'). \)

Therefore \( IL^= (G) = IL^= (HV(G)) \), by Corollary 4.4.8.

**Corollary 4.5.12** Every algebraic semantics can be obtained from a class of presheaves. In particular, the general algebraic semantics \( \mathcal{A}\mathcal{E} \) is generated by the class of all presheaves.

### 4.6 Morphisms of presheaves

**Lemma 4.6.1** Let \( \mathcal{F}_1, \mathcal{F}_2 \) be two inhabited disjoint presheaves over a locale \( \Omega \), \( g : \mathcal{F}_1 \rightarrow \mathcal{F}_2 \) a morphism (in the sense of Lemma 4.5.7). Then \( g \) is a strong morphism from \( HV(\mathcal{F}_1) \) to \( HV(\mathcal{F}_2) \) (in the sense of Definition 4.4.17).

**Proof**

Let \( HV(\mathcal{F}_1) = (\Omega, F_1^*, E_1) \). Then

\[ E_1(a, b) = \bigvee \{ u \mid u \leq |a| \land |b|, a = b \land u \}, \]

\[ E_2(g(a), g(b)) = \bigvee \{ u \mid u \leq |g(a)| \land |g(b)|, g(a) = g(b) \land u \} \]
(see Proposition 4.5.8). Since \( g \) is a morphism, we have:

\[
|g(a)| = |a|, \quad g(a)\|u = g(a)\|u, \quad g(b)\|u = g(b)\|u.
\]

So \( u \in X_1(a, b) \) implies

\[
g(a)\|u = g(a)\|u = g(b)\|u = g(b)\|u,
\]

and thus \( u \in X_2(g(a), g(b)) \). Hence \( X_1(a, b) \subseteq X_2(g(a), g(b)) \), and so

\[
E_1(a, b) \leq E_2(g(a), g(b)).
\]

On the other hand,

\[
X_1(a, a) = \{u \mid u \leq |a|\},
\]

and thus

\[
E_1(a, a) = |a|;
\]

similarly,

\[
E_2(g(a), g(a)) = |g(a)|;
\]

hence

\[
E_1(a, a) = E_2(g(a), g(a))
\]

holds, and the proof is completed. \( \Box \)

So we obtain

**Proposition 4.6.2** There exist functors

\[
HV: \text{Psh}(\Omega) \rightarrow \Omega-\text{Hvs}
\]

for any locale \( \Omega \);

\[
MV: \text{Psh}(\Omega) \rightarrow \Omega-\text{mvs}
\]

for any basic locale \( \Omega \),

such that \( HV(\mathcal{F}) \), \( MV(\mathcal{F}) \) are defined as above, and for a strong morphism \( f \),

\[
HV(f) = MV(f) = f.
\]

The converse to Proposition 4.6.2 is in general false.

**Example 4.6.3** Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be inhabited disjoint presheaves shown at Fig. 4.1. Let \( g: \mathcal{F}_1^* \rightarrow \mathcal{F}_2^* \) be the map such that

\[
g(a) = a, \quad g(a\|u) = b, \quad g(a\|x) = a\|x
\]

for \( x = v \) or \( w \). Then \( g \) is not a morphism of presheaves. On the other hand, \( g \) is a strong morphism of the corresponding H.v.s. (Definition 4.4.17), since

\[
E_2(a, b) = u = E_1(a, a\|u).
\]
Proposition 4.6.4 Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two inhabited disjoint presheaves over a locale $\Omega$. Every strong morphism $f : HV(\mathcal{F}_1) \to HV(\mathcal{F}_2)$ (Definition 4.4.17) is a morphism of presheaves if $\mathcal{F}_2$ satisfies the following condition:

$$\forall v \in \Omega \ \forall b, c \in \mathcal{F}_2(v) \ (E_2(b, c) = v \Rightarrow b = c).$$

Moreover, every strong H.v.s.-morphism $f : HV(\mathcal{F}_1) \to HV(\mathcal{F}_2)$ is a morphism of presheaves $\mathcal{F}_1 \to \mathcal{F}_2$ iff for any strong H.v.s.-morphism $g : HV(\mathcal{F}_1) \to HV(\mathcal{F}_2)$ the following holds:

$$(*) \ \forall u > v \ \forall a \in g(\mathcal{F}_1(u)) \forall b \in \mathcal{F}_2(v) \ (E_2(a, b) = v \Rightarrow a|v = b).$$

**Proof** (If.) Let $f$ be a morphism of H.v.s. Let us show that $f(x|v) = f(x)|v$ for $x \in F_1(u), v < u$. Take $a = f(x), b = f(x|v)$. Then $E_2(a, b) \geq E_1(x, x|v) = v$, and thus $E_2(a, b) = v$ and $a|v = b$, by the condition $(*)$ applied to $f$.

(Only if.) Assume that every strong H.v.s.-morphism $HV(\mathcal{F}_1) \to HV(\mathcal{F}_2)$ is a morphism of presheaves $\mathcal{F}_1 \to \mathcal{F}_2$. Suppose there exists an H.v.s.-morphism $g : HV(\mathcal{F}_1) \to HV(\mathcal{F}_2)$ that does not satisfy $(*)$, i.e. for some $u > v, x \in \mathcal{F}_1(u), b \in \mathcal{F}_2(v)$ we have $E_2(a, b) = v, a|v \neq b$ for $a = f(x) \in \mathcal{F}_2(u)$. By our assumption, $g$ is a morphism of presheaves, and thus $g(x|v) = a|v$. Consider the map $f : \mathcal{F}_1 \to \mathcal{F}_2$ defined as follows

$$f(y) := \begin{cases} 
    b & \text{if } y = x|v, \\
    g(y) & \text{otherwise}.
\end{cases}$$

Then $f$ is not a morphism of presheaves, since $f(x|v) = b$, but $f(x)|v = a|v \neq b$. On the other hand, $f$ is a strong morphism of H.v.s.

In fact, let us show that $E_1(y, z) \leq E_2(f(y), f(z))$ for $y, z \in \mathcal{F}_1$. It is sufficient to consider the case $y \neq z = x|v$. Then

$$E_1(y, z) \leq E_2(g(y), a|v) = E_2(g(y), a|v) \land E_2(a|v, b) \leq E_2(g(y), b) = E_2(f(y), f(z)),$$
since
\[ E_2(g(u), a|v) \leq E_2(a|v, a|v) = v = E_2(a, b) \land E_2(a|v, a) \leq E_2(a|v, b). \]

**Definition 4.6.5** Let \( \mathcal{F}_1, \mathcal{F}_2 \) be (inhabited) disjoint presheaves over a locale \( \Omega \). A pre-morphism from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) is a map \( g : \mathcal{F}_1^* \to \mathcal{F}_2^* \) preserving extents, i.e. such that
\[
(1) \quad |g(a)| = |a| \text{ for any } a.
\]
A pre-morphism \( g \) is called a p-morphism if it satisfies
\[
(3) \quad \text{for any } u \in \Omega, \ b \in \mathcal{F}_2(u)
\]
\[ u = \bigvee \{v \leq u \mid \exists a \in \mathcal{F}_1^* (v \leq |g(a)| \land b|v = g(a)|v)\} \]
A p\(^{-}\)-morphism is a p-morphism satisfying
\[
(4) \quad \text{if } a_1 \in \mathcal{F}_1(u_1), \ a_2 \in \mathcal{F}_1(u_2), \text{ then for any } w
\]
\[ w \leq u_1 \land u_2 \land g(a_1)|w = g(a_2)|w \]
\[ \Rightarrow w \leq \bigvee \{u \mid u \leq u_1 \land u_2 \land a_1|u = a_2|u\}; \]
\[
(5) \quad \text{if } a \in \mathcal{F}_1(u), \ v \leq u, \text{ then}
\]
\[ v = \bigvee \{w \leq v \mid g(a)|w = g(a|v)|w\}. \]
A p-embedding is a p-morphism satisfying (4).

Note that
\[
(2) \quad g(a|u) = g(a)|u
\]
implies (5); (3) holds for every surjective pre-morphism; (4) holds for every injective morphism.

The definition of a p\(^{=}\)-morphism of presheaves is rather complicated, but it exactly corresponds to the definition of a p-morphism of H.v.s. (Definition 4.4.9). More precisely, the following holds.

**Lemma 4.6.6**

(1) A map \( g : \mathcal{F}_1^* \to \mathcal{F}_2^* \) is a p\(^{=}\)-morphism of presheaves iff \( g \) is a strong p\(^{=}\)-morphism of corresponding H.v.s. \( g : HV(\mathcal{F}_1) \to HV(\mathcal{F}_2) \).

(2) If there exists a p-embedding \( g : \mathcal{F}_1 \to \mathcal{F}_2 \) (respectively, p\(^{-}\)-morphism) of disjoint presheaves, then \( IL(\mathcal{F}_1) \subseteq IL(\mathcal{F}_2) \) (respectively, \( IL^=(\mathcal{F}_1) \subseteq IL^=(\mathcal{F}_2) \)).

(Cf. Lemma 4.4.7 stating the converse inclusion: \( IL^=(\mathcal{F}_2) \subseteq IL^=(\mathcal{F}_1).\))

**Proof** Let us check items (i)–(iii) from Definition 4.4.9.

(i) \[ E_1(a) = E_2(g(a)) \iff |a| = |g(a)| \iff (1). \]
4.6. MORPHISMS OF PRESHEAVES

(ii) \[ E_2(b) \leq \bigvee_{a \in D_1} E_2(b, g(a)) \]
\[ \iff |b| \leq \bigvee_{a \in D_1} \big\{v \mid v \leq |b| \land |g(a)|, \ b|v = g(a)|v\big\} \iff (3). \]

(iii) \[ E_2(g(a_1), g(a_2)) \leq E_1(a_1, a_2) \iff \]
\[ \iff \bigvee \{v \mid v \leq |g(a_1)| \land |g(a_2)|, \ g(a_1)|v = g(a_2)|v\} \leq \]
\[ \leq \bigvee \{u \mid u \leq |a_1| \land |a_2|, a_1|u = a_2|u\} \]
\[ \iff \forall v \leq |g(a_1)| \land |g(a_2)| \land g(a_1)|v = g(a_2)|v \iff \]
\[ \Rightarrow v \leq \bigvee \{u \mid u \leq |a_1| \land |a_2|, a_1|u = a_2|u\} \iff (4). \]
\[ E_1(a_1, a_2) \leq E_2(g(a_1), g(a_2)) \iff \]
\[ \iff \bigvee \{u \mid u \leq |a_1| \land |a_2|, a_1|u = a_2|u\} \leq \]
\[ \leq \bigvee \{v \mid v \leq |g(a_1)| \land |g(a_2)|, g(a_1)|v = g(a_2)|v\} \iff \]

(♯) \[ \forall u \ (u \leq |a_1| \land |a_2| \land a_1|u = a_2|u \Rightarrow \]
\[ \Rightarrow u \leq \bigvee \{v \mid v \leq |g(a_1)| \land |g(a_2)|, g(a_1)|v = g(a_2)|v\}. \]

This condition (♯) implies (5). In fact, let \( a \in \mathcal{F}_1(u), \ v \leq u \). Take \( a_1 = a, \ a_2 = a|v\); then \( a_1|v = a_2|v \). Thus
\[ v \leq \bigvee \{w \mid w \leq v, \ g(a)|w = g(a)|w\} \]
(recall that \( |g(a)| = u \geq v = |g(a)|v \), by (1)).

On the other hand, (5) implies (♯). Indeed, assume that \( a_1 \in \mathcal{F}_1(u_1), a_2 \in \mathcal{F}_2(u_2), \ u \leq u_1 \land u_2, \ a_1|u = a_2|u \). Take the sets
\[ X_i := \{w \mid w \leq u, \ g(a_i)|w = g(a_i)|u\} \]
for \( i = 1, 2 \). Then by (5),
\[ u = \left( \bigvee X_2 \right) \land \left( \bigvee X_2 \right) = \bigvee \{w_1 \land w_2 \mid w_1 \in X_1, \ w_2 \in X_2\} \]
(recall that \( \Omega \) is a complete Heyting algebra). And if \( v = w_1 \land w_2, \ w_i \in X_i \),
then
\[ v \leq u \leq u_1 \land u_2 = |g(a_1)| \land |g(a_2)|, \]
\[ (a_1)|v = (g(a_1)|w_1)|v = (g(a_1)|u|w_1)|v = g(a_1)|u|v = g(a_2)|u|v \]
\[ = (g(a_2)|u|w_2)|v = (g(a_2)|u|w_2)|v = g(a_2)|v. \]

Note that a \( p^{(\neq)} \)-morphism is not necessarily a morphism (because the condition (2) does not always hold). On the other hand, (1) and (2) implies (5); in fact, \( g(a)|v = g(a)|v = g(a)|v \).

Also note that every surjective pre-morphism is a \( p \)-morphism: the condition (3) holds if \( b = g(a) \) for some \( a \in \mathcal{F}_1^* \). Every isomorphism (i.e. a bijective morphism) is a \( p^{\approx} \)-morphism; in fact, (4) follows from (2) and the bijectivity.
4.7 Sheaves

In this and the next section we look more closely at topological semantics. First let us show that in this case presheaves can be replaced with sheaves.

Recall (cf. [Godement, 1958]) that a sheaf (over a locale $\Omega$) is a presheaf $\mathcal{F}$ satisfying two conditions:

(F1) if $u = \bigvee_{i \in I} u_i$, $a, b \in \mathcal{F}(u)$ and $(a|u_i) = (b|u_i)$ for all $i \in I$, then $a = b$;

(F2): if $u = \bigvee_{i \in I} u_i$, $a_i \in \mathcal{F}(u_i)$ and $(a_i|u_i \wedge u_j) = (a_j|u_i \wedge u_j)$ for all $i, j \in I$, then

$$\exists a \in \mathcal{F}(u) \forall i \in I (a|u_i) = a_i.$$ 

Every disjoint presheaf $\mathcal{F}$ over a topological space $(W, \sqsubset)$ gives rise to the canonical sheaf $\tilde{\mathcal{F}}$ defined as follows.

For $x \in W$ consider the set

$$\mathcal{F}_0(x) := \{ a \in \mathcal{F}^* \mid x \in |a| \}$$

with the equivalence relation

$$(a \equiv_x b) := x \in E(a, b).$$

The equivalence class $a_x := (a/ \equiv_x)$ is called the germ of $a$ at $x$. The set $\mathcal{F}(x) := (\mathcal{F}_0(x)/ \equiv_x)$ of all the germs at $x$ is called the stalk (or the fibre) of $\mathcal{F}$ at $x$. We define $\tilde{\mathcal{F}}(u)$ as the set of maps from $u$ to $\mathcal{F}^*$ sending every $x \in u$ to some germ at $x$. More precisely,

$$\tilde{\mathcal{F}}(u) := \{ f \in \prod_{x \in u} \mathcal{F}(x) \mid \forall x \in u \exists v \subseteq u \exists a \in \mathcal{F}(v) (x \in v \& \forall y \in v f(y) = a_y) \},$$

and for $v \subseteq u$, let

$$\tilde{\mathcal{F}}(u, v)(f) := (f \restriction v),$$

the restriction of the map $f$ from $u$ to $v$.

The following fact is well-known [Godement, 1958]:

**Proposition 4.7.1**

- $\tilde{\mathcal{F}}$ is a sheaf;

- the family of mappings $\alpha_u : \mathcal{F}(u) \to \tilde{\mathcal{F}}(u)$ such that

  $$\alpha_u(a) = f \iff \forall x \in u f(x) = a_x$$

  is a morphism of presheaves;

- $\alpha$ is isomorphic iff $\mathcal{F}$ is a sheaf.
For $a \in F^*$ let us denote 
\[ \tilde{a} := \alpha_{|a|}(a). \]

**Lemma 4.7.2** Let $F$ be a presheaf over a topological space, $\tilde{F}$ its canonical sheaf. Let $HV(F) = (\Omega, F^*, E)$, $HV(\tilde{F}) = (\Omega, \tilde{F}^*, \tilde{E})$.

Then for any $a, b \in F^*$ $E(a, b) = \tilde{E}(\tilde{a}, \tilde{b})$.

**Proof** Let $\Omega$ be the Heyting algebra of the given topological space. Then by Definition 4.3.4,

\[
x \in E(a, b) \iff \exists u \in \Omega \ (x \in u \land u \leq |a| \cap |b| \land a|u = b|u) \\
\iff \exists u \in \Omega \ (x \in u \land u \leq |a| \cap |b| \land \forall z \in u a_z = b_z) \\
\iff \exists u \in \Omega \ (x \in u \land u \leq |a| \cap |b| \land \tilde{a}|u = \tilde{b}|u) \\
\iff x \in \tilde{E}(\tilde{a}, \tilde{b}).
\]

**Lemma 4.7.3** Consider the same $F$, $\tilde{F}$ as in the previous lemma. Let $g : F^* \rightarrow \tilde{F}^*$ be the map such that $g(a) = \tilde{a}$ for $a \in F^*$. Then $g$ is a strong $p^-$-morphism $HV(F) \rightarrow HV(\tilde{F})$.

**Proof** By Lemma 4.7.2 and because $g$ is surjective (see remarks after Definition 4.4.9).

**Proposition 4.7.4** $IL^= (F) = IL^= (\tilde{F}), \ ML^= (F) = ML^= (\tilde{F})$

**Proof** Follows from the previous lemma, by Lemma 4.4.10 and Corollary 4.4.8; recall that in the modal S4-case $p^=$-morphisms are just the same as in the intuitionistic case.

**Corollary 4.7.5** $TE$ is generated by the class of all sheaves over topological spaces.

**Question 4.7.6** Is the semantics $AE$ generated by the class of all sheaves over locales (or complete modal algebras)?

### 4.8 Fibrewise models

In this section we show that validity in a presheaf $F$ over a topological space can be defined via a forcing relation between points and formulas, similarly to Kripke semantics. Instead of Kripke models, we can use ‘fibrewise models’, which are collections of classical models on fibres of $F$. As in Definition 3.2.2, a *system of domains* over a non-empty set $W$ (whose elements are called *possible worlds*, or *points*) is a pair $(W, D)$, where $D$ is a family of non-empty sets $D = (D_u)_{u \in W}$. A *classical valuation* in a non-empty set $V$ is a map $\xi$ from $PL$ to relations on $V$ such that $\xi(P) \subseteq V^\circ$, whenever $P \in PL^*$ ($V^0$ is treated as a certain singleton). A *modal valuation* in a system of domains $(W, D)$ is a family $(\eta_u)_{u \in W}$, where
$\eta_u$ is a classical valuation in $D_u$ such that $\eta_u(p) \subseteq \{u\}$ for every proposition letter $p$. A (modal) model over $(W, D)$ is a triple $(W, D, \eta)$, where $\eta$ is a valuation in $(W, D)$.

**Definition 4.8.1** Let $\mathcal{F}$ be a presheaf over a topological space $(W, \Box)$. We associate with $\mathcal{F}$ the system of domains $\mathcal{F}^+ := (W, D)$ with $D_u = \mathcal{F}(u)$ (the fibre at $u$). A fibrewise model over $\mathcal{F}$ is a model over $\mathcal{F}^+$.

**Definition 4.8.2** For a fibrewise model $M = (W, D, \eta)$ we define the (modal) forcing relation $M, u \Vdash A$ (or briefly: $u \Vdash A$) between $u \in W$ and a $D_u$-sentence $A$:

- $M, u \Vdash P^0_k$ iff $u \in \eta_u(P^0_k)$;
- $M, u \Vdash P^m_k(a)$ iff $a \in \eta_u(P^m_k)$ (for $m > 0$);
- $M, u \Vdash a = b$ iff $a$ equals $b$;
- $M, u \not\Vdash \bot$;
- $M, u \Vdash B \lor C$ iff $M, u \Vdash B$ or $M, u \Vdash C$;
- $M, u \Vdash B \land C$ iff $M, u \Vdash B \& M, u \Vdash C$;
- $M, u \Vdash B \supset C$ iff $M, u \not\Vdash B$ or $M, u \Vdash C$;
- $M, u \Vdash \exists x A$ iff $\exists a \in D_u$ $M, u \Vdash [a/x]A$;
- $M, u \Vdash \forall x A$ iff $\forall a \in D_u$ $M, u \Vdash [a/x]A$;
- $M, u \Vdash \Box A((a_1)_u, \ldots, (a_n)_u)$ iff there exist an open $U$ such that $u \in U$, $U \subseteq |a_1| \cap \ldots \cap |a_n|$ and $\forall v \in U$ $M, v \Vdash A((a_1)_v, \ldots, (a_n)_v)$.

Let us check that forcing is well-defined. This has to be done only for the last item. More precisely, we have to show that if

**Lemma 4.8.3** Let $M = (\mathcal{F}, D, \eta)$ be a fibrewise model. Consider the algebra $\Omega = MA(W, \Box)$ and define the map $\check{\eta} : AF_{\mathcal{F}^+} \rightarrow \Omega$ as follows:

$\check{\eta}(A(a)) = \{u \in E(a) \mid u \Vdash A(a)\}$,

where $FV(A) \subseteq x$, $A(a) = [a/x]A$. Then $\check{\eta}$ is a valuation on $\mathcal{F}$ and (*) is true for every $A(a) \in MF_{\mathcal{F}^+}$.

**Proof** $\check{\eta}$ is a valuation since

$\check{\eta}(P(a)) \cap E(a, b)) = \{w \in E(a, b) \mid M, w \Vdash P(a)\} \subseteq \{w \in E(b) \mid w \Vdash P(b)\} = \check{\eta}(P(b))$

(here $a_u = b_u$ for $u \in E(a, b)$, by the definition of germs). The second claim is proved by induction on the length of $A$. Let us consider two cases.
4.9. EXAMPLES OF ALGEBRAIC SEMANTICS

(i) \( w \in \hat{\eta}(a = b) \iff w \in E(a, b) \iff \exists U \ni w(a|U = b|U) \iff a_w = b_w \iff w \models a = b; \)

(ii) \( w \in \hat{\eta}(\Box A(a)) \iff \exists U \ni w \iff \exists U \ni w \forall v \in U (v \models A(a)) \)

\[^{(*)}\]

It is clear that \((*) \Rightarrow (**)\). Conversely, assume \((**).\) Since \((a_w = b_w),\) there exists an open \(V \subseteq U\) containing \(w\) such that \((\forall v (a|V = b|V));\) thus \((\forall v \in V a_v = b_v),\) and so \(v \models A(b_v),\) by \((**).\)

**Proposition 4.8.4** A modal formula is valid in \(F\) iff it is true in every fibrewise model over \(F.\)

**Proof** It is sufficient to show that every valuation \(\varphi\) in \(F\) equals \(\hat{\eta}\) for some fibrewise valuation \(\eta.\) In fact, define \(\eta\) by

\[ \eta_u(P) := \{a_u \mid u \in \varphi(P(a))\}. \]

\(\eta\) is well-defined, since \(a_u = b_u\) implies \(x \in E(a, b),\) and therefore \(u \in \varphi(P(a))\) iff \(u \in \varphi(P(b)),\) by Definition 4.2.1.

4.9 Examples of algebraic semantics

**Definition 4.9.1** A presheaf \(F\) over a locale \(\Omega\) is called constant if all its domains are the same (usually, non-empty) and all its restriction maps are identity functions. A constant presheaf over \(\Omega\) with the domain \(D\) is denoted by \(C(\Omega, D).\) A presheaf \(F\) has a constant domain (or briefly, is a CD-presheaf) if every \(F(u, v)\) is bijective (for \(u \geq v).\)

It is clear that every constant presheaf is a sheaf. It is also clear that every CD-presheaf \(F\) over \(\Omega\) is isomorphic to the constant sheaf \(C(\Omega, \mathcal{F}(1)).\)

One can easily see that for a CD-m.v.s. \((\Omega, D)\) the corresponding presheaf constructed in Proposition 4.5.11 is isomorphic to \(C(\Omega, D).\) Using these observations, we obtain:

**Proposition 4.9.2** The following classes correspond to equal semantics:

1. the class of constant presheaves;
2. the class of CD-presheaves;
3. the class of CD-m.v.s. (or H.v.s.).

**Remark 4.9.3** Note that \(HV(C(\Omega, D)) \neq (\Omega, D).\) In fact, the domain of \(HV(C(\Omega, D))\) contains not only the individuals \(a \in D\) with \(E(a) = 1,\) but also the restrictions \(a|u\) such that \(E(a|u) = E(a, a|u).\)
Thus we obtain the algebraic semantics with constant domains denoted by $\mathcal{A}$. Historically, it was introduced earlier than the general semantics $\mathcal{AE}$ [Rasiowa and Sikorski, 1963, Ch. X, §15, Ch XI], [Takano, 1987].

Similarly to Lemma 3.10.25 we have:

**Lemma 4.9.4** The semantics $\mathcal{T}$ and $\mathcal{A}$ do not satisfy (CP).

**Proof** Consider the Kripke sheaf $\Theta_0$ from the proof of Proposition 3.10.24. From the proof of Lemma 3.10.25 we know that $\mathbf{ML}(\Theta_0)$ is an intersection of $\mathcal{CK}$-complete logics. So it suffices to show that $\mathbf{ML}(\Theta_0) \not\subset \mathcal{A}$; in fact, it is easily seen that $F \vDash (C' \lor K')$ implies $F \vDash C'$ or $F \vDash K'$ for every $CD$-m.v.s. $F$, similarly to Proposition 3.10.24. ■

**Definition 4.9.5** A presheaf $\mathcal{F}$ is called monic if all $\mathcal{F}(u, v)$ (for $u \geq v$) are injective. An m.v.s. $F = (\Omega, D, E)$ is called monic if

$$E(a, b) \in \{0, (E(a) \cap E(b))\} \text{ for all } a, b \in D;$$

similarly for $H$-v.s.

**Lemma 4.9.6** If an m.v.s. $F$ is monic, then the associated presheaf $\mathcal{F}$ (Proposition 4.5.11) is monic.

**Proof** It is sufficient to show that $[v, a] = [v, b]$ implies $[u, a] = [u, b]$, whenever $v \leq u \leq E(a) \cap E(b)$. But $[v, a] = [v, b]$ (i.e. $v \leq E(a, b)$) only if $E(a, b) \neq 0$, and so, $E(a, b) = E(a) \cap E(b)$, and thus $u \leq E(a, b)$ by our assumption. ■

**Lemma 4.9.7** Let $\mathcal{F}$ be an inhabited disjoint monic presheaf. Then the m.v.s. $\mathbf{MV}(\mathcal{F})$ is monic.

**Proof** Assume that $a, b \in \mathcal{F}^*$, $E(a, b) \neq 0$, $u = |a| \cap |b|$. Then $a|v = b|v$ for some $v \leq u$, and $a|u = b|u$ (since $\mathcal{F}$ is monic). Thus

$$E(a) \cap E(b) = u = \bigcup \{v \leq u \mid (a|v) = (b|v)\} = E(a, b).$$

■

From Lemmas 4.9.6 and 4.9.7 we obtain:

**Proposition 4.9.8** The classes of monic m.v.s and monic presheaves generate the same semantics.

This semantics is denoted by $\mathcal{MA}$.

Note that the semantics $\mathcal{MK}$ introduced in Section 3.4 is generated by monic presheaves over Kripke spaces.

Let us state some simple logical properties of monic m.v.s, which will be used later on. Recall that $CE$ denotes the formula $(x \neq y) \supset \Box(x \neq y)$, $DE$ denotes the formula $(x = y) \lor (x \neq y)$. 

Lemma 4.9.9  **CE** is valid in every monic m.v.s.

**Proof**  For a valuation \( \varphi \) we have:

\[
(\varphi(a \neq b) = (E(a) \cap E(b)) - E(a, b),
\]

so \( \varphi(a \neq b) \) is open, being either \( E(a) \cap E(b) \) or \( 0 \).

**Corollary 4.9.10**  **DE** is valid in every monic H.v.s. (see 1.4 and Lemma 2.2.2).

The converse is not true: an H.v.s. validating **DE** may be not monic.

**Corollary 4.9.11**  \( \Lambda^{=c} \subseteq C_{\mathcal{MA}}(\Lambda) \) if \( \Lambda \) is an m.p.l.,

\( \Lambda^{=d} \subseteq C_{\mathcal{MA}}(\Lambda) \) if \( \Lambda \) is an s.p.l.

**Proposition 4.9.12**  Let \( \Lambda \) be a propositional logic, \( QA \) its quantified version (see. Section 2.4). Then \( C_{\mathcal{MA}}(QA) = C_A(QA) \).

**Proof**  Assume that an m.v.s. \( F = (\Omega, D, E) \) separates some sentence \( B \in MF \) from \( QA \); let us find a monic m.v.s. with the same property. Consider \( F' = (\Omega, D, E') \), where

\[
E'(a, b) = \begin{cases} E(a) & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}
\]

It is clear that \( F \) and \( F' \) validate the same propositional formulas, that is \( QA \subseteq ML(F') \). There exists a valuation \( \varphi \) in \( F \) such that \( \varphi(B) \neq 1 \), and \( \varphi \) is also a valuation in \( F \), since \( a \neq b \) implies \( \varphi(P(b)) \cap E'(a, b) = 0 \leq \varphi(P(a)) \), and \( \varphi(P(a)) \leq E(a) = E'(a) \).

**Corollary 4.9.13**  If the quantified version \( QA \) of a modal propositional logic \( \Lambda \) is \( \Lambda \)-complete, then \( QA^{=c} \) is a conservative extension of \( QA \), i.e. \( QA = (Q\Lambda^{=c})^\circ \) (cf. Section 2.9).

We can also define corresponding restrictions of neighbourhood and Kripke semantics.

**Definition 4.9.14**  \( T := A \cap TE \) (neighbourhood semantics with constant domains),

\( \mathcal{MT} := MA \cap TE \) (mono-neighbourhood semantics),

\( \mathcal{MK} := MA \cap KE \) (mono-Kripke semantics).

Note that \( \mathcal{MT} \) corresponds to monic presheaves over neighbourhood frames.
Chapter 5

Metaframe semantics

5.1 Preliminary discussion

In this chapter we study some further variations of Kripke semantics. In all these semantics we keep to the main idea that the truth value of a formula depends on a possible world and a formula $\Box A$ is true at some world iff $A$ is true at every accessible world.

Kripke bundles

Let us first consider $S_4$-frames. Our starting point is Kripke sheaf semantics. Recall (Section 3.5) that a Kripke sheaf over a propositional $S_4$-frame $F$ is a system of domains $D = (D_u | u \in F)$, together with transition maps $\rho_{uv} : D_u \to D_v$ parametrised by pairs $(u,v) \in R$; $\rho_{uv}(a)$ is an 'inheritor' of an individual $a \in D_u$ in the world $v$. In Kripke sheaves inheritors of different individuals can collapse in some accessible world. But can we give more freedom to individuals allowing them to have several inheritors in the same accessible world (perhaps, in their own)? Following this idea we come to Kripke bundles [Shehtman and Skvortsov, 1990], where transition maps $\rho_{uv}$ are replaced with 'inheritance' (or 'counterpart') relations $\rho_{uv} \subseteq D_u \times D_v$; $a \rho_{uv} b$ is read as 'b is an inheritor (or a counterpart) of a in v' (or a v-inheritor of a). If the domains $D_u$ are disjoint, we can equivalently use the global inheritance relation $\rho \subseteq D^+ \times D^+$ on the set of all individuals $D^+$ and define $\rho_{uv}$ as $\rho \cap (D_u \times D_v)$.

Here we assume that the relation $\rho$ is reflexive and transitive — which is analogous to the properties of transition maps from Definition 3.6.1.

Models in Kripke bundles are defined in the same way as in Kripke sheaves. Forcing is defined by induction, but with a special care about the $\Box$-case.

The first idea is to extend the definition of forcing from Kripke sheaves to Kripke bundles as follows. For a $D_u$-sentence $A$ put

$$\forall \quad M, u \models \Box A \text{ iff for any } v \in R(u) \text{ every } v\text{-version of } A \text{ is true at } M, v.$$
In Kripke sheaves $v$-versions of $D_u$-sentences are obtained by replacing every occurrence of every $a \in D_u$ with $a|_v$, the unique inheritor of $a$ in $v$. In Kripke bundles inheritors may be not unique, so we can define a $v$-version of $A$ as a result of replacing every occurrence of every individual with some of its $v$-inheritor. But this definition gives too much freedom, because then we can refute some theorems of $\mathbf{QK}^= \!$ (and $\mathbf{QK}$).

To see this, consider the formula $B := \forall x (x = x)$ and the following Kripke bundle.
Then (for any model $M$)

$$M, u \not\models \Box(a = a),$$

because $a = a$ has a false $v$-version $b_1 = b_2$. Hence $M, u \not\models B$.

The same argument applies to the formula without equality

$$C := \forall x \Box(P(x) \supset P(x)).$$

It is refuted at $M, u$ for any model $M$ such that $M, v \models P(b_1)$ and $M, v \not\models P(b_2)$. In fact, then $P(a) \supset P(a)$ has a false $v$-version $P(b_1) \supset P(b_2)$.

In these examples undesirable $v$-versions of $D_u$-sentences are obtained by replacing different occurrences of the same individual $a$ with its different inheritors $b_1, b_2$. There is the simplest ‘way out’ — to forbid such a splitting of individuals within the same $D_u$-sentence. So we adopt the following definition:

$$(\nabla)$$

a $v$-version of a $D_u$-sentence $A$ is a $D_v$-sentence obtained by replacing all occurrences of every $a \in D_u$ in $A$ with some its $v$-inheritor $a'$.

In other words, $v$-versions of $A$ are of the form $f : A$ for maps $f : \text{Const}(A) \longrightarrow D_v$, where $\text{Const}(A)$ is the set of all constants from $A$, such that $a \rho_f a$ for any $a \in \text{Const}(A)$.

These maps are ‘local analogues’ of the maps $\rho_{uv}$ in Kripke sheaves; the latter are defined on the whole $D_u$.

It seems natural to write the condition $(\nabla)$ (which is analogue of $(\ldots)$) in a form similar to Definition 3.6.4.

$$(\nabla')$$

$$M, u \models \Box A(a_1, \ldots, a_n) \text{ iff } \forall v \in R(u) \forall b_1, \ldots, b_n \left( \left( \bigwedge_{i=1}^n a_i \rho_{uv} b_i \Rightarrow M, v \models A(b_1, \ldots, b_n) \right) \right).$$

But such a ‘definition’ is ambiguous. In fact, as we know (Section 2.4) the same $D_u$-sentence can be presented as $A(a_1, \ldots, a_n) := [a_1, \ldots, a_n/x_1, \ldots, x_n] A$ for different formulas $A(x_1, \ldots, x_n)$. For example,

$$(a = a) = [a/x] (x = x) = [a, a/x, y] (x = y).$$

Now consider the same Kripke bundle model $M$ as above. By applying $(\nabla')$ we obtain

$$M, u \not\models [a, a/x, y] \Box(x = y),$$

but

$$M, u \models [a/x] \Box(x = x).$$

However the ambiguity of $(\nabla')$ disappears after fixing one of the presentations of a $D_u$-sentence. For example, if we use only maximal generators,1 $\Box(a = a)$ is regarded as $[a/x] \Box(x = x)$, so $M, u \models (a = a)$ from this viewpoint. This

---

1Recall that a maximal generator is obtained by replacing all occurrences of a constant with the same variable, cf. Section 2.4.
restricted version of \((\forall')\) exactly corresponds to the combination of \((\forall)\) and \((\forall\forall)\).

Although the use of maximal generators and the condition \((\forall\forall)\) seem ad hoc, later on we shall see that this is sufficient for (and closely related to) soundness.

There is another natural way to avoid ambiguity in definition \((\forall')\) — instead of evaluating \(D_u\)-sentences, we can evaluate formulas under variable assignments. This approach is well-known in classical logic. In full detail and in a more general context it will be considered in Section 5.9. Here we only sketch the main idea.

The truth value of a formula \(A(x)\) with a list of parameters \(x\) at a world \(u\) is defined under a 'finite assignment' (or a \(D_u\)-transformation) \([x \mapsto a]\), where \(a\) is a tuple in \(D_u\) (cf. 2.4). Then \((\forall')\) changes to

\[
(\forall'') \quad M, u \models \Box A(x) [x \mapsto a] \text{ iff } \\
\forall v \in R(u) \forall b_1 \ldots b_n (\forall i a_i \rho_{uv} b_i \Rightarrow M, v \models A(x) [x \mapsto b]).
\]

Now the formulas \(\Box(x = x), \Box(P(x) \supset P(x))\) become true — in fact, e.g.

\[
M, u \models \Box(x = x) [x \mapsto a]
\]

is equivalent to

\[
\forall v \in R(u) \forall b (a \rho_{uv} b \Rightarrow M, v \models x = x [x \mapsto b]).
\]

But soundness still remains a problem. Viz., consider the same Kripke bundle on Fig. 5.2. Then according to \((\forall'')\), the \(\text{QK}^=\)-theorem

\[
A := (x = y \supset \Box(x = y))
\]

is refuted. In fact (for any model \(M\))

\[
M, u \models x = y [x, y \mapsto a, a],
\]

but

\[
M, u \not\models \Box(x = y) [x, y \mapsto a, a],
\]

since

\[
M, v \not\models \Box(x = y) [x, y \mapsto b_1, b_2].
\]

A similar example exists in modal logics without equality. Consider the following \(\text{QK}\)-theorem:

\[
\Box P(x, x) \supset \exists y \Box P(x, y),
\]

which is a substitution instance of axiom \((Ax13)\) from 2.6.10. This formula is refuted in the same Kripke bundle in the model \(M\) such that

\[
M, u \models P(x, x) [x \mapsto a], \\
M, v \models P(x, y) [x, y \mapsto b_i, b_j] \text{ iff } i = j.
\]
In fact,
\[ M, u \models \Box P(x, x) [x \mapsto a], \]
since
\[ M, u \models P(x, x) [x \mapsto a], \]
\[ M, v \models P(x, x) [x \mapsto b_i]. \]

But
\[ M, u \not\models \Box P(x, y) [x, y \mapsto a, a], \]
since
\[ M, v \not\models P(x, y) [x, y \mapsto b_1, b_2]. \]

These counterexamples, together with the soundness theorem (Proposition 5.2.12), justify the truth definition stated in (\( \nabla \)), (\( \nabla \nabla \)) or its equivalent (\( \nabla' \)) using maximal generators.

**Functor semantics**

*C-sets* are another generalisation of Kripke sheaves, where we allow for many transition maps between the same possible worlds. This happens, because a quasi-ordered set is replaced with an arbitrary category \( \mathcal{C} \). More precisely, let \( \mathcal{C} \) be a category with the set of objects \( W \). Consider the frame \( F = (W, R) \), where \( uRv \) iff there exists a morphism from \( u \) to \( v \), i.e. iff the set \( \mathcal{C}(u, v) \) of morphisms from \( u \) to \( v \) is non-empty. \( F \) is called the *frame representation* of \( \mathcal{C} \). A \( \mathcal{C} \)-set is defined as a functor from \( \mathcal{C} \) to \( \text{SET} \); it can be regarded as a system of domains \((F, D)\) together with a family of functions respecting composition and identity, cf. Definition 3.6.1. Obviously, Kripke sheaves over \( F \) are nothing but \( \text{Cat}F \)-sets.

In a \( \mathcal{C} \)-set, the \( v \)-inheritors of an individual \( d \in D_u \) are \( \rho_\alpha d \) for all \( \alpha \in \mathcal{C}(u, v) \).

Given a model \( M \) in a \( \mathcal{C} \)-set \((F, D, \rho)\), we define forcing, so that
\[ M, u \models \Box B(d_1, \ldots, d_n) \text{ iff} \]
\[ \forall v \in R(u) \forall \alpha \in \mathcal{C}(u, v) M, v \models B(\rho_\alpha d_1, \ldots, \rho_\alpha d_n). \]

Similarly to Kripke sheaves, in \( \mathcal{C} \)-sets we may not worry about different presentation of a \( D_u \)-sentence as \( B(d_1, \ldots, d_n) \), because \( \rho_\alpha \) are functions. A \( v \)-inheritor of a \( D_u \)-sentence is obtained by replacing all individuals \( a \in D_u \) with their images \( \rho_\alpha a \) under the same transition map \( \rho_\alpha, \alpha \in \mathcal{C}(u, v) \).

**Validity**

Validity in a Kripke bundle or in a \( \mathcal{C} \)-set is defined again as the truth in all models. But there is another problem: the set of valid formulas may be not substitution closed! To show this, consider the category \( \mathcal{C} \) with a single object\(^2\)

---

\(^2\)I.e. its frame representation \( F \) is a reflexive singleton.
$u$ and two arrows: $id$ and $\gamma$ such that $\gamma \circ \gamma = \gamma$ (and certainly, $id \circ \gamma = \gamma \circ id = \gamma$; $C$ is just a two-element monoid). Let $F$ be a $C$-set in Fig. 5.3 with $D_u = \{a, b\}$, $\rho_\gamma(a) = \rho_\gamma(b) = b$.

![Figure 5.3.](image)

Then the formula $p \triangleright \Box p$ is valid in $F$ (since $F \vDash p \triangleright \Box p$), but $P(x) \triangleright \Box P(x)$ is not. In fact, consider a model in which $u \models P(a), u \nvdash P(b)$; then $u \nvdash \Box P(a)$, because $u \nvdash P(\rho_\gamma a)$.

**Exercise 5.1.1** Let us call a formula $A$ intuitionistically valid in a $C$-set (or in a Kripke bundle) if $AT$ is valid. Show that the formula $p \lor \neg p$ is intuitionistically valid in the same $C$-set as above, whereas $P(x) \lor \neg P(x)$ is not intuitionistically valid.

**Exercise 5.1.2** Consider the Kripke bundle $G$ with the single domain $D_u = \{a, b\}$ such that $\rho_{uu} = \{(a, a), (b, b), (a, b)\}$, see Fig. 5.4. Show that the formula $P(x) \triangleright \Box P(x)$ is not valid and $P(x) \lor \neg P(x)$ is not intuitionistically valid in $G$.

So it turns out that the set of all formulas (modal or intuitionistic) that are valid in a given Kripke bundle or a $C$-set, is not necessarily a predicate logic. To obtain a sound semantics, we replace validity with the notion of strong validity — a formula is said to be strongly valid if all its substitution instances are valid.

At first glance, two new semantics seem independent. But actually the semantics of $C$-sets is stronger. Furthermore, their natural combination (the ‘$C$-bundle semantics’) is strongly equivalent to $C$-sets. $C$-bundles are defined analogously to $C$-sets, with the following difference — we replace functions $\rho_\alpha : D_u \rightarrow D_v$ with relations $\rho_\alpha \subseteq D_u \times D_v$ parametrised by morphisms $\alpha \in C(u, v)$; the clause $(\forall')$ is modified as follows:

$$(\forall^+) \quad M, u \models \Box A(a_1, \ldots, a_n) \text{ iff } \forall v \in R(u) \forall \gamma \in C(u, v) \forall b_1, \ldots, b_n \left( \forall_i a_i \rho_\gamma b_i \Rightarrow M, v \models A(b_1, \ldots, b_n) \right),$$
5.2. Kripke bundles

Now let us turn to precise definitions for the polymodal case.

Definition 5.2.1 An \((N\text{-modal})\) Kripke bundle over an \(N\text{-modal propositional frame} F = (W, R_1, \ldots, R_N)\) is a triple \(\mathcal{F} = (F, D, \rho)\), in which \(D = (D_u \mid u \in F)\) is a system of domains over \(F\) and \(\rho = (\rho_{iuv} \mid uR_iv, \ 1 \leq i \leq N)\) is a family of relations \(\rho_{iuv} \subseteq D_u \times D_v\) such that for any \(i, u, v\)

\[ (\sharp_1) \ uR_iv \Rightarrow \forall a \in D_u \ \exists b \in D_v \ a\rho_{iuv}b, \]

i.e. \(\text{dom}(\rho_{iuv}) = D_u\); this means that every individual has inheritors in all accessible worlds. The frame \(F\) is called the base of \(\mathcal{F}\), and the domain \(D_u\) the fibre at \(u\).

For the 1-modal case we use the notation \(\rho_{uv}\) rather than \(\rho_{1uv}\).

Definition 5.2.2 An intuitionistic Kripke bundle is a 1-modal Kripke bundle \((F, D, \rho)\) over an \(S4\)-frame \(F = (W, R)\) satisfying the following two conditions:

\( (\sharp_2) \) every \(\rho_{uv}\) is reflexive;

\( (\sharp_3) \) if \(uRvRw\), then \(\rho_{uv} \circ \rho_{vw} \subseteq \rho_{uw} \).
**Definition 5.2.3** An intuitionistic Kripke bundle \((F, D, \rho)\) is called a Kripke quasi-sheaf if every \(\rho_{uu}\) is the identity function on \(D_u\), i.e. if

\[ \forall a, b \in D_u \ (a \rho_{uu} b \iff a = b). \]

Obviously, every Kripke sheaf is a 1-modal Kripke bundle, in which all \(\rho_{uv}\) are functions. It is also clear that in the intuitionistic (and in the S4-) case every Kripke sheaf is a Kripke quasi-sheaf.

The other way round, Lemma 3.6.3 shows that every Kripke bundle, in which \(\rho_{iuu}\) are functions satisfying ‘coherence conditions’, gives rise to a Kripke sheaf. In a sense, Kripke sheaves are exactly Kripke bundles of this kind.

As mentioned in the previous section, a Kripke bundle can be presented in another equivalent form.

**Lemma 5.2.4**

1. Let \(\mathcal{F} = (F, D, \rho)\) be a Kripke bundle, in which the domains \(D_u\) are disjoint. Consider the total domain

\[ D^1 := \bigcup \{D_u \mid u \in W\} \]

with relations

\[ \rho_i := \bigcup \{\rho_{iuu} \mid u R_i v\} \]

for \(1 \leq i \leq N\).

Then we obtain the propositional frame of individuals (also called the total frame of \(\mathcal{F}\)):

\[ F_1 = (D^1, \rho_1, \ldots, \rho_N). \]

Let \(\pi : D^1 \to W\) be the map such that \(\pi(a) = u\) for \(a \in D_u\). Then \(\pi\) is a \(p\)-morphism \(F_1 \Rightarrow F\). Moreover, for \(N = 1\), \(\mathcal{F}\) is an intuitionistic Kripke bundle, iff \(F_1\) is an S4-frame.

2. Conversely, a \(p\)-morphism of modal frames

\[ \pi : F_1 = (W', \rho_1, \ldots, \rho_N) \Rightarrow F = (W, R_1, \ldots, R_N) \]

gives rise to a system of disjoint domains \(D_u = \pi^{-1}(u)\) for \(u \in W\) and a family of relations \(\rho_{iuu} = \rho_i \cap (D_u \times D_v)\) for \(u R_i v\), \(1 \leq i \leq N\), which forms a modal Kripke bundle.

For the case \(N = 1\), the corresponding Kripke bundle is intuitionistic iff both \(F_1, F\) are S4-frames.

Sometimes it is convenient to denote the base \(F = (W, R_1, \ldots, R_N)\) of a Kripke bundle \(\mathcal{F}\) by \(F_0 = (D^0, R_0^1, \ldots, R_0^N)\) and the frame of individuals \(F_1\) by \((D^1, R_1^1, \ldots, R_N^1)\).

Thus we come to the following equivalent definition of Kripke bundles:

\[ ^3\] As in the case of Kripke sheaves, cf. Section 3.5.
5.2. KRIPKE BUNDLES

Definition 5.2.5 A Kripke bundle (modal or intuitionistic) is a $p$-morphism between two propositional frames (resp., modal or intuitionistic).

The choice between two equivalent definitions of Kripke bundles is a matter of convenience.

It is clear that quasi-sheaves correspond to $p$-morphisms of $S_4$-frames $\pi : F_1 \to F_0$ such that $F_1 \upharpoonright \pi^{-1}(u)$ is a discrete frame, i.e. a quasi-sheaf is a Kripke bundle with discrete fibres.

Exercise 5.2.6 Show that Kripke sheaves correspond to coverings, i.e. to Kripke bundles with the unique lift property:

$$\pi(a)Rv \Rightarrow \exists b \ (a \rho b \& \pi(b) = v).$$

Definition 5.2.7 A valuation $\xi$ in a (modal) Kripke bundle $F = (F, D, \rho)$ is just a (modal) valuation in its system of domains $D = (D_u \mid u \in W)$, cf. Definition 3.2.4. The pair $M = (F, \xi)$ is called a Kripke bundle model.

The inductive definition of forcing is more delicate than for Kripke frames or Kripke sheaves, because (as explained in Section 5.1) in the clause for $\Box$ we should explicitly indicate all the individuals occurring in $A$ and use the presentation of $D_u$-sentences described in Lemma 2.4.4.

Forcing in Kripke bundle models is quite similar to Kripke frames or Kripke sheaves; the only difference is in the $\Box$-clause:

Definition 5.2.8 Let $F = (F, D, \rho)$ be a Kripke bundle, $F = (W, R_1, \ldots, R_N)$, $M = (F, \xi)$ a Kripke bundle model. We define the forcing relation $M, u \models A$ between worlds $u \in F$ and $D_u$-sentences (with equality) by induction:

- $M, u \models P^0_k$ iff $u \in \xi_u(P^0_k)$;
- $M, u \models P^m_k(a)$ iff $a \in \xi_u(P^m_k)$ (for $m > 0$);
- $M, u \models a = b$ iff $a$ equals $b$;
- $M, u \not\models \bot$;
- $M, u \models B \lor C$ iff $(M, u \models B$ or $M, u \models C)$;
- $M, u \models B \land C$ iff $M, u \models B \& M, u \models C$;
- $M, u \models B \supset C$ iff $(M, u \not\models B$ or $M, u \models C)$;
- $M, u \models \exists x A$ iff $\exists a \in D_u M, u \models [a/x] A$;
- $M, u \models \forall x A$ iff $\forall a \in D_u M, u \models [a/x] A$.
- $M, u \models \Box_i [a_1, \ldots, a_n/x_1, \ldots, x_n] B_0$ iff $\forall v \in R_i(u) \forall b_1, \ldots, b_n \in D_v (\forall j a_j \rho_{iuv} b_j \Rightarrow M, v \models [b_1, \ldots, b_n/x_1, \ldots, x_n] B_0)$ where $FV(B_0) = \{x_1, \ldots, x_n\}$ and $a_1, \ldots, a_n \in D_u$ are distinct.
So in the latter case we present a $D_u$-sentence $\square_i B$ as $\square_i [a/x] B_0$ for a formula $B_0$ and an injective $[x \mapsto a]$, recall that $B_0$ is called a maximal generator of $B$ and it is unique up to renaming of $x$. This clause can be equivalently presented in the form discussed in 5.1:

$$(\forall) \quad M, u \models \square A \iff \text{for any } v \in R(u) \text{ every } v\text{-version of } A \text{ is true at } M, v.$$ 

In particular, for a sentence $B$ we have:

$$M, u \models \square_i B \iff \forall v \in R_i(u) \ M, v \models B,$$

exactly as in standard Kripke semantics.

**Definition 5.2.9** A (modal) predicate formula is called true in a Kripke bundle model if its universal closure $\forall A$ is true at every world of this model. A formula is called valid in a Kripke bundle $\mathbb{F}$ if it is true in every model over $\mathbb{F}$.

As above, the sign $\models$ denotes the truth in a model and the validity in a Kripke bundle.

The following is trivial (cf. Lemma 3.2.20(1)):

**Lemma 5.2.10** For a Kripke bundle model $M$ and a modal formula $A(x_1, \ldots, x_n)$ $M \models A(x_1, \ldots, x_n)$ iff $\forall u \in M \forall a_1, \ldots, a_n \in D_u \ M, u \models A(a_1, \ldots, a_n)$.

Recall that $A(a_1, \ldots, a_n)$ denotes $[a_1, \ldots, a_n/x_1, \ldots, x_n] A(x_1, \ldots, x_n)$, cf. 2.4.

Consider the set of valid formulas

$$\text{ML}_{\mathbb{F}}^{(=)} := \{ A \in MF_{\mathbb{F}}^{(=)} \mid \mathbb{F} \models A \}.$$

This set is closed under MP, Gen and $\square$-introduction, cf. 3.2.27. But it may be not substitution closed (cf. Exercise 5.1.2), so we introduce the following notion.

**Definition 5.2.11** A formula $A \in MF_{\mathbb{F}}^{(=)}$ is called strongly valid in a Kripke bundle $\mathbb{F}$ (respectively, strongly valid with equality) in $\mathbb{F}$ if all its $MF_{\mathbb{F}}^{(=)}$-substitution instances are valid in $\mathbb{F}$; notation: $\mathbb{F} \models^+ A$ (resp., $\mathbb{F} \models^{+ =} A$).

Let

$$\text{ML}^{(=)}(\mathbb{F}) := \{ A \in MF_{\mathbb{F}}^{(=)} \mid \mathbb{F} \models^+ A \}.$$

Then we have

**Proposition 5.2.12**

1. $\text{ML}^{(=)}(\mathbb{F})$ is an m.p.l.$(=)$;

2. The strong validity of a formula $A$ follows from the validity of all its $m$-shifts $A^m$, i.e.

$$\text{ML}^{(=)}(\mathbb{F}) = \{ A \in MF_{\mathbb{F}}^{(=)} \mid \forall m \mathbb{F} \models A^m \}.$$
Speaking informally, Proposition 5.2.12(2) means that $A$ is strongly valid iff it is valid with arbitrarily many extra parameters.

The proof is postponed until Section 5.13.

More exactly, this is a particular case of a general soundness theorem for metaframe semantics, cf. Section 5.12. However, let us sketch a general plan of the proof.

At the first stage we prove (2). Recall that substitution instances of a formula $A$ are the strict substitution instances of its $m$-shifts $A^m$. So it is sufficient to check the following

Claim. $ML^m(\mathcal{F})$ is closed under strict substitutions.

The proof is very similar to the substitution closedness of Kripke sheaves (Proposition 3.6.17) in the particular case of strict substitutions.

In the proof of (1), the strong validity of axioms, due to (2), follows from the validity of their shifts. The latter is proved in a straightforward way. For propositional axioms it is easier to use proposition 5.3.7 (see below), which in its turn, follows from (2).

The substitution closedness of $ML^m(\mathcal{F})$ is obvious. The closedness under $\mathcal{M}$, $\Box$-introduction follows from the same property of $ML^m(\mathcal{F})$ (again, due to (2)). For example, for MP note that $(A \supset B)^n = A^n \supset B^n$, so $B^n \in ML^m(\mathcal{F})$ whenever $A^n, (A \supset B)^n \in ML^m(\mathcal{F})$.

Thanks to 5.2.12(1), we may call $ML^m(\mathcal{F})$ the modal logic of $\mathcal{F}$. Also note that $ML^m(\mathcal{F})$ is the largest modal logic contained in $ML^m(\mathcal{F})$ (so to say, the substitution interior of $ML^m(\mathcal{F})$).

Let us point out again that for Kripke sheaves the notions of validity and strong validity are equivalent, because the set of valid formulas is already substitution closed. Exercise 5.1.2 shows that this is not always the case in Kripke bundles.

Exercise 5.2.13 Give an example of a 1-modal Kripke bundle, in which $\rho_{uv}$ are functions, but the set of valid formulas is not substitution-closed.

We also remark that the notion of strong validity a priori depends on the language. In fact, it might happen that $\mathcal{F} \models SA$ for any $MF_N$-substitution $S$, but not for any $MF^m_N$-substitution. But 5.2.12(2) shows that this is impossible, so $ML^m(\mathcal{F})$ is conservative over $ML(\mathcal{F})$.

Definition 5.2.14 For a class $\mathcal{C}$ of Kripke bundles we define

$$ML^m(\mathcal{C}) := \bigcap \{ ML^m(\mathcal{F}) \mid \mathcal{F} \in \mathcal{C} \}.$$ 

Definition 5.2.15 Kripke bundle semantics $KB_N$ is generated by the class of all $N$-modal Kripke bundles. $KB_N$-complete logics are called Kripke bundle complete.
**Definition 5.2.16** (Cf. Definition 3.2.32) If $F$ is a propositional Kripke frame, $\mathcal{KB}(F)$ denotes the class of all Kripke bundles based on $F$. The logic $\mathbf{ML}^\langle\rangle$ $\mathcal{KB}(F)$ is called the Kripke bundle modal logic (with equality) determined over $F$. Similarly we define $\mathcal{KB}(\mathcal{C})$ for a class of propositional frames $\mathcal{C}$ and the Kripke bundle modal logic determined over $\mathcal{C}$.

### 5.3 More on forcing in Kripke bundles

Now let us present an arbitrary Kripke bundle as a collection of propositional Kripke frames. This presentation is more convenient from the technical viewpoint, and it will lead us to a more general notion of a metaframe.

**Definition 5.3.1** Let $D = (D_u \mid u \in W)$ be a system of disjoint domains over a propositional modal frame $F = (W, R_1, \ldots, R_N)$. For $n \geq 0$ we define the $n$th level of $D$ as the set $D^n := \bigcup\{D^n_u \mid u \in W\}$.

We also identify $D^1_u$ and $D_u$ (strictly speaking, they are not equal), so that our new definition of $D^1$ corresponds to that in Lemma 5.2.4. Also note that $D^0 = W$, since $D^0_u = \{u\}$, see Definition 3.2.4.

Note that every valuation $\xi = (\xi_u)_{u \in W}$ in a system of disjoint domains is associated with the following function $\xi^+$ sending $n$-ary predicate letters to subsets of $D^n$:

$$\xi^+(P^n_k) := \bigcup_{u \in W} \xi_u(P^n_k).$$

In particular, we have

$$\xi^+(P^0_k) \subseteq W.$$

Recall that for any function $\theta$ such that $\theta(P^n_k) \subseteq D^n$, there exists a valuation $\xi$ such that $\theta = \xi^+$:

$$\xi_u(P^n_k) := \theta(P^n_k) \cap D^n_u.$$

To define accessibility relations on the sets $D^n$, we first recall the notation from Lemma 5.2.4:

$$\rho_i := \bigcup\{\rho_{iuv} \mid uR_iv\}.$$

Recall the subordination relation between $n$-tuples (see Introduction):

$$a \text{ sub } b := \forall j, k (a_j = a_k \Rightarrow b_j = b_k).$$

**Definition 5.3.2** Let $\mathbb{F} = (F, D, \rho)$ be a modal Kripke bundle over $F = (W, R_1, \ldots, R_N)$. For $n > 0$, $1 \leq i \leq N$, we define the relation on $D^n$:

$$aR^n_i b \text{ iff } \forall j \ a_j \rho_i b_j \ & \ & \text{ & a sub } b.$$

So in particular, $R^1_i = \rho_i$. For $n = 0$ we put $R^0_i := R_i$.

The frame $F_n := (D^n, R^n_1, \ldots, R^n_N)$ is called the $n$th level of $\mathbb{F}$.  

Exercise 5.3.3 Show that if $\rho_i$ is transitive, then $R^n_i$ is also transitive.

Again in the 1-modal case we write $R^n$ rather than $R^n_i$.

Now the inductive clause for $\square_iB$ from the definition of forcing can be rewritten in the following form.

Lemma 5.3.4 Under the conditions of Definition 5.2.8, let $B$ be an $N$-modal formula with $FV(B) \subseteq r(x)$, $|x| = n$. Then for any $u \in F$ and $a \in D^n_u$

$$M, u \Vdash \square_iB(a) \quad \text{iff} \quad (*) \forall v \in R_i(u) \forall b \in D^n_v (aR^n_i b \Rightarrow M, v \Vdash B(b))$$

where as usual, $B(a)$ denotes $[a/x]B$.

Proof ('Only if.') Suppose $M, u \Vdash \square_iB(a)$. As explained after Definition 5.2.8, this means that every $v$-version of $B(a)$ is true at $M, v$ (for any $v \in R_i(u)$), and $v$-versions are obtained according to $(\land \forall)$. To check $(*)$, let us show that for $aR^n_i b$, $B(b)$ is a $v$-version of $B(a)$.

In fact, every occurrence of $a_k$ in $B(a)$ either replaces an occurrence of $x_k$ in $B$ or replaces an occurrence of $x_j$ in $B$ for $j \neq k$ — if $a_j = a_k$. In the first case an occurrence of $a_k$ in $B(a)$ corresponds to an occurrence of $b_k$ in $B(b)$. Similarly, in the second case an occurrence of $a_j = a_k$ in $B(a)$ corresponds to an occurrence of $b_j$ in $B(b)$, and $b_j = b_k$, since $aR^n_i b$.

('If.') Suppose $(*)$, and consider a presentation of $B(a)$ as $[c/x]B_0$ with a distinct $c$. Then $B_0$ is a maximal generator of $B(a) = [a/y]B(y)$, so by 2.4.5(2), $B_0$ is obtained by a variable substitution from $B$, i.e. $B_0 = [x \cdot \sigma/y]B$ for some $\sigma \in \Sigma_{mn}$, where $|x| = n$, $|y| = m$.

To show that $M, u \Vdash \square_i[c/x]B_0$, consider $v \in R_i(u)$ and $d \in D^n_v$ such that $\forall k c_k \rho_i d_k$. Then

$$[d/x]B_0 = [d/x][x \cdot \sigma/y]B = [d \cdot \sigma/y]B.$$

Also $(c \cdot \sigma)$ sub $(d \cdot \sigma)$, since $c_{\sigma(i)} = c_{\sigma(j)}$ implies $\sigma(i) = \sigma(j)$ and thus $d_{\sigma(i)} = d_{\sigma(j)}$ (remember that $c$ is distinct). Therefore $(c \cdot \sigma)R^n_i(d \cdot \sigma)$; hence $M, v \Vdash [d \cdot \sigma/y]B$ by $(*)$, i.e. $M, v \Vdash [d/x]B_0$ by $(**)$. Eventually $M, u \Vdash \square_i[c/x]B_0$ by Definition 5.2.8.  

Since the world $u$ is fully determined by a tuple $a \in D^n$, we can drop $u$ from the notation of forcing and just write $M \Vdash A(a)$ (or $M \Vdash [a/x]A$, to be more precise). Now $(*)$ becomes almost the same as in the propositional case:

$$M \Vdash \square_iB(a) \quad \text{iff} \quad \forall b \quad (aR^n_i b \Rightarrow M \Vdash B(b)).$$

Since by definition of $R^n_i$, $a$ is subordinate to $b$, the truth value of $\square_iB(a)$ is well-defined for any tuple of individuals $a = (a_1, \ldots, a_n)$ in the same world; some of $a_i$'s may coincide.
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So $F$ corresponds to the sequence of propositional frames $F_n$, in which $F_0$ is the basic propositional frame. A Kripke bundle model $M = (F, \xi)$ is then associated with the family of propositional Kripke models $(M_n)_{n \in \omega}$, where $M_n = (F_n, \xi^n)$ and for any $k$, $\xi^n(P_k^0) := \xi^+(P_k^0)$. $F_n$ (respectively, $M_n$) is called the $n$-level of $F$ (respectively, $M$).

This presentation of Kripke bundles allows us to describe the propositional fragment $ML^{(\omega)}(F)$ of the predicate logic $ML^{(\omega)}(F)$.

Lemma 5.3.5 Let $M = (F, \xi)$ be an $N$-modal Kripke bundle model, and let $M_n = (F_n, \xi^n)$ be the $n$-level of $M$. Then for any $N$-modal propositional formula $A$ and for any $a \in D^n$

$$M_n, a \models A \iff M \models A^n(a).$$

Proof Easy, by induction on the length of $A$. The case $A = \Box_i B$ readily follows from Lemma 5.3.4.

Lemma 5.3.6 Let $F$ be an $N$-modal Kripke bundle, $A$ a propositional $N$-modal formula. Then for any $n \geq 0$

$$F \models A^n \iff F_n \models A.$$

Proof Lemma 5.3.5 implies that for any model $M$ over $F$,

$$M_n \models A \iff M \models A^n.$$

In fact, note that by 5.2.10,

$$M \models A^n \iff \forall u \in M \forall a \in D^*_u M, u \models A^n(a)$$

iff $M_n \models A$ by 5.3.5.

Hence

$$F \models A^n \iff \exists \xi (F, \xi) \models A^n \iff \forall \xi \left( F_n, \xi^n \right) \models A.$$

Finally note that every valuation $\theta$ in $F_n$ is $\xi^n$ for some valuation $\xi$ in $F$. In fact, one can put $\xi^+(P_k^n) := \theta(P_k^n)$ for any $k$ and define arbitrary values of $\xi^+$ on other predicate letters. Hence $\forall \xi \left( F_n, \xi^n \right) \models A \iff F_n \models A$.

Proposition 5.3.7 For an $N$-modal Kripke bundle $F$ and an $N$-modal propositional formula $A$, $F \models A$ iff $\forall n F_n \models A$, i.e.

$$ML^{(\omega)}(F) = \bigcap_{n \in \omega} ML(F_n).$$

Proof Follows from Lemma 5.3.6 and Proposition 5.2.12.
5.4  Morphisms of Kripke bundles

Definition 5.4.1 Let $F = (F, D, \rho)$, $G = (G, D', \rho')$ be Kripke bundles. A pair $f = (f_0, f_1)$ is called an equality-morphism (briefly, $=$-morphism) from $F$ to $G$ (notation: $f : F \rightarrow G$) if

1. $f_0 : F \rightarrow G$ is a morphism of propositional frames,
2. $f_1 : F_1 \rightarrow G_1$ is a morphism of propositional frames,
3. $f_1$ is a fibrewise bijection, i.e. every $f_{1u} := f_1 \upharpoonright D_u$ is a bijection between $D_u$ and $D'_{f_0(u)}$.

$f_0$ is called the world component, $f_1$ the individual component of $(f_0, f_1)$. An $=$-morphism $(f_0, f_1)$ is called

- a $p=$-morphism if $f_0$ is surjective (and thus, a $p$-morphism);
- an isomorphism if $f_0$, $f_1$ are isomorphisms.

It is clear that Definition 3.3.1 for Kripke frames is a particular case of 5.4.1. From the definition one can easily see that the following diagram commutes (where $\pi$, $\pi'$ are the p-morphisms corresponding respectively to $F$, $G$).

```
      D
    /  \  /
  f_0 /   \ f_1
     /     /
    D'     \\ D'
    /  \  /
  f_0' /   \ f_1'
     /     /
    D'     \\ D'
```

Exercise 5.4.2 Show that in Definition 5.4.1 the condition (i) follows from (ii) and (iii).

Exercise 5.4.3 Show that if $(f_0, f_1)$ is an $=$-morphism and $f_0$ is an isomorphism, then $f_1$ is also an isomorphism.

Lemma 5.4.4 If $f : F \rightarrow F'$ is an $=$-morphism of Kripke bundles, then the map $f_n$ sending $a$ to $f \cdot a$ is a morphism $F_n \rightarrow F'_n$.

Proof  By definition, the statement holds for $n = 0, 1$.

Let us show that $f_n$ respects subordination. In fact, suppose $a \leq b$; then

$f_1(a_i) = f_1(a_j)$ implies $a_i = a_j$ (since $f$ is a fibrewise bijection), and hence $b_i = b_j$ (due to $a \leq b$), $f_1(b_i) = f_1(b_j)$. Thus $(f \cdot a)\leq (f \cdot b)$.

The other way round, $(f \cdot a)\leq (f \cdot b)$ implies $a \leq b$; the reader can show this in a similar way.

Now the monotonicity of $f_n$ ($n > 1$) follows easily. In fact, if $a R_k b$, then for every $i$, $a_i R_k b_i$, and thus $f_1(a_i) R_k f_1(b_i)$, which implies $(f \cdot a) R_k^\ast (f \cdot b)$, since $f$ respects subordination.
To show the lift property for $f_n \ (n > 1)$, assume $(f \cdot a)R^n_k c$. Then for every $i$, $f_1(a_i)R_k c_i$, and thus by the lift property of $f_1$, there exists $b_i$ such that $a_i R_k b_i$ and $c_i = f_1(b_i)$. Put $b := (b_1, \ldots, b_n)$; then a sub $b$. In fact, since $f$ is a fibrewise bijection, we have

$$a_i = a_j \iff f_1(a_i) = f_1(a_j)$$

and

$$b_i = b_j \iff c_i = c_j.$$ 

Since $(f \cdot a) sub c$, we also have

$$f_1(a_i) = f_1(a_j) \Rightarrow c_i = c_j.$$ 

Thus $a_i = a_j$ implies $b_i = b_j$. So we obtain $aR^n_k b$ and $f_n(b) = c$. 

\begin{definition}
Let $f = (f_0, f_1) : \mathbb{F} \longrightarrow \mathbb{G}$. For Kripke bundle models $M = (\mathbb{F}, \xi)$, $M' = (\mathbb{F}', \xi')$ we say that $f$ is an $\equiv$-morphism from $M$ to $M'$ (notation: $f : M \longrightarrow^\equiv M'$) if

$$M, u \models P(a) \iff M', f_0(u) \models P(f_1 \cdot a).$$

for any $P \in PL^n$, $a \in D^n_u$, $n > 0$; or in another notation:

$$a \in \xi^+(P) \iff (f_1 \cdot a) \in \xi'^+(P),$$

and also

$$M, u \models P \iff M', f_0(u) \models P$$

for any $P \in PL^0$.

Then we obtain an analogue of Lemma 3.3.11

\begin{lemma}
If $(f_0, f_1) : M \longrightarrow^\equiv M'$ then for any $u \in M$ and for any $D_u$-sentence $A$

$$M, u \models A \iff M', f_0(u) \models f_1 \cdot A,$$

where $f_1 \cdot A$ is obtained from $A$ by replacing every $c \in D_u$ with $f_1(c)$. $D_u$-

\end{lemma}

\begin{proof}
By induction. Let us consider only the $\Box$-case, i.e. $A = \Box_i B(a)$, where $a \in D^n_u$. We also assume that $n > 0$; the easy case $n = 0$ is left for the reader.

So suppose $M, u \not\models \Box_i B(a)$, $a \in D^n_u$, and let us show that

$$M', f_0(u) \not\models \Box_i B(f_1 \cdot a).$$

We have $v \in R_i(u)$, $b \in D^n_u$ such that $M, v \not\models B(b)$ and $aR^n_i b$. By the induction hypothesis, we obtain $M', f_0(v) \not\models B(f_1 \cdot b)$. We also have $(f_1 \cdot a)R^n_i (f_1 \cdot b)$ by Lemma 5.4.4, and $f_0(u)R_i f_0(v)$, since $f_0$ is a morphism. Hence $M', f_0(u) \not\models \Box_i B(f_1 \cdot a)$.

\end{proof}

\begin{footnote}
Cf. Section 5.3.
\end{footnote}
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For the converse, suppose $M', f_0(u) \not\models \Box_i B(f_1 \cdot a)$. Then there exist $w$ and $c \in D^i_{u'}$ such that

$$(f_1 \cdot a)R_i^n c, f_0(u)R_i'w, M', w \not\models B(c).$$

By Lemma 5.4.4, $c = f_1 \cdot b$ for some $b \in R_i^n(a)$, and (say, from $a_1R_i b_1$) it easily follows that $uR_i v$, where $v$ is the world of $b$. Hence $M, v \not\models B(b)$ by the induction hypothesis, and thus $M, u \not\models \Box_i B(a)$.

Lemma 5.4.7 Let $(f_0, f_1) : \mathbb{F} \rightarrow= \mathbb{G}$, and let $M'$ be a Kripke bundle model over $\mathbb{G}$. Then there exists a model $M$ over $\mathbb{F}$ such that $(f_0, f_1) : M \rightarrow= M'$.

Proof If $M' = (G, \psi')$, one can take

$$\psi^+(P) := \{a \mid f_1 \cdot a \in \psi'^+(P)\}$$

for $P \in \mathbb{P} \mathbb{L}^n$, $n > 0$, and

$$\psi^+(P) := \{u \mid f_0(u) \in \psi'^+(P)\}$$

for $P \in \mathbb{P} \mathbb{L}^0$.

Hence we obtain an analogue of Proposition 3.3.13.

Proposition 5.4.8 If there exists a $\rho$-morphism $f : \mathbb{F} \rightarrow= \mathbb{F}'$, then $\mathbb{M} \mathbb{L}_\rho^\mathbb{F}(\mathbb{F}) \subseteq \mathbb{M} \mathbb{L}_\rho^\mathbb{F}'(\mathbb{F}')$ and thus, $\mathbb{M} \mathbb{L}_\rho^\mathbb{F}(\mathbb{F}) \subseteq \mathbb{M} \mathbb{L}_\rho^\mathbb{F}'(\mathbb{F}')$.

Proof Let $(f_0, f_1) : \mathbb{F} = (F, D, \rho) \rightarrow= \mathbb{F}' = (F', D', \rho')$, and assume that $\mathbb{F}' \not\models A(x_1, \ldots, x_n)$. Then for some $u' \in F'$, $a_1, \ldots, a_n \in D_{u'}^i$, for some model $M'$ over $\mathbb{F}'$ we have

$$M', u' \not\models A(a_1, \ldots, a_n).$$

By the previous lemma, there exists a model $M$ over $\mathbb{F}$ such that

$$(f_0, f_1) : M \rightarrow M'.$$

Since $f_0$ and all $f_{1u}$ are surjective, there exist $u, a_1, \ldots, a_n$, such that

$$u' = f_0(u), a'_1 = f_{1u}(a_1), \ldots, a'_n = f_{1u}(a_n),$$

and thus by Lemma 5.4.6, we obtain $M, u \not\models A(a_1, \ldots, a_n)$.

Therefore, $\mathbb{F}' \not\models A$ implies $\mathbb{F} \not\models A$.

A particular case of an $\rho$-morphism is an inverse image morphism obtained by the pullback construction as follows.

Lemma 5.4.9 Let $\mathbb{F} = (F, D, \rho)$ be an $N$-modal Kripke bundle, $f : G \rightarrow= F$ a morphism of propositional frames, and put

$$X := \{(u', a) \mid u' \in G, a \in D^1, f(u') = \pi(a)\},$$
where $\pi : F_1 \rightarrow F_0$ corresponds to $\mathbb{F}$. Consider the relations on $X$ defined as follows.

$$(u', a)\rho'_i(v', b) \iff u'R_i'v' \& a\rho_ib,$$

where $\rho_i$ and $R'_i$ are the relations respectively in $F^1$ and $G$. Let also

$$\pi'(u', a) := u', \ \varphi(u', a) := a.$$

Then

1. $\pi' : (X, \rho'_1, \ldots, \rho'_N) \rightarrow F'$;
2. $(f, \varphi) : G \rightarrow= \mathbb{F}$, where $G$ corresponds to $\pi'$.

**Proof** In all the cases the monotonicity is obvious.

To check the lift property for $\pi'$, suppose $u'R_i'v'$. Then $f(u')R_i'f(v')$ by monotonicity, and thus there exists $b$ such that $a\rho_ib$ and $\pi(b) = f(v')$. Hence $(v', b) \in X$, $\pi'(v', b) = b$, $(u', a)\rho'_i(v', b)$ as required.

It is clear that $\varphi$ is a fibrewise bijection; in fact,

$$a \in D_{f(u')} \iff \pi(a) = f(u') \iff (u', a) \in X \iff (u', a) \in D'_{u'}.$$

Checking the lift property for $\varphi$ is left to the reader.

For the Kripke bundle $G$ constructed in Lemma 5.4.9 we shall use the notation $f_*\mathbb{F}$ and say that it is obtained by *change of the base along $f$.

A particular case of this construction is a generated subbundle. Explicitly it is defined as follows.

**Definition 5.4.10** Let $F \upharpoonright V$ be a generated subframe of a propositional frame $F$, $\mathbb{F} = (F, D, \rho)$ a Kripke bundle over $F$. Then we define

$$\mathbb{F} \upharpoonright V := (F', D', \rho'),$$

where $F' = F \upharpoonright V$, $D' = D \upharpoonright V$, $\rho'$ is an appropriate restriction of $\rho$, i.e. $\rho'_{uv} = \rho_{uv}$ for $(u, v) \in R_i \cap (V \times V)$. $\mathbb{F} \upharpoonright V$ is called a generated subbundle of $\mathbb{F}$ (more precisely, the restriction of $\mathbb{F}$ to $V$).
Lemma 5.4.11 Let \( j : F \upharpoonright V \rightarrow F \) be the inclusion morphism, \( \varphi : D^1 \rightarrow D^1 \) the inclusion map. Then \( (j, \varphi) : F \upharpoonright V \rightarrow F \).

Proof An easy exercise. \hfill \blacksquare

Exercise 5.4.12 Show that \( F \upharpoonright V \) is isomorphic to \( j_* F \).

In particular, if \( F \upharpoonright V \) is a cone \( F \uparrow u \), then \( F \upharpoonright V \) is also called the cone of \( F \) and denoted by \( F \uparrow u \).

Lemma 5.4.13

\[
\text{ML}^{=} (F) = \bigcap \{ \text{ML}^{=} (F \uparrow u) \mid u \in F \}.
\]

Proof Follows readily from Lemmas 5.4.6 and 5.4.7; an exercise for the reader (cf. the proof of Lemma 1.3.26). \hfill \blacksquare

Lemma 5.4.14 Let \( F \) be a Kripke bundle, \( F \upharpoonright V \) its generated subbundle. Then

\[
\text{ML}^{=} (F) \subseteq \text{ML}^{=} (F \upharpoonright V).
\]

Proof Follows for the previous lemma and the observation that every cone in \( F \upharpoonright V \) is a cone in \( F \). \hfill \blacksquare

Finally let us show that \( G = f_* F \) gives rise to the pullback (or ‘coamalgam’) diagram:

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & F \\
\downarrow{\pi'} & & \downarrow{\pi} \\
G & \xrightarrow{f} & F
\end{array}
\]

In precise terms, this means the following.

Lemma 5.4.15 Let \( f : G \rightarrow F \) be a morphism of propositional frames, \( F \) a Kripke bundle over \( F \), \( \pi : F_1 \rightarrow F \) the associated p-morphism, \( F = f_* F \) and \( (f, \varphi) : G \rightarrow F \) the \( = \)-morphism described in 5.4.9. Also let \( (f, \psi) : H \rightarrow F \) be an arbitrary \( = \)-morphism from a Kripke bundle \( H \) over \( G \) to \( F \).

Then

(1) there exists a unique map \( g : G_1 \rightarrow F_1 \) such that the following diagram (where \( \tau \) is associated with \( H \)) commutes.
(2) $g$ is an isomorphism between $H_1$ and $G_1$.

**Proof**  
(i) The diagram commutes iff for any $x \in H$, $\pi'(g(x)) = \tau(x)$ and $\varphi(g(x)) = \psi(x)$, i.e. iff $g(x) = (\tau(x), \psi(x))$; note that $(\tau(x), \psi(x)) \in G_1$ since $\pi \cdot \psi = f \cdot \tau$. So $g$ is well-defined.

(ii) We can consider the category $\mathcal{P}$ of Kripke bundle morphisms of the form $(f, \psi) : H \to F$, where $H$ is an arbitrary Kripke bundle over $G$. A morphism (in $\mathcal{P}$) from $(f, \psi) : H \to F$ to $(f', \psi') : H' \to F$ is defined as a map $h$, for which the following diagram (where $\tau' : H_1 \to G$ is associated with $H'$) commutes.

The assertion (i) shows that our specific $(f, \varphi)$ is a terminal object of $\mathcal{P}$. But actually the above argument applies to any $(f, \varphi) : G \to F$ from $\mathcal{P}$.

In fact, we only need to find a unique $y \in G_1$ such that $\pi'(y) = \tau(x) \& \varphi(y) = \psi(x)$. So it suffices to show that

$$\exists y \in G_1(\pi'(y) = u \& \varphi(y) = a)$$

whenever $f(u) = \pi(a)$. But this holds since $\varphi$ is a fibrewise bijection between $\pi'^{-1}[u]$ and $\pi^{-1}[f(u)]$. 


Thus all objects in \( P \) are terminal, and all morphisms are invertible. So the map \( g \) from (i) is bijective. To complete the proof of (ii), it remains to show the monotonicity of \( g \); the monotonicity \( g^{-1} \) follows by the same argument. So let \( S_i \) be the relations in \( H_1 \). Then for any \( y, z \in H_1 \)

\[
yS_i z \text{ implies } \psi(y) \rho_i \psi(z), \text{ i.e. } \varphi(g(y)) \rho_i \varphi(g(z)),
\]

and similarly

\[
yS_i z \Rightarrow \pi'(g(y))R'_i \pi'(g(z)).
\]

Thus

\[
yS_i z \Rightarrow g(y)\rho'_i g(z)
\]

by definition (see Lemma 5.4.9).

5.5 Intuitionistic Kripke bundles

Now let us consider the intuitionistic case. First note the following:

Lemma 5.5.1 If \( F \) is an intuitionistic Kripke bundle, then all the \( F_n \) are \( S_4 \)-frames.

Proof An exercise.

Definition 5.5.2 A valuation \( \xi \) in an intuitionistic Kripke bundle \( F \) is called intuitionistic if every set \( \xi^+(P^n_k) = \bigcup \{ \xi_u(P^n_k) \mid u \in W \} \) is stable in \( F_n \), i.e. if \( R^n(\xi^+(P^n_k)) \subseteq \xi^+(P^n_k) \). The model \( (F, \xi) \) is also called intuitionistic in this case.

Intuitionistic forcing in intuitionistic Kripke bundle models is defined via modal forcing similarly to the case of Kripke frames (Definition 3.2.12):

Definition 5.5.3 Let \( M \) be an intuitionistic model over an intuitionistic Kripke bundle \( F = (F, D, \rho) \), \( F = (W, R), u \in M \), \( A \) an intuitionistic \( D_u \)-sentence. Then we put

\[
M, u \vdash A := M, u \vdash A^T.
\]

Now we obtain an analogue of Lemma 3.2.13:

Lemma 5.5.4 Under the conditions of Definition 5.5.3

1. \( M, u \vdash B \iff M, u \vdash B \) (for \( B \) atomic);
2. \( M, u \vdash B \land C \iff (M, u \vdash B \land M, u \vdash C) \);
3. \( M, u \vdash B \lor C \iff (M, u \vdash B \lor M, u \vdash C) \);
4. \( M, u \vdash B(a) \supset C(a) \iff \forall v \in R(u) \forall b \in D^n_v (aR^n b \land M, v \vdash B(b) \Rightarrow M, v \vdash C(b)) \);
(5) $M, u \models \exists x B$ iff $\exists a \in D_u \ M, u \models [a/x] B$;

(6) $M, u \models \forall x B(x, a)$ iff

\[ \forall v \in R(u) \ \forall d \in D_v \ \forall b \in D_v^n \ (aR^n b \Rightarrow M, v \models B(d, b)) \]

(7) $M, u \models \neg B(a)$ iff $\forall v \in R(u) \ \forall b \in D_v^n \ (aR^n b \Rightarrow M, v \models B(b))$;

(8) $M, u \models a \neq b$ iff $a$ does not equal $b$.

Here we assume that in all cases we evaluate $D_u$-sentences at $u$. In (4), (6) and (7) we also assume that $a \in D_u^n$ is a list of individuals, which replaces a list of variables containing the parameters respectively of $B \supset C$ or $\forall x B$, $B$.

Proof Let us consider only the case (4); the others are an exercise for the reader. In the fixed model $M$ we have

\[
\begin{align*}
&u \models B(a) \supset C(a) \text{ iff } \\
u \models (B(a) \supset C(a))^T = \Box(B(a)^T \supset C(a)^T) \text{ by Definitions 5.5.3, 2.11.1} \\
&\text{iff } \forall v \in R(u) \ \forall b \in D_v^n \ (aR^n b \Rightarrow v \models B(b)^T \supset C(b)^T) \text{ by Lemma 5.3.4} \\
&\text{iff } \forall v \in R(u) \ \forall b \in D_v^n \ (aR^n b \text{ and } v \models B(b) \Rightarrow v \models C(b)) \text{ by Definitions 5.5.3, 5.2.8.}
\end{align*}
\]

As in the modal case, we can drop $u$ from the notation; so e.g. (4), (6) are written as follows:

(4) $M \models (B \supset C)(a)$ iff $\forall b(aR^n b \text{ and } M \models B(b) \Rightarrow M \models C(b))$;

(6) $M \models \forall x B(x, a)$ iff $\forall d \forall b \ ((db) \in D^{n+1} \text{ and } aR^n b \Rightarrow M \models B(d, b))$.

Now we obtain some facts similar to those proved in Section 3.2.

Lemma 5.5.5 Let $M$ be the same as in Lemma 5.5.4. Then for any $D^1$-sentence $A(a)$

$M \models A(a) \text{ and } aR^n b \Rightarrow M \models A(b)$.

Proof By Definition 5.5.3 and Lemma 5.3.4.

Definition 5.5.6 Let $\mathbb{F}$ be an intuitionistic Kripke bundle; $M$ a Kripke bundle model over $\mathbb{F}$. The pattern of $M$ is the Kripke bundle model $M_0$ over $\mathbb{F}$ such that for any $u \in \mathbb{F}$ and any atomic $D_u$-sentence $A$

$M_0, u \models A$ iff $M, u \models \Box A$.

It is clear that the model $M_0$ is intuitionistic.

Lemma 5.5.7 Under the conditions of Definition 5.5.6, we have for any $u \in \mathbb{F}$, for any $D_u$-sentence $A$

$M_0, u \models A$ iff $M, u \models A^T$. 

5.5. INTUITIONISTIC KRPKE BUNDLES

Proof By induction, similar to Lemma 3.2.15. Let us consider only the case $A = B(a) \supset C(a)$. We have

$M_0, u \vdash A$ iff

$\forall v \in R(u) \forall b \in D_v^n (aR^n b \& M_0, v \vdash B(b) \Rightarrow M_0, v \vdash C(b))$ (by Lemma 5.5.4)

if $\forall v \in R(u) \forall b \in D_v^n (aR^n b \& M, v \vdash B(b)^T \Rightarrow M, v \vdash C(b)^T)$

(by the induction hypothesis)

iff $M, u \vdash (B(a)^T \supset C(a)^T)$ (by Lemma 5.3.4 and Definition 5.2.8).

The latter formula is $A^T$ by Definition 2.11.1.

Lemma 5.5.8 Let $M$ be an intuitionistic Kripke bundle model with the accessibility relation $R$, $A(x)$ an intuitionistic formula with all its parameters in the list $x, |x| = n$. Then for any $u \in M$

$M, u \vdash \forall x A(x) \iff \forall v \in R(u) \forall a \in D_v^n M, u \vdash A(a)$.

Proof The claim is similar to Lemma 3.2.18, but the proof is different; now it is based on soundness (Proposition 5.2.12).

By Definition 3.2.12, in the given model $M$ we have:

$u \vdash \forall x A(x) \iff u \vdash (\forall x A(x))^T$.

By Lemma 2.11.7, in QS4 the latter formula is equivalent to $\square \forall x A(x)^T$, and thus

$u \vdash (\forall x A(x))^T \iff u \vdash \square \forall x A(x)^T$ (by Proposition 5.2.12)

iff $\forall v \in R(u) \forall a \in D_v^n v \vdash A(a)^T$ (by Definition 5.2.8 and Lemma 5.2.10)

iff $\forall v \in R(u) \forall a \in D_v^n v \vdash A(a)$ (by Definition 5.5.3).

$\square$

Exercise 5.5.9 Let $M$ be an intuitionistic Kripke bundle model with the accessibility relation $R$, $u \in M$, $A(x, y)$ an intuitionistic formula with $FV(A) \subseteq xy$, $|x| = n, |y| = m$. Then for any $u \in M, c \in D_u^m$

$M, u \vdash \forall x A(x, c) \iff \forall v \in R(u) \forall a \in D_v^n \forall d \in D_v^n (cR^n d \Rightarrow M, u \vdash A(a, d))$.

Definition 5.5.10 An intuitionistic predicate formula $A$ is called

- true in an intuitionistic Kripke bundle model $M$ (notation: $M \vDash A$) if $\forall A$ is true at every world of $M$;

- valid in an intuitionistic Kripke bundle $\mathcal{F}$ (notation: $\mathcal{F} \vDash A$) if it is true in all intuitionistic Kripke models over $\mathcal{F}$;

- strongly valid in an intuitionistic Kripke bundle $\mathcal{F}$ (notation: $\mathcal{F} \vDash^+ A$) if all its substitution instances are valid in $\mathcal{F}$.
Lemma 5.5.11 $M \models A(x_1,\ldots,x_n)$ iff $\forall u \in M \forall a_1,\ldots,a_n \in D_u \ M,u \models A(a_1,\ldots,a_n)$.

Proof Similar to Lemma 3.2.20.

Proposition 5.5.12 Let $\mathbb{F}$ be an S4-based Kripke bundle, $A \in I\mathbb{F}^\approx$. Then

1. $\mathbb{F} \models A$ iff $\mathbb{F} \models A^T$;

2. the following three assertions are equivalent:

   (a) $\mathbb{F} \models A$,
   (b) $\forall m \mathbb{F} \models A^m$,
   (c) $\mathbb{F} \models A^T$.

Proof

1. Let $A = A(x)$, $|x| = n$.

   (Only if.) Assume $\mathbb{F} \models A$. Let $M$ be a Kripke bundle model over $\mathbb{F}$. By Lemma 5.5.11, we have to show $M,u \models A^T(a)(= A(a)^T)$ for any $u \in \mathbb{F}$, $a \in D_u^n$. By Lemma 5.5.7 we have:

   $M_0,u \models A(a)$ iff $M,u \models (a)^T$.

   By Lemma 5.5.11, $\mathbb{F} \models A$ implies $M_0,u \models A(a)$, and thus we obtain $\mathbb{F} \models A^T$.

   (If.) Assume $\mathbb{F} \models A^T$. Then for any intuitionistic model $M$, any $u \in M$, $a \in D_u^n$ we have $M,u \models A(a)^T(= A^T(a))$ by Lemma 5.5.11, i.e. $M,u \models A(a)$. Hence $\mathbb{F} \models A$, by Lemma 5.5.11.

2. (a)⇒(b) is obvious.

(b)⇒(c). Assume $\mathbb{F} \models A^m$, i.e. $\mathbb{F} \models (A^m)^T$. Since $(A^m)^T = (A^T)^m$, we obtain $\mathbb{F} \models A^T$ by Proposition 5.2.12.

(c)⇒(a). Assume $\mathbb{F} \models A^T$, and for an arbitrary $I\mathbb{F}^\approx$-substitution $S$ let us show $\mathbb{F} \models SA$, i.e. $\mathbb{F} \models (SA)^T$. By Lemma 2.11.5, $(SA)^T \equiv S^T(A^T)$ is a QS4$^\approx$-theorem, and thus it is valid in $\mathbb{F}$, by Proposition 5.2.12. By assumption, $\mathbb{F} \models S^T(A^T)$; thus we obtain $\mathbb{F} \models (SA)^T$. Therefore $\mathbb{F} \models A^+$.

Proposition 5.5.13 For an intuitionistic Kripke bundle $\mathbb{F}$ the set of strongly valid formulas

$$\mathit{IL}^{(=)}(\mathbb{F}) := \{A \in I\mathbb{F}^{(=)} \mid \mathbb{F} \models A\}$$

is a superintuitionistic predicate logic (with or without equality), moreover,

$$\mathit{IL}^{(=)}(\mathbb{F}) = T\mathit{ML}^{(=)}(\mathbb{F}).$$

Proof Similar to Proposition 3.2.29. By Proposition 5.5.12, $\mathbb{F} \models A$ iff $\mathbb{F} \models A^T$, which means $A \in \mathit{IL}^{(=)}(\mathbb{F})$ iff $A^T \in \mathit{ML}^{(=)}(\mathbb{F})$. Thus $\mathit{IL}^{(=)}(\mathbb{F}) = s(\mathit{ML}^{(=)}(\mathbb{F}))$. ■
Definition 5.5.14 The set \( \text{IL}(\equiv)(\mathcal{F}) \) is called the superintuitionistic logic of \( \mathcal{F} \) (respectively, with or without equality).

Now we easily obtain intuitionistic analogues of some assertions from the previous section.

Lemma 5.5.15 Let \((f_0, f_1) : M \rightarrow M'\) for intuitionistic Kripke bundle models \(M, M'\). Then for any \(u \in M\) and for any intuitionistic \(D_u\)-sentence \(B\)

\[ M, u \models B \text{ iff } M', f_0(u) \models f_1 \cdot B, \]

Proof From Lemma 5.4.6, Definition 5.5.3 and the observation that \(f_1 \cdot B^T = (f_1 \cdot B)^T\).

Proposition 5.5.16 If \(\mathcal{F}\) and \(\mathcal{G}\) are intuitionistic Kripke bundles and \(\mathcal{F} \rightarrow \mathcal{G}\), then \(\text{IL}(\equiv)(\mathcal{F}) \subseteq \text{IL}(\equiv)(\mathcal{G})\) and \(\text{IL}(\equiv)(\mathcal{F}) \subseteq \text{IL}(\equiv)(\mathcal{G})\).

Proof The first statement is an exercise for the reader. For the second, note that \(\mathcal{F} \rightarrow \mathcal{G}\) implies \(\text{ML}(\equiv)(\mathcal{F}) \subseteq \text{ML}(\equiv)(\mathcal{G})\) by Proposition 5.4.8; hence \(s(\text{ML}(\equiv)(\mathcal{F})) \subseteq s(\text{ML}(\equiv)(\mathcal{G}))\), i.e. \(\text{IL}(\equiv)(\mathcal{F}) \subseteq \text{IL}(\equiv)(\mathcal{G})\), by Proposition 5.5.13.

Lemma 5.5.17 For an intuitionistic Kripke bundle \(\mathcal{F}\)

\[ \text{IL}(\equiv)(\mathcal{F}) = \bigcap \{ \text{IL}(\equiv)(\mathcal{F} \upharpoonright u) \mid u \in F \}. \]

Proof Follows readily from Lemma 5.4.13, Proposition 5.5.13, and the equality \(s \left( \bigcap L_i \right) = \bigcap s(L_i)\).

Lemma 5.5.18 Let \(\mathcal{F}\) be an intuitionistic Kripke bundle, \(\mathcal{F} \upharpoonright V\) its generated subbundle. Then

\[ \text{IL}(\equiv)(\mathcal{F}) \subseteq \text{IL}(\equiv)(\mathcal{F} \upharpoonright V). \]

Proof Similar to Lemma 5.4.14.

For the intuitionistic case the notion of a Kripke bundle can be slightly extended.

Definition 5.5.19 An intuitionistic Kripke quasi-bundle is a quasi-\(p\)-morphism \(\pi : F_1 \rightarrow F_0\) between \(\mathsf{S4}\)-frames, cf. Definition 1.4.18.\(^5\)

A Kripke quasi-bundle can be presented equivalently as a triple \(\mathcal{F} = (F_0, D, \rho)\) with a system of domains \(D = (D_u \mid u \in W)\) and a family of relations \(\rho = (\rho_{uv} \mid uRv)\) satisfying (\(\#_2\)), (\(\#_3\)) from Definition 5.2.2 and

\((\#_1')\) \(uRv \Rightarrow \forall a \in D_u \exists v' \approx_R v \exists b \in D_v \alpha\rho_{uv'b} \).

\(^5\)Recall that quasi-\(p\)-morphisms are intuitionistic analogues of \(p\)-morphisms, cf. Chapter 1.
However one should be careful about using quasi-bundles. Since an intuitionistic quasi-bundle is not necessarily a modal bundle, it is not a surprise that the semantics of quasi-bundles is not sound for modal logics. But it is not sound for intuitionistic logic either! To obtain soundness, we have to reduce the class of quasi-bundles, see Section 5.17 for details.

**Exercise 5.5.20** Try to give an inductive definition of intuitionistic forcing in a Kripke quasi-bundle (intuitionistic valuations are defined similarly to Kripke bundles). Note that the definition of forcing is not so straightforward.

A familiar truth definition for a formula $\exists x A$ is irrelevant in this case, as the following counterexample shows.

**Example 5.5.21** Consider a quasi-bundle $\mathbb{F}$ pictured at Fig. 5.5. $b_1 \in D_{u_1}$ does not have inheritors in $u_2$, but $b_1$ is its own inheritor in $u_1 \approx_R u_2$. So $\mathbb{F}$ is still a quasi-bundle.

\[ \begin{array}{c}
\text{Figure 5.5. } R \text{ is universal; } \rho \text{ is universal on } \{a_1, a_2\}. \\
\end{array} \]

Consider an intuitionistic model $M = (\mathbb{F}, \xi)$ such that $\xi^+(P) = \{b\}$. Putting

\[ M, u \models \exists x P(x) \iff \exists c \in D_u M, u \models P(c) \]

we obtain

\[ M, u_1 \models \exists x P(x), \]

but

\[ M, u_2 \not\models \exists x P(x). \]

So since $u_1Ru_2$, there is no truth-preservation for $\exists x P(x)$.

But as we shall soon see (cf. Lemma 5.14.15), truth-preservation (or 'monotonicity') is necessary for soundness.
So here is an alternative definition:

\[ M, u \models \exists x B(x, a) \iff \exists v \approx u \exists d \in D_v \exists b \in D_v^n(b \approx^n a \& M, v \models B(d, b)). \quad (*) \]

Due to the quasi-lift property (1.4.18) of quasi-bundles, (*) can be rewritten as

\[ M, u \models \exists x B(x, a) \iff \exists vRu \exists d \in D_v \exists b \in D_v^n(b R^n a \& M, v \models B(d, b)). \quad (**) \]

Note that this definition for the \( \exists \)-case is dual to the clause for the \( \forall \)-case in Lemma 5.5.4(6).

The modified definition provides the desired truth-preservation property, as we shall see later on.

**Lemma 5.5.22** Let \( M = (F, \xi) \) be an intuitionistic Kripke bundle model, and let \( \xi^n \) be a propositional valuation in its \( n \)-th level \( F_n \), \( n \geq 0 \) such that \( \xi^n(p_k) = \xi^+(P^n_k) \) for any \( k \geq 0 \). Let \( M_n = (F_n, \xi^n) \) be the corresponding propositional Kripke model. Then every \( M_n \) is intuitionistic, and for any intuitionistic propositional formula \( A \) and for any \( a \in M_n \)

\[ M_n, a \models A \iff M \models A^n(a). \]

**Proof** \( M_n \) is intuitionistic, according to Definition 5.5.2.

\[ M_n, a \models A \iff M_n, a \models A^T \quad \text{(by Definition 1.4.1)} \]
\[ \iff M \models (A^T)^n(a) = (A^n(a))^T \quad \text{(by Lemma 5.3.5)} \]
\[ \iff M \models A^n(a) \quad \text{(by Definition 5.5.3)}. \]

**Lemma 5.5.23** Let \( F \) be an intuitionistic Kripke bundle, \( A \) a propositional intuitionistic formula. Then for any \( n \geq 0 \)

\[ F \models A^n \iff F_n \models A. \]

**Proof** \( F \models A^n \iff F \models (A^n)^T = (A^T)^n \) (by Proposition 5.5.12) if \( F_n \models A^T \) (by Lemma 5.3.6) if \( F_n \models A \) (by Lemma 1.4.7).

**Proposition 5.5.24** For an intuitionistic Kripke bundle \( F \),

\[ \text{IL}_k^{=} (F) = \bigcap_{n \in \omega} \text{IL}(F_n). \]

**Proof** \( \forall n \ F_n \models A \iff \forall n \ F_n \models A^T \) (by Lemma 1.4.7) if \( F \models A^T \) (by Proposition 5.3.7) if \( F \models A \) (by Proposition 5.5.12).

Let us also make some remarks on Kripke quasi-sheaves.

**Lemma 5.5.25** Let \( F \) be a quasi-sheaf over an (intuitionistic) frame \( F \). Assume that \( u \approx_R v \) in \( F \), i.e. \( u, v \) are in the same cluster. Then \( \rho_{uv} \) is a bijection between \( D_u \) and \( D_v \) with the converse \( \rho_{vu} \).
Proof First, note that \( b \rho_{uv} a \) implies \( b \rho a \). In fact, by Definition 5.2.1, there exists \( a' \) such that \( b \rho_{uv} a' \). Then by transitivity, \( b \rho a' \), and thus \( a = a' \), since \( F \) is a quasi-sheaf. Obviously, the same argument shows that \( b \rho a \) implies \( b \rho_{uv} a \), and thus \( \rho_{uv} = \rho_{uv}^{-1} \). It remains to show that both \( \rho_{uv} \), \( \rho_{vu} \) are functions, and again we can check this only for \( \rho_{uv} \). So assume \( a_0 \rho_{uv} a_1, a_0 \rho_{uv} a_2 \); then \( a_1 \rho_{uv} a_0 \), and thus by transitivity, \( a_1 \rho_{uv} a_2 \), which implies \( a_1 = a_2 \), by Definition 5.2.3. ■

So, similarly to the case of Kripke sheaves (Section 3.5), we can factorise a quasi-sheaf \( F \) and obtain a quasi-sheaf over the partially ordered frame \( F^- \):

**Lemma 5.5.26** Let \( F = (F, D, \rho) \) be the same as in Lemma 5.5.25, with \( F = (W, R) \) and consider the corresponding frame of individuals \( F^1 = (D^1, R^1) \); let \( \pi : F^1 \to F \) be the associated \( p \)-morphism. Let \( F^- = (D^-, R^-) \), \( (F^1)^- = ((D^1)^-, (R^1)^-) \) be their skeletons (Definition 1.3.39). Then

1. The map \( \pi^- : a^- \mapsto \pi(a^-) \) is a \( p \)-morphism (and thus defines a Kripke bundle \( F^- \)).

2. \( F^- \) is a Kripke quasi-sheaf.

3. There exists a \( p=-\)-morphism \( F \to F^- \).

**Proof** (i) \( \pi \) is well-defined. In fact, \( a \approx_{R^1} b \) only if \( \pi(a) \approx_R \pi(b) \), due to the monotonicity of \( \pi \).

Next, \( a^- (R^1)^- b^- \) iff \( a R^1 b \) (by Definition 1.3.39), only if \( \pi(a) R \pi(b) \) (since \( \pi \) is \( p \)-morphism), iff \( \pi(a)^- R^- \pi(b)^- \) (by Definition 1.3.39).

To check the lift property for \( \pi^- \), suppose \( \pi^- R^- v^- \), i.e. \( \pi(a) R v \), by Definition 1.3.39. Since \( \pi \) is a \( p \)-morphism, there exists \( b \) such that \( a R^1 b \) and \( \pi(a) = v \), and thus \( a^- (R^1)^- b^- \), \( v^- = \pi(b)^- = \pi^- (b^-) \).

(ii) Suppose \( \pi^- (a^-) = \pi^- (b^-) \), i.e. \( \pi(a) \approx_{R^1} \pi(b) \) and also \( (a^-) (R^1)^- (b^-) \), i.e. \( a R^1 b \). To obtain \( a^- = b^- \), we have to show that \( b R^1 a \).

By the lift property, there exists \( a' \) such that \( b R^1 a' \) and \( \pi(a') = \pi(a) \). But then \( a R^1 a' \) by transitivity, and since \( F \) is a Kripke quasi-sheaf, it follows that \( a' = a \). Hence \( b R^1 a \).

(iii) Let us show that \( (f_0, f_1) : F \longrightarrow F^- \), where \( f_0(u) := u^- \), \( f_1(a) := a^- \).

In fact, these maps are propositional \( p \)-morphisms, by Lemma 1.3.40, so it remains to show that \( (f_0, f_1) \) is a fibrewise bijection, i.e. that \( f_1 \) restricted to \( D_u \) is a bijection.

To show the injectivity, suppose \( \pi(a) = \pi(b) = u \), but \( a \neq b \). Then \( a^- \neq b^- \), since in Kripke quasi-sheaves different individuals in the same fibre are not related.

For the surjectivity, note that an arbitrary element of \( (\pi^-)^{-1}(u^-) \) is of the form \( b^- \), with \( u^- = \pi^- (b^-) = \pi(b)^- \). Let us find \( a \in D_u \) such that \( a^- = b^- \).

Let \( v := \pi(b) \); then \( u \approx_R v \), and by Lemma \( \rho_{vu} \) is a bijection between \( D_u \) and \( D_v \), whose converse is \( \rho_{uv} \). Now take \( a := \rho_{vu} \); it follows that \( a \approx_{R^1} b \), and \( a \in D_u \).

**Definition 5.5.27** \( F^- \) is called the skeleton of \( F \).
5.5. INTUITIONISTIC Kripke BUNDLES

So we obtain an analogue of Lemma 3.7.18:

**Lemma 5.5.28** $\text{IL(}=\text{(F)} = \text{IL(}=\text{(F(}=\text{)})$ for a Kripke quasi-sheaf $\text{F}$.

**Proof** The inclusion $\text{IL(}=\text{(F)} \subseteq \text{IL(}=\text{(F(}=\text{)})$ follows from Lemma 5.5.26 and Proposition 5.5.6. To prove the converse, suppose $M = (\text{F}, \varphi)$ is an intuitionistic model, $A$ is an intuitionistic sentence and $M \not\models A$, and let us show $\text{F(}=\text{)} \not\models A$.

Let $f : \text{F} \to \text{F(}=\text{)}$ (Lemma 5.5.26), and let us define a valuation $\psi$ in $\text{F(}=\text{)}$ such that

$$\psi^+(P) = \{ f \cdot a \mid a \in \varphi^+(P) \}.$$

for any predicate letter $P$. Then $\psi$ is an intuitionistic valuation. In fact, suppose $c = f \cdot a \in \psi^+(P)$, $c(R^=)n d$, and let us show that $d \in \psi^+(P)$.

By Lemma 5.4.4, there exists $b \in R^n(a)$ such that $d = f \cdot b$ (Fig. 5.6). Then from $aR^n b$ we obtain $b \in \varphi^+(P)$. Thus by definition, $d \in \psi^+(P)$.

![Figure 5.6.](image)

So $\psi$ is intuitionistic, and we also have $f : (\text{F}, \varphi) \to (\text{F(}=\text{)}, \psi)$.

In fact, by definition, $a \in \varphi^+(P)$ only if $f \cdot a \in \psi^+(P)$, and it remains to show the converse.

Suppose $f \cdot a \in \psi^+(P)$, i.e. $f \cdot a = f \cdot b$ for some $b \in \varphi^+(P)$. Then $a \approx b$ (by the definition of $f$), and thus $a \in \varphi^+(P)$, since $\varphi$ is intuitionistic.

By Lemma 5.4.6 we obtain (for a given sentence $A$ and for any world $u$):

$$M, u \models A \iff (\text{F(}=\text{)}, \psi), u^\sim \models A.$$  

Since $M \not\models A$ by our assumption, we conclude that $\text{F(}=\text{)} \not\models A$.

Thus in the intuitionistic case it is sufficient to consider Kripke quasi-sheaves only over posets.

On the other hand, for Kripke bundles analogous reductions are not always possible, because the domains of equivalent worlds may be of different cardinality. So we have to consider Kripke bundles over arbitrary quasi-ordered sets. Lemma 5.5.16 shows that we can use only cones, but this does not simplify anything, because a quasi-ordered cone may still have several equivalent roots with non-equivalent domains.
5.6 Functor semantics

In Chapter 4 we defined validity in a presheaf over a locale and showed that the presheaf semantics is equivalent to algebraic semantics. Functor semantics is based on an alternative definition of validity in a presheaf (over an arbitrary category), which is mainly due to Ghilardi, Makkai and Reyes. In this section we define this semantics in a more general polymodal setting.

Let us begin with a notion of a precategory. This is actually a coloured multigraph, or a labelled transition system. Similarly to a category (cf. 3.4.3), it consists of objects (or points) and morphisms (arrows), but a priori without composition, and without special identity morphisms. Every morphism is coloured.

**Definition 5.6.1** An $N$-precategory is a tuple $C = (X, Y, o, t, cr)$, where $X$, $Y$ are non-empty classes; $o : Y \rightarrow Y$ and $cr : Y \rightarrow \{1, \ldots, N\}$ are functions. $X$ is called the class of objects (notation: $ObC$), and $Y$ is the class of morphisms (notation: $Mor C$). $o(f)$, $t(f)$, $cr(f)$ are called respectively the origin, the target, and the colour of $f$. An $i$-morphism is a morphism of colour $i$. We also use the following notation:

- $Mor_i(C) := cr^{-1}(i)$ (the class of all $i$-morphisms),
- $C(u, v) := \{f \mid o(f) = u, t(f) = v\}$ (the class of all morphisms from $u$ to $v$),
- $C_i(u, v) := Mor_i(C) \cap C(u, v)$ (the class of all $i$-morphisms from $u$ to $v$).

We shall usually consider small precategories, where $X, Y$ are sets. In this case the Kripke frame $(W, R_1, \ldots, R_N)$ is called the frame representation of $C$ (notation: $FR(C)$) if $W = ObC$ and $R_i = \{(u, v) \mid C_i(u, v) \neq \emptyset\}$.

Note that a 1-precategory is nothing but a multigraph. It is also clear that a category is a 1-precategory expanded by identity morphisms and composition of consecutive morphisms (cf. Section 3.6); its frame representation is an $\mathbf{S}_4$-frame.

Every 1-precategory $C$ can be extended to a category $C^*$; this construction is similar to reflexive transitive closure of a binary relation. Namely, morphisms in $C^*$ are paths (sequences of consecutive morphisms) in $C$:

$$u_0 \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} u_2 \rightarrow \ldots \rightarrow u_{n-1} \xrightarrow{\alpha_n} u_n$$

(where $u_0, \ldots, u_n$ may be not all different), and we also add all identity morphisms $1_u$ (as new).

The composition of morphisms in $C^*$ is defined as the join of sequences:

$$(u \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_n} v) \circ (v \xrightarrow{\beta_1} \ldots \xrightarrow{\beta_n} w) := u \xrightarrow{\alpha_1} \ldots \xrightarrow{\beta_n} w$$

and of course we define

$$1_u \circ \varphi := \varphi \circ 1_v := \varphi$$

for any $\varphi \in C^*(u, v)$.

If a (small) category $C$ has a single object, then $Mor C^*$ is the free monoid generated by the set $Mor C$.
Definition 5.6.2 Let $C$ be a (small) $N$-precategory with the set of objects $W$. A $C$-preset is a triple $\mathfrak{F} = (C, D, \rho)$, in which $D = (D_u \mid u \in W)$ is a system of domains on $W$ and $\rho = (\rho_\alpha \mid \alpha \in \text{Mor} C)$ is a family of functions indexed by morphisms of $C$, such that $\rho_\alpha : D_u \rightarrow D_v$ for $\alpha \in C(u, v)$.

The objects of $C$ are called possible worlds of $\mathfrak{F}$, and the elements of $D_u$ are called individuals at $u$. If $d \in D_u$, $\alpha \in C(u, v)$, the image $\rho_\alpha(d)$ is called the $\alpha$-inheritor of the individual $d$ in the world $v$.

Note that an individual $d \in D_u$ can have different $\alpha$-inheritors in the same world $v$ for different morphisms $\alpha$.

Definition 5.6.3 Let $C$ be a category based on a precategory $C_0$, i.e. $C = (C_0, \circ, 1)$. A $C_0$-preset $\mathfrak{F}$ is called a $C$-set (more precisely, an intuitionistic $C$-set) if it preserves composition and identity morphisms:

\begin{align*}
(iv) \quad & \rho_\alpha \circ \beta = \rho_\alpha \circ \rho_\beta; \\
(v) \quad & \rho_{1_u} = \text{id}_{D_u};
\end{align*}

for any $\alpha, \beta \in \text{Mor} C$, $u \in \text{Ob} C$.

Let also give an alternative definition of $C$-presets.

Definition 5.6.4 A prefunctor $F : C \rightarrow C'$ from an $N$-precategory $C$ to an $N$-precategory $C'$ is a map sending objects of $C$ to objects of $C'$ and morphisms of $C$ to morphisms of $C'$ and preserving colours, origins and targets: if $f \in C_i(u, v)$, then $F(f) \in C'_i(F(u), F(v))$.

We also extend this definition to the case when $C$ is an $N$-precategory and $C'$ is a $1$-precategory: if $f \in C_i(u, v)$, then $F(f) \in C'_i(F(u), F(v))$.

So we see that a $C$-preset $(C, D, \rho)$ is nothing but a prefunctor $F : C \rightarrow \text{SET}$ such that $F(u) = D_u$, $F(\alpha) = \rho_\alpha$. Conversely, every prefunctor $F : C \rightarrow \text{SET}$ such that every $F(u)$ is non-empty, can be considered as a preset.

On the other hand, a preset $\mathfrak{F} = (C, D, \rho)$ is also associated with the precategory of individuals $C^1(\mathfrak{F})$ such that

\begin{align*}
\text{Ob} C^1(\mathfrak{F}) := D^1 := \bigcup_{u \in \text{Ob} C} D_u, \\
C^1(\mathfrak{F})_{i}(a, b) := \{(a, b, \alpha) \mid b = \rho_\alpha(a), \alpha \in \text{Mor} i C\}.
\end{align*}

We also obtain an etale prefunctor $E : C^1(\mathfrak{F}) \rightarrow C$ such that $E(a) = u$ whenever $a \in D_u$, and $E(a, b, \alpha) = \alpha$.

Proposition 5.6.5

(1) Let $F : C \rightarrow \text{SET}$ be a $C$-preset, and let $E : C^1(\mathfrak{F}) \rightarrow C$ be the corresponding etale prefunctor. Then $E$ has the following properties

\footnote{We may assume that $\text{Ob} C \cap \text{Mor} C = \emptyset$; otherwise, $C$ can be replaced with an isomorphic category having this property.}
(i) \( \forall u \in \text{Ob} \mathcal{C} \ \exists a \in \text{Ob} \mathcal{C}^1(\mathcal{F}) \ E(a) = u \) (surjectivity on objects);
(ii) \( \forall a \in \text{Ob} \mathcal{C}^1(\mathcal{F}) \ \forall \alpha \in \text{Mor} \mathcal{C} \ (E(a) = o(\alpha) \Rightarrow \exists f \ (\mathcal{F}(f) = \alpha \land o(f) = a)) \) (the unique lift property).

(2) Let \( \mathcal{G} : \mathcal{C}' \leadsto \mathcal{C} \) be a prefunctor which is surjective on objects and has the unique lift property. Then \( \mathcal{C}' \) is isomorphic to \( \mathcal{C}^1(\mathcal{F}) \) for some \( \mathcal{C} \)-preset \( \mathcal{F} \).

Proof

(1) (i) \( \mathcal{F} \) is surjective on objects since every set \( \mathcal{F}(u) \) is non-empty.
   (ii) Suppose \( \alpha \in \mathcal{C}(u, v) \) and \( a \in D_u \). Then \( E(f) = \alpha \land o(f) = a \) holds iff \( f = (a, b, \alpha) \), where \( b = \rho_\alpha(a) \).

(2) For an object \( u \) put \( D_u := \mathcal{F}(u) := \mathcal{G}^{-1}[u] \).
   For a morphism \( \alpha \in \mathcal{C}(u, v) \), put

\[ \mathcal{F}(\alpha) := \rho_\alpha := \{(a, b) \mid \exists f \ (o(f) = a \land t(f) = b) \land \mathcal{G}(f) = \alpha \} \]

The precategories \( \mathcal{C}' \) and \( \mathcal{C}^1(\mathcal{F}) \) are isomorphic. In fact, they have the same objects, and there is a bijection between \( \mathcal{C}^1(\mathcal{F})(a, b) \) and \( \mathcal{C}'(a, b) \) sending \( (a, b, \alpha) \) to the the unique lift of \( \alpha \) beginning at \( a \), i.e. to \( f \) such that \( \mathcal{G}(f) = \alpha, o(f) = a \).

If \( \mathcal{C} \) is category, \( \mathcal{F} : \mathcal{C} \leadsto \text{SET} \) is a \( \mathcal{C} \)-set, then we can also make \( \mathcal{C}^1(\mathcal{F}) \) a category by putting

\[ (a, b, \alpha) \circ (b, c, \beta) := (a, c, \alpha \circ \beta) ; \ 1_a := (a, a, 1_a) \] for \( a \in D_u \).

Lemma 5.6.6 If \( \mathcal{F} \) is a \( \mathcal{C} \)-set, then \( E : \mathcal{C}^1(\mathcal{F}) \leadsto \mathcal{F} \) is a functor.

Proof

In fact, \( E((a, b, \alpha) \circ (b, c, \beta)) = \alpha \circ \beta = E(a, b, \alpha) \circ E(b, c, \beta) \), and \( E(1_a) = 1_{E(a)} \).
Definition 5.6.7 For a propositional $N$-frame $F = (W, R_1, \ldots, R_N)$ we define the associated $N$-precategory $C = \text{Cat}_0 F$ as follows:

$$\text{Ob} C := W,$$

$$C_i(u, v) := \begin{cases} \{(i, u, v)\} & \text{if } u R_i v, \\ \emptyset & \text{otherwise.} \end{cases}$$

From definitions it is clear that every Kripke sheaf based on a frame $F$ is a $\text{Cat}_0 F$-preset; if $F$ is an S4-frame, we obtain a $\text{Cat}_F$-preset (where $\text{Cat}_F$ is the category defined in Section 3.5.3).

Remark 5.6.8 Note that if $F$ is not an S4-frame, it may happen that a $\text{Cat}_0 F$-preset is not a Kripke sheaf, because it may not satisfy the coherence conditions $(1)^*$, $(2)^*$ from Lemma 3.6.3.

Definition 5.6.9 (cf. Definitions 5.2.7, 5.2.8). A (modal) valuation in a $C$-preset $F$ is a valuation in the system of domains $D = (F(u) \mid u \in \text{Ob} C)$. A model over $F$ is a pair $(F, \xi)$, where $\xi$ is a valuation in $F$.

The forcing relation $M, u \models A$ between a world $u \in W$ and a $D_u$-sentence (with equality) $A$ is defined by induction:

- $M, u \models P_0^k$ iff $u \in \xi u(P_0^k)$;
- $M, u \models P^m_k(a)$ iff $a \in \xi u(P^m_k)$ (for $m > 0$);
- $M, u \models a = b$ iff $a$ equals $b$;
- $M, u \not\models \bot$;
- $M, u \models B \lor C$ iff $(M, u \models B \text{ or } M, u \models C)$;
- $M, u \models B \land C$ iff $M, u \models B \land M, u \models C$;
- $M, u \models B \supset C$ iff $(M, u \not\models B \text{ or } M, u \models C)$;
- $M, u \models \square_i B(a)$ iff $\forall v \in R_i(u) \forall \alpha \in C_i(u, v) M, v \models B(\rho_\alpha \cdot a)$;
- $M, u \models \exists x A$ iff $\exists a \in D_u M, u \models [a/x] A$;
- $M, u \models \forall x A$ iff $\forall a \in D_u M, u \models [a/x] A$.

Here as usual in the $\square$-case $B(a)$ means $[a/x] B$, and we assume that $FV(B) = x$.

Note that the truth condition for $\square_i[a/x] B$ obviously extends to the case when $FV(B) \subseteq x$.

The following definition is similar to the case of Kripke bundles.

Definition 5.6.10 A (modal) predicate formula $A$ is called
• true in a C-preset model M if its universal closure \( \forall A \) is true at every world of M;

• valid in a C-preset F if A is true in every model over F;

• strongly valid in a C-preset F if all substitution instances of A are valid in F.

We use the same signs as above: \( \vdash \) for truth and validity, \( \vdash^+ \) for strong validity.

The following is a trivial consequence of the definitions:

**Lemma 5.6.11** For a C-preset model M and a modal formula A(x)

\[ M \models A(x) \text{ iff } \forall u \in M \forall a \in D^n_u \quad M, u \models A(a). \]

For a C-preset F let

\[
\begin{align*}
\mathcal{ML}^{(=)}(F) & := \{ A \in MF_N^{(=)} \mid F \models A \}, \\
\mathcal{ML}^{(=)}(F) & := \{ A \in MF_N^{(=)} \mid F \models^+ A \}.
\end{align*}
\]

The proof of the following soundness result is postponed until Section 5.13 (Proposition 5.13.2).

**Proposition 5.6.12** For a C-preset F:

1. \( \mathcal{ML}^{(=)}(F) \) is a modal predicate logic;

2. \( \mathcal{ML}^{(=)}(F) = \{ A \in MF_N^{(=)} \mid \forall m \ F \models A^m \} \).

**Definition 5.6.13** For N-modal predicate logics we introduce the functor semantics \( \mathcal{FS}_N \) generated by presets over N-precategories.

Similarly to Kripke bundles, we introduce levels for C-presets.

**Definition 5.6.14** Let \( F = (C, D, \rho) \) be a C-preset over an N-precategory C. Let us define relations \( R^n_i \) on \( D^n \) for \( n > 0, 1 \leq i \leq N \):

\[ a R^n_i b \text{ iff } \exists \gamma \in Mor_i C \ \rho_\gamma \cdot a = b. \]

Also let \( R^0_i := R_i \), \( F^n := (D^n, R^n_1, \ldots, R^n_N) \).

The nth level of F is the frame \( F^n := (D^n, R^n_1, \ldots, R^n_N) \). We again abbreviate \( R^n_i \) to \( R^n \) in the 1-modal case.

From the definition of forcing 5.6.9 we readily obtain

**Lemma 5.6.15** Let \( C \) be an N-precategory, \( M = (F, \xi) \) a model over a C-preset \( F = (C, D, \rho) \). Let B be an N-modal formula with \( FV(B) \subseteq x \), \( |x| = n \). Then for any \( u \in F \) and \( a \in D^n_u \)

\[ M, u \models \Box_i B(a) \text{ iff } \forall u \in R_i(u) \forall b \in D^n_v \quad (a R^n_i b \Rightarrow M, v \models B(b)). \]
Lemma 5.6.16 If $C$ is a category and $F$ is a $C$-set, then every $F_n$ is an intuitionistic propositional frame.

Proof An exercise.

Definition 5.6.17 A valuation $\xi$ in a $C$-set $F$ is called intuitionistic if every set $\xi^+(P^m_k) = \bigcup \{\xi_u(P^m_k) \mid u \in W\}$ is stable in $F_n$, i.e. if $R^n(\xi^+(P^m_k)) \subseteq \xi^+(P^m_k)$. The model $(F, \xi)$ is also called intuitionistic.

Definition 5.6.18 Let $M$ be an intuitionistic model over a $C$-set $F = (C, D, \rho)$, $u \in M$, $A$ an intuitionistic $D_u$-sentence. Then we put

$$M, u \models A : = M, u \models A^T.$$

Lemma 5.6.19 Under the conditions of Definition 5.6.18

1. $M, u \models B$ iff $M, u \models B$ (for $B$ atomic);
2. $M, u \models B \land C$ iff $(M, u \models B \land M, u \models C)$;
3. $M, u \models B \lor C$ iff $(M, u \models B$ or $M, u \models C)$;
4. $M, u \models B(a) \supset C(a)$ iff $\forall v \in R(u) \forall \mu \in C(u, v) (M, v \models B(\rho_\mu \cdot a) \Rightarrow M, v \models C(\rho_\mu \cdot a))$;
5. $M, u \models \forall x B(x, a)$ iff $\forall v \in R(u) \forall \mu \in C(u, v) \forall c \in D_v M, v \models B(c, \rho_\mu \cdot a)$;
6. $M, u \models \exists x B$ iff $\exists a \in D_y M, u \models [a/x] B$;
7. $M, u \models \neg B(a)$ iff $\forall v \in R(u) \forall \mu \in C(u, v) M, v \not\models B(\rho_\mu \cdot a)$;
8. $M, u \models a \neq b$ iff $a$ does not equal $b$.

Proof Similar to 5.5.4, with obvious changes.

Lemma 5.6.20 Let $M$ be the same as in Lemma 5.6.19. Then for any $D^1$-sentence $A(a)$

$$M \models A(a) \land a R^n b \Rightarrow M \models A(b).$$

Definition 5.6.21 Let $F$ be a $C$-set, $M$ a model over $F$. The pattern of $M$ is the model $M_0$ over $F'$ such that for any $u \in F$ and any atomic $D_u$-sentence $A$

$$M_0, u \models A \text{ iff } M, u \models \Box A.$$

The model $M_0$ is intuitionistic, and similarly to Lemma 5.5.7, we obtain

Lemma 5.6.22 For any $u \in F$, for any $D_u$-sentence $A$

$$M_0, u \models A \text{ iff } M, u \models A^T.$$
Lemma 5.6.23 Let $M$ be an intuitionistic model over a $C$-set with the accessibility relation $R$, $A(x)$ an intuitionistic formula, $|x| = n$. Then for any $u \in M$

$M, u \models \forall x A(x)$ iff $\forall v \in R(u) \forall a \in D^n_u M, u \models A(a)$.

Proof Similar to Lemma 5.5.8 using soundness (5.6.12).

Definition 5.6.24 A formula $A \in IF(=)$ is called

- true in an intuitionistic $C$-set model $M$ (notation: $M \models A$) if $\forall A$ is true at every world of $M$;
- valid in a $C$-set $F$ (notation: $F \models A$) if it is true in all intuitionistic models over $F$;
- strongly valid in a $C$-set $F$ (notation: $F \models^+ A$) if all its $IF(=)$-substitution instances are valid in $F$.

Lemma 5.6.25 For an intuitionistic $C$-set model $M$ and an intuitionistic formula $A$

$M \models A(x)$ iff $\forall u \in M \forall a \in D^n_u M, u \models A(a)$.

Proposition 5.6.26 Let $F$ be a $C$-set, $A \in IF^=$. Then

1. $F \models A$ iff $F \models A^T$.
2. The following three assertions are equivalent:
   - (a) $F \models^+ A$;
   - (b) $\forall m F \models A^m$;
   - (c) $F \models^+ A^T$.

Proof Cf. Proposition 5.5.12.

Proposition 5.6.27 For a $C$-set $F$ the set

$$IL(=)(F) := \{ A \in IF(=) \mid F \models^+ A \}$$

is a superintuitionistic predicate logic (with or without equality), and

$$IL(=)(F) =^T ML(=)(F).$$

Definition 5.6.28 The set $IL(=)(F)$ is called the superintuitionistic logic of $F$. 
5.7 Morphisms of presets

Now let us define truth-preserving morphisms of presets.

**Definition 5.7.1** (Cf. Proposition 5.6.5.) A prefuctor $\Phi : D \rightsquigarrow E$ has the lift property if

$$\forall a \in ObD \forall \mu \in MorE (\Phi(a) = o(\mu) \Rightarrow \exists f \in MorD \Phi(f) = \mu).$$

The reader can see that this is a generalisation of the lift property for propositional frame morphisms.

**Definition 5.7.2** Let $F$ be a $C$-preset, $F'$ a $C'$-preset. A pair $\gamma = (\Phi, \Psi)$ is called an $\equiv$-morphism from $F$ to $F'$ (notation: $\gamma : F \rightarrow= F'$) if

1. $\Phi : C \rightsquigarrow C'$ is a prefuctor with the lift property;
2. $\Psi = (\Psi_u \mid u \in ObC)$ is a family of bijections

$$\Psi_u : F(u) \rightarrow F'(\Phi(u));$$

3. $\Psi$ respects morphisms, i.e. for any $\mu \in C(u, v)$ the following diagram commutes:

\[
\begin{array}{ccc}
F(v) & \xrightarrow{\Psi_v} & F'(\Phi(v)) \\
\downarrow F(\mu) & & \downarrow F'(\Phi(\mu)) \\
F(u) & \xrightarrow{\Psi_u} & F'(\Phi(u))
\end{array}
\]

$\gamma$ is called a $p=\equiv$-morphism if it also has the property

4. $\Phi$ is surjective on objects.

The above conditions (2), (3) mean that $\Psi$ is a functor morphism ('natural transformation') from $F$ to $F' \cdot \Phi$.

**Definition 5.7.3** For $\gamma = (\Phi, \Psi) : F \rightarrow= F'$, $n \geq 0$, we define the map $\gamma_n : F_n \rightarrow F'_n$ as follows:

$$\gamma_0(u) := \Phi(u), \ \gamma_n(a) := \Psi_u \cdot a$$

whenever $a \in F(u)^n, \ n > 0$.

**Lemma 5.7.4** $\gamma_n : F_n \rightarrow= F'_n$. 
Proof We consider only the case $n > 0$, and assume $N = 1$ to simplify notation. To prove the monotonicity, suppose $aR^nb$, i.e.

$$\exists \mu \in C(u,v) \forall i \ b_i = F(\mu)(a_i).$$

Then by 5.7.2(3),

$$\Psi_v(b_i) = \Psi_v(F(\mu)(a_i)) = F'(\Phi(\mu))(\Psi_u(a_i)).$$

Thus

$$\Psi_v \cdot b = F'(\Phi(\mu)) \cdot (\Psi_u \cdot a),$$

and so by definition, $\gamma_n(a)(R')^n\gamma_n(b)$.

To check the lift property, suppose $a \in F_n$, $a' = \Psi_u \cdot a$, $a'R^nb'$, $a' \in F'(u')$, $b' \in F'(v')$. Then

$$\exists \lambda \in C'(u',v') \ b' = F'(\lambda) \cdot a'.$$

Since $\Phi$ has the lift property, there exist $v \in ObC$ and $\mu \in C(u,v)$ such that $\lambda = \Phi(\mu)$. Then put

$$b := F(\mu) \cdot a.$$

By definition, $aR^nb$, and we also claim that $b' = \gamma_n(b) = \Psi_v \cdot b$ (Fig. 5.8).
In fact, let $b'' = \Psi_v \cdot b$. By 5.7.1, the following diagram commutes:

\[
\begin{array}{ccc}
F(\mu) & \xrightarrow{\Psi_v} & F'(\lambda) \\
\downarrow^b & & \downarrow^{b'} \\
F(u) & \xrightarrow{\Psi_u} & F'(u)
\end{array}
\]

Thus $b'' = F'(\lambda) \cdot a' = b'$.

**Definition 5.7.5** Let $\gamma : F \rightarrow= G$ be an $=$-morphism of presets. $\gamma$ is called an $=$-morphism from a model $M = (F, \xi)$ to $M' = (F', \xi')$ (notation: $f : M \rightarrow= M'$) if

\[M, u \vDash P(a) \iff M', \gamma_0(u) \vDash P(\gamma_n(a))\]

for any $P \in PL^n$, $a \in F(u)^n$, $n > 0$; and also

\[M, u \vDash P \iff M', \gamma_0(u) \vDash P\]

for any $P \in PL^0$.

**Lemma 5.7.6** If $\gamma : M \rightarrow= M'$ for preset models $M = (F, \xi)$, $M' = (F', \xi')$, then for any $u \in M$ and for any $F(u)$-sentence $A$

\[M, u \vDash A \iff M', \gamma_0(u) \vDash \gamma_1 \cdot A.\]

If the models are intuitionistic, then for any intuitionistic $F(u)$-sentence $A$

\[M, u \vDash A \iff M', \gamma_0(u) \vDash \gamma_1 \cdot A.\]

**Proof** By an obvious modification of the proofs of 5.4.6, 5.5.15 based on Lemma 5.7.4; an exercise for the reader.

**Lemma 5.7.7** Let $\gamma : F \rightarrow= G$, and let $M'$ be a model over $G$. Then there exists a model $M$ over $F$ such that $\gamma : M \rightarrow= M'$.

**Proof** The same as for Lemma 5.4.7.

**Proposition 5.7.8** If there exists a $p=$-morphism $F \rightarrow= F'$, then $ML^= (F) \subseteq ML^= (F')$ and thus, $ML^= (F) \subseteq ML^= (F')$. Similarly, for the intuitionistic case: $IL^= (F) \subseteq IL^= (F')$ and $IL^= (F) \subseteq IL^= (F')$.

**Proof** Cf. Propositions 5.4.8, 5.5.16.
Let us now construct inverse image morphisms.

**Lemma 5.7.9** Let $F$ be a $C$-preset, $E : C' \Rightarrow C$ the corresponding etale prefunctor, $G : D \Rightarrow C$ an arbitrary prefunctor with the lift property. Consider a precategory $D'$ with the class of objects

$$\{(u, a) \mid u \in \text{Ob}D, \ a \in \text{Ob}C', \ G(u) = E(a)\}$$

and with

$$D'((u, a), (v, b)) := \{(\mu, f) \mid \mu \in D(u, v), \ f \in C'(a, b), \ G(\mu) = E(f)\}.$$ 

Then

1. There exists an etale prefunctor $E' : D' \Rightarrow D$ such that $E'(u, a) = u$, $E'(\mu, f) = \mu$.
2. There exists a prefunctor with the lift property, $G' : D' \Rightarrow C'$ such that $G'(u, a) = a$, $G'(\mu, f) = f$.
3. The following diagram commutes:

\[
\begin{array}{ccc}
D' & \xrightarrow{G'} & C' \\
\downarrow{E'} & & \uparrow{E} \\
D & \xrightarrow{G} & C
\end{array}
\]

4. If $F'$ is a $D$-preset corresponding to $E'$, then there exists $(G, \Psi) : F' \Rightarrow F$ such that $\Psi_a(u, a) = a$.

5. If $C$, $D$ are categories, $F$ is a $C$-set and $G$ is a functor, then $D'$ becomes a category with the composition

$$(\mu, f) \circ (\nu, g) := (\mu \circ \nu, f \circ g)$$

and $F'$ becomes a $D$-set.

**Proposition 5.7.10** Every $\Rightarrow$-morphism of presets $(\Phi, \Psi) : F' \Rightarrow F$ is the composition of the inverse image morphism $\Theta : \Phi_*F \Rightarrow F$ and a prefunctor isomorphism (over the same precategory) $\Xi : F' \Rightarrow \Phi_*F$.

**Definition 5.7.11** Let $C$, $D$ be $N$-precategories. $D$ is called a full subprecategory of $C$ (notation: $D \subseteq_0 C$) if

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9Cf. Lemma 5.4.9.
5.7. MORPHISMS OF PRESETS

(1) \( \text{Ob} D \subseteq \text{Ob} C \),

(2) \( \forall u, v \in \text{Ob} D \forall i \leq N \ C_i(u, v) = D_i(u, v) \).

If \( C, D \) are categories and \( D \subseteq C \), then \( D \) is called a full subcategory of \( C \) (notation: \( D \subseteq C \)) if

(3) for any consecutive morphisms \( f, g \) in \( D \)

\[ f \circ g \ (\text{in} \ D) = f \circ g \ (\text{in} \ C) \].

Definition 5.7.12 A full sub(pre)category \( D \) of a (pre)category \( C \) is called a conic sub(pre)category (notation: \( D \subseteq (0) C \)) if

\[ \forall u \in \text{Ob} D \forall v (C^*(u, v) \neq \emptyset \Rightarrow v \in \text{Ob} D) \].

Here is a typical example.

Definition 5.7.13 For a (pre)category \( C \) and \( u \in \text{Ob} C \) the cone generated by \( u \) (notation: \( C^\uparrow u \)) is defined as the full sub(pre)category with the class of objects \( \{v | C^*(u, v) \neq \emptyset \} \).

Thus every object of \( C^\uparrow u \) is accessible from \( u \) via a path of morphisms.

Definition 5.7.14 Let \( F \) be a \( C \)-preset, \( D \subseteq C \). The restriction of \( F \) to \( D \) (notation: \( F \uparrow D \)) is the \( D \)-preset taking the same values as \( F \) on objects and morphisms of \( D \).

For \( u \in \text{Ob} C \) we define the cone of \( F \) generated by \( u \): \( F^\uparrow u := F \uparrow (C^\uparrow u) \).

Note that if \( F \) is a \( C \)-set and \( D \subseteq C \), then obviously, \( F \uparrow D \) is a \( D \)-set.

Lemma 5.7.15 Let \( F \) be a \( C \)-preset, \( D \subseteq C \). Let \( J : D \rightarrow C \) be the inclusion prefunctor (sending every object and every morphism of \( D \) to itself). Also let \( \Psi = (id_{F(u)} | u \in \text{Ob} D) \). Then

\[ (J, \Psi) : F \uparrow D \longrightarrow F \].

Proof Obviously, \( J \) has the lift property and \( \Psi \) respects morphisms.

Lemma 5.7.16 (1) For a \( C \)-preset \( F \)

\[ \text{ML}(=)(F) = \bigcap \{ \text{ML}(=)(F \uparrow u) | u \in F \} \].

(2) For a \( C \)-set \( F \)

\[ \text{IL}(=)(F) = \bigcap \{ \text{IL}(=)(F \uparrow u) | u \in F \} \].

Proof Along the same lines as Lemmas 5.4.13, 5.5.17.

Lemma 5.7.17 (1) If \( F \) is a \( C \)-preset, \( D \subseteq C \), then

\[ \text{ML}(=)(F) \subseteq \text{ML}(=)(F \uparrow D) \].

(2) If \( F \) is a \( C \)-set, \( D \subseteq C \), then

\[ \text{IL}(=)(F) \subseteq \text{IL}(=)(F \uparrow D) \].

Proof Easy, from 5.7.16.
5.8 Bundles over precategories

Let us now combine two generalisations of Kripke sheaves: Kripke bundles and presets.

**Definition 5.8.1** A modal C-bundle over an N-precategory C is a triple \( F = (C, D, p) \), in which

- \( D = (D_u \mid u \in W) \) is a system of domains;
- \( \rho = (\rho_\alpha \mid \alpha \in \text{Mor}C) \) is a family of relations parametrised by morphisms of C;
- \( \rho_\alpha \subseteq D_u \times D_v, \ \text{dom } \rho_\alpha = D_u \text{ for } \alpha \in C(u,v) \).

An intuitionistic C-bundle over a category C is a modal C-bundle satisfying two extra conditions:

1. \( \rho_{1_u} \) is reflexive (on \( D_u \)) for \( u \in W \),
2. \( \rho_\alpha \circ \rho_\beta \subseteq \rho_{\alpha \circ \beta} \),

*cf. the corresponding conditions for intuitionistic Kripke bundles in Definition 5.2.2.*

The notion of a C-bundle generalises both Kripke bundles and C-sets. In fact, C-sets are just C-bundles with functions \( \rho_\alpha \), and Kripke bundles based on a frame \( F \) are exactly Cat_0F-bundles (where Cat_0F is the precategory from Definition 5.6.7).

But it turns out that the semantics of C-bundles is strongly equivalent (in the terminology of Section 2.16) to the semantics of C-sets, as we show below. Hence it follows that the semantics of Kripke bundles KB is included in the functor semantics FS. But actually, FS is stronger than KB, as we shall see later on (Section 5.10).

To define forcing in C-bundles, we first extend Definition 5.3.2 to this case:

**Definition 5.8.2** For a C-bundle \( F = (C, D, \rho) \) put

\[ aR^n b \iff a \text{ sub } b \& \exists f \in \text{Mor}C \ \forall j \ a_j \rho_f b_j \]

for \( a, b \in D^n, \ n > 0 \).

Also put \( R^0_i := R_i \), the corresponding relation in \( FR(C) \).

**Definition 5.8.3** A (modal) valuation in a C-bundle \( F = (C, D, \rho) \) is a valuation in the system of domains D. A model over \( F \) is a pair \( (F, \xi) \), where \( \xi \) is a valuation in \( F \).

Now let us give a preliminary definition of forcing.
Definition 5.8.4 (cf. Definitions 5.2.7, 5.2.8, 5.6.9). The forcing relation \( M, u \models A \) between a world \( u \in W \) and a \( D_u \)-sentence (with equality) \( A \) is defined by induction:

- \( M, u \models P_k^0 \) iff \( u \in \xi_u(P_k^0) \);
- \( M, u \models P_k^m(a) \) iff \( a \in \xi_u(P_k^m) \) (for \( m > 0 \));
- \( M, u \models a = b \) iff \( a \) equals \( b \);
- \( M, u \not\models \bot \);
- \( M, u \models B \lor C \) iff \( (M, u \models B \) or \( M, u \models C) \);
- \( M, u \models B \land C \) iff \( M, u \models B \) & \( M, u \models C \);
- \( M, u \models B \supset C \) iff \( (M, u \not\models B \) or \( M, u \models C) \);
- \( M, u \models \Box_i B(a) \) iff \( \forall v \in R_i(u) \forall b \in D_v^n (aR^n_i b \Rightarrow M, v \models B(b)) \) (for \( |a| = n \));
- \( M, u \models \exists x A \) iff \( \exists a \in D_u M, u \models [a/x] A \);
- \( M, u \models \forall x A \) iff \( \forall a \in D_u M, u \models [a/x] A \).

Again in the \( \Box \)-case \( B(a) \) means \( [a/x] B \), and we assume that \( FV(B) = x \).

But this definition should be justified. In fact, as we know, the presentation of the same formula in a form \( B(a) \) is not unique. So we should prove that forcing does not really depend on this presentation. To show this, we find a precategory corresponding to the same ‘metaframe’ relations \( R^n_i \).

Proposition 5.8.5 Let \( \mathbb{F} = (C, D, \rho) \) be a \( C \)-bundle over an \( N \)-precategory \( C \). Then there exists an \( N \)-precategory \( C' \) and a \( C' \)-preset \( \mathbb{F}' \) such that

- \( FR(C) = FR(C') \);
- \( D_u = F'(u) \) for any \( u \in ObC \);
- \( R^n_i = R'^n_i \) for all \( n \in \omega, 1 \leq i \leq N \) (where \( R^n_i \) and \( R'^n_i \) are the nth level accessibility relations in \( \mathbb{F} \) and \( \mathbb{F}' \) respectively).

Moreover, if \( \mathbb{F} \) is an intuitionistic \( C \)-bundle over a category \( C \) (in particular a Kripke bundle over an \( \mathbf{S}_4 \)-frame, then \( C' \) is also a category and \( \mathbb{F}' \) is a \( C' \)-set.

Proof Put

\[
C'_i(u, v) := \{ f \text{ is a function } D_u \to D_v \mid \exists \gamma \in C_i(u, v) f \subseteq \rho_\gamma \},
\]

and let \( \rho'_i := f \) in \( \mathbb{F}' \). So we replace every relation \( \rho_\gamma \subseteq D_u \times D_v \) with the set of all functions from \( D_u \) to \( D_v \) included in this relation.
Let us show that \( R^m_i = R^n_i \). In fact, suppose \( a R^m_i b \); then \( a \) sub \( b \), and there exists \( f \in C'_i(u,v) \) such that \( b = \rho'_f \cdot a = f \cdot a \), i.e. for all \( j \), \( b_j = f(a_j) \). Since \( f \in C'_i(u,v) \), there exists \( \gamma \in C_i(u,v) \) such that \( f \subseteq \rho_\gamma \); thus \( a_j \rho_\gamma b_j \), which implies \( a R^n_i b \).

The other way round, suppose \( a R^n_i b \). Then the set \( g = \{(a_1, b_1), \ldots, (a_n, b_n)\} \) is a function (since \( a \) sub \( b \)), and \( g \subseteq \rho_\gamma \) for some \( \gamma \in C_i(u,v) \). This \( g \) can be prolonged to a total function \( f : D_u \rightarrow D_v \) such that \( f \subseteq \rho_\gamma \), since \( \text{dom} \rho_\gamma = D_u \) by Definition 5.8.1. Hence \( b = f \cdot a = \rho'_f \cdot a \), which implies \( a R^m_i b \).

If \( C \) is category, then \( C' \) also becomes a category after we define \( \circ \) as the composition of functions and \( 1_{D_a} \) as the identity function \( \text{id}^D \). In fact, if \( f \subseteq \rho_\alpha \) and \( g \subseteq \rho_\beta \), then \( f \circ g \subseteq \rho_\alpha \circ \rho_\beta \subseteq \rho_{\alpha \circ \beta} \) (by Definition 5.8.1); \( \text{id}_{D_a} \subseteq \rho_1 \), since \( \rho_1 \) is reflexive by 5.8.1.

**Remark 5.8.6** The crucial point of this construction is 'local functionality' in the definition of forcing for a Kripke bundle. This corresponds to the choice of distinct individuals \( a_1, \ldots, a_n \) in the inductive clause for \( \Box \), or to the conjunct \( a \) sub \( b \) in the definition of \( R^n_i \).

**Remark 5.8.7** Note that the argument fails for intuitionistic Kripke quasi-bundles, because the totality of \( \rho_{uv} \) is not guaranteed in this case.

Therefore a model \( M = (\mathcal{F}, \xi) \) corresponds to the model \( M' = (\mathcal{F}', \xi) \), and we can define \( M, u \models A \) as \( M', u \models A \). Then we obtain the properties described in Definition 5.8.4, and we can further define validity in \( \mathcal{F} \) and the logic

\[
\text{ML}^{(\equiv)}(\mathcal{F}) := \text{ML}^{(\equiv)}(\mathcal{F}')
\]

in the modal case and

\[
\text{IL}^{(\equiv)}(\mathcal{F}) := \text{IL}^{(\equiv)}(\mathcal{F}')
\]

in the intuitionistic case.

Hence we obtain the semantics of \( C \)-bundles, which is obviously strongly equivalent to the functor semantics, and the following

**Corollary 5.8.8** \( \mathcal{KB}_N \preceq \mathcal{FS}_N \), \( \mathcal{KB}_{\text{int}} \preceq \mathcal{FS}_{\text{int}} \).

**Proof** In fact, for a Kripke bundle \( \mathcal{F} \) with the base \( F \) there exists a \( \text{Cat}_0 F \)-preset \( \mathcal{F}' \) such that \( \text{ML}^{(\equiv)}(\mathcal{F}) = \text{ML}^{(\equiv)}(\mathcal{F}') \).

**5.9 Metaframes**

The definition of forcing in Kripke bundles and \( C \)-sets via forcing in propositional frames \( F_n \) motivates a further generalisation of Kripke semantics. So far the relations \( R^n_i \) were derived from other relations or functions. Now let us consider arbitrary metaframes; they are defined as sequences of propositional frames of tuples, without special requirements for accessibility relations.
Definition 5.9.1 Let $F_0 = (W, R_1, \ldots, R_N)$ be a propositional Kripke frame, $D$ a system of disjoint (nonempty) domains over $W$.

Let $F_n = (D^n, R^n_1, \ldots, R^n_N)$, $n > 0$, be propositional Kripke frames such that $D^n = \bigcup_{u \in W} D^n_u$. Then the pair $F = ((F_n)_{n \in \omega}, D)$ is called an $N$-metaframe based on $F_0$. As usual, $W$ is called the set of possible worlds and $D^1$ the set of individuals of $F$. $F_n$ is called the $n$-level of $F$.

For $a \in D^n_u$ we denote the domain $D_u$ by $D(a)$.

We shall often identify a metaframe $F$ with the sequence $(F_n)_{n \in \omega}$, i.e. consider it as a graded propositional frame. Note that the worlds of $F_n$ are $n$-tuples of individuals, and the original worlds from $W$ are ‘0-tuples’. Thus we may define

$$D^0_u := \{u\}, \quad D^0 := W, \quad R^0_i := R_i.$$

We say that an $n$-tuple $b$ is an $i$-inheritor (counterpart) of an $n$-tuple $a$ if $aR^n_i b$.

As usual, $i(=1)$ is not indicated in the notation if $N=1$.

By default we assume that an arbitrary metaframe and all its components are denoted as in 5.9.1.

Definition 5.9.2 A (modal) valuation in an $N$-metaframe $F$ is a valuation in its system of domains. A (modal) metaframe model over $F$ is a pair $M = (F, \xi)$, where $\xi$ is a (modal) valuation in $F$.

So a valuation in $F$ is a graded propositional valuation; $n$-ary predicate letters are evaluated in $F$ as propositional letters in $F_n$.

Our goal is now to define forcing in metaframe models. This definition is not a straightforward generalisation of the corresponding definitions in functor and Kripke bundle semantics. The reason is that in arbitrary metaframes it is impossible to define forcing for $D_u$-sentences in the natural way — this will be shown later on.

Remember that every $D_u$-sentence is obtained by replacing parameters with $D_u$-individuals in a formula $A(x)$, or by applying a $D_u$-transformation $[x \mapsto a]$ to $A(x)$, cf. Definition 2.4.1.

This transformation itself (not only its result $A(a)$ ‘forgetting’ the original $A(x)$ and $[x \mapsto a]$) is essential in our definition. Moreover, we define forcing for a formula $A(x)$ with respect to an ‘ordered assignment’ $(x, a)$ (arranging a transformation in a certain order).

In the classical case (and in all our earlier Kripke-type semantics) the two versions of the truth definition (with $D$-sentences or variable assignments) are equivalent. The original Tarski’s definition is given in terms of assignments.

So we begin with

Definition 5.9.3 An ordered assignment (at a world $u$) in a metaframe $F = ((F_n)_{n \in \omega}, D)$ is a pair $(x, a)$, where $x$ is a distinct list of variables, $a \in D^n_u$. If the list $x$ is empty, then by definition, $a$ is just the world $u$. 
CHAPTER 5. METAFRAME SEMANTICS

We define forcing in a model $M$ as a relation $M, (x, a) \vdash A$ between ordered assignments and formulas. It can also be regarded as a relation between tuples $a \in D^n$ and pairs $(A, x)$, so we use the alternative notation $M, a \vdash A [x]$. This is read as 'a forces a pair $(A, x)$', or 'a
formal expression $A[x]$'.\(^\text{10}\) It is convenient to
give such a definition for the case when $FV(A) \subseteq r(x)$.

Let us make some remarks about notation. Remember that according to the Introduction, for a tuple $a = (a_1, \ldots, a_n)$ and $\sigma : I_m \rightarrow I_n$, we denote

$$a \cdot \sigma = (a_{\sigma(1)}, \ldots, a_{\sigma(m)})$$

and

$$a - a_i = \widehat{a_i} = a \cdot \delta^n_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).$$

The same notation applies to lists of variables; thus in Definition 5.9.4, $x \cdot \sigma$
means $(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ and $x - x_i$ means $x \cdot \delta^n_i$ for a list of variables $x = (x_1, \ldots, x_n)$, $\sigma : I_m \rightarrow I_n$.

However the case $m = 0$ is special; then $\sigma = \emptyset$ is the empty map, and we define

$x \cdot \emptyset$ as the empty list,
$a \cdot \emptyset$ as the world of $a$ (i.e. $a \cdot \emptyset = u \leftrightarrow a \in D^n_u$).

Note that every atomic formula $P^m_k (y)$ with $r(y) \subseteq r(x)$, $|x| = n$, has the form
$P^m_k (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$, or $P^m_k (x \cdot \sigma)$, for some $\sigma \in \Sigma_{mn}$.

\textbf{Definition 5.9.4} For a metaframe model $M$ and a modal formula (with equality) $A$ we define forcing $M, a \vdash A [x]$ under an ordered assignment $(x, a)$ (such that $FV(A) \subseteq r(x)$)\(^\text{11}\) by induction:

1. $M, a \not\vdash \bot [x]$;
2. $M, a \vdash P^m_j (x \cdot \sigma) [x]$ iff $(a \cdot \sigma) \in \xi^+ (P^m_j)$ (for $m > 0$);
3. $M, a \vdash P^0_j [x]$ iff $u \in \xi^+ (P^0_j)$ (for $a \in D^n_u$);
4. $M, a \vdash (x_i = x_j) [x]$ iff $a_i = a_j$;
5. $M, a \vdash B \supset C [x]$ iff $(M, a \not\vdash B [x]$ or $M, a \vdash C [x])$;
6. $M, a \vdash B \lor C [x]$ iff $(M, a \vdash B [x]$ or $M, a \vdash C [x])$;
7. $M, a \vdash B \land C [x]$ iff $(M, a \vdash B [x]$ & $M, a \vdash C [x])$;
8. $M, a \vdash \Box_i B [x]$ iff $\forall b (a R^n_i b \Rightarrow M, b \vdash B [x])$;
9. if $y \not\in x$, $a \in D^n_u$, then
   $M, a \vdash \exists y B [x]$ iff $\exists c \in D_u M, (ac) \vdash B [xy]$,
   $M, a \vdash \forall y B [x]$ iff $\forall c \in D_u M, (ac) \vdash B [xy]$.

\(^\text{10}\)The notation $A[x]$ should not be mixed up with $A(x)$ denoting a formula with parameters in $x$.
\(^\text{11}\)If $x$ is empty, $a$ is a possible world $u$, this is written as $M, u \vdash A [\cdot]$, or just $M, u \vdash A$.\n
5.9. METAFRAMES

(10) \( M, a \models \exists x_i B [x] \iff M, a - a_i \models \exists x_i B [x - x_i]; \)
\( M, a \models \forall x_i B [x] \iff M, a - a_i \models \forall x_i B [x - x_i]. \)

So by Definition 5.9.4, we obtain:

- \( M, a \models \neg B [x] \iff M, a \not\models B [x] \);
- \( M, a \models \Diamond_i B [x] \iff \exists v \in W \exists b \in D^n \ (aR^n_i b \& M, b \models B [x]). \)

Also note that if \( D \) is a system of domains over \( W, m, n > 0 \), then every map \( \sigma \in \Sigma_{mn} \) gives rise to the map \( \pi_\sigma : D^n \to D^m \) sending \( a \) to \( a \cdot \sigma \), and it is clear that \( \pi_\sigma \) maps every \( D^n \) to \( D^m \) exactly as in the Introduction.

For the empty map \( \emptyset_n \in \Sigma_{0n} \) we can define the map \( \pi_{\emptyset_n} : D^n \to D^0 = W \) sending every tuple to its world; sometimes we use a simpler notation and write \( \pi_\emptyset \) rather than \( \pi_{\emptyset_n} \).

In a metaframe the maps \( \pi_\sigma \) are called jlections. They have the properties mentioned in Lemma 0.0.1:

\[ \pi_\sigma \cdot \pi_\tau = \pi_{\tau \cdot \sigma} \text{ for } \sigma \in \Sigma_{km}, \tau \in \Sigma_{mn}; \]

if \( \sigma \in \Upsilon_n \), then \( \pi_\sigma \) is a permutation of \( D^n \) and \( \pi_{\sigma^{-1}} = (\pi_\sigma)^{-1} \).

Lemma 5.9.5 Let \( \mathcal{F} = ((F_n)_{n \in \omega}, D) \) be an \( N \)-metaframe, \( \sigma \in \Sigma_{mn} \). Then \( \pi_\sigma[D^n] = \{ a \in D^m \mid \sigma \text{ sub } a \}. \)

Proof Follows from Lemma 0.0.2 applied to every \( D_u \).

Lemma 5.9.6 Let \( \mathcal{F} \) be a metaframe, in which not all domains are one-element. Then for \( \sigma \in \Sigma_{mn} \), \( \sigma \) is injective iff \( \pi_\sigma : D^n \to D^m \) is surjective.

Proof Follows from Lemma 0.0.3.

We also use notation from the Introduction:

\[ \pi^n_i := \pi_\delta^n, \pi^n_\sigma := \pi_\sigma^n, \pi^n_\sigma := \pi_\sigma^n; \]

recall that

\[ \pi^n_1(a_1, \ldots, a_n) = (a_1, \ldots, a_n, a_n) \text{ for } n > 0, \]
\[ \pi^n_1(a) = a - a_i \text{ for } a \in D^n, n > 0, \pi^n_+(a) = a - a_{n+1} \text{ for } a \in D^{n+1}, n \geq 0. \]

Remembering that \( \sigma^0_+ = \emptyset_1 \in \Sigma_{01} \), we also have \( \pi^0_+(a_1) = u \) for \( a_1 \in D_u \).

We shall use the same notation \( \pi_\sigma \) for lists of variables; in particular, \( \pi_\sigma(x_1, \ldots, x_n) \) is the empty list.

Note that in all the clauses except (8), (10), Definition 5.9.4 is essentially the same as in earlier Kripke-type semantics. For example, consider Kripke bundles. Our new definition (5.9.4) corresponds to the old one (5.2.8) as follows.

\[ \text{Here similarly to } 5.3.2, \sigma \text{ sub } a \text{ denotes the property } \forall i, j (\sigma(i) = \sigma(j) \Rightarrow a_i = a_j). \]
\( M, a \vDash A[x] \) (in the new sense) is equivalent to \( M, u \vDash [a/x]A \) (in the old sense), where \( u \) is the world of \( a \). So 5.9.4(2)

\[
M, a \vDash P_k^m(x \cdot \sigma) \mid x \Leftrightarrow a \cdot \sigma \in \xi^+ (P_k^m)
\]

becomes

\[
M, u \vDash [a/x]P_k^m(x \cdot \sigma)(= P_k^m(a \cdot \sigma)) \Leftrightarrow a \cdot \sigma \in \xi^+ (P_k^m)
\]

exactly as in 5.2.8.

For evaluating a formula \( A \) we use assignments \((x, a)\) with \( r(x) \supseteq FV(A) \). The requirement \( r(x) = FV(A) \) is insufficient, e.g. in clause (5), because it may be that \( FV(B \supseteq C) \neq FV(B), FV(C) \).

We have to be careful about the quantifier clauses. In fact, if \( r(x) \supseteq FV(\exists y B) \), then either \( y \not\in r(x) \) or \( y \in r(x) \). In the first case we obtain the clause (9), which is clear. But if \( y \in r(x) \), the clause (9) is inapplicable, because \((xy, ac)\) is not a correct assignment — \( y \) is repeated twice. So before adding \( y \) to \( x \), we should eliminate it from \( x \) and its value from \( a \). This leads us to clause (10).

Sometimes it is convenient to join (9) and (10) in a single clause. To do this, let us introduce some more notation.

For a (distinct) list of variables \( x \) and \( y \in Var \) we put

\[
x - y := \begin{cases} 
  x - x_i & \text{if } y = x_i, \\
  x & \text{if } y \not\in x.
\end{cases}
\]

and

\[
x|\|y := (x - y)y.
\]

These lists can be obtained by corresponding transformations, viz. put

\[
\varepsilon_{x - y} := \begin{cases} 
  \delta^n_i & \text{if } y = x_i, \\
  id_n & \text{if } y \not\in x,
\end{cases}
\]

\[
\varepsilon_{x|\|y} := (\varepsilon_{x - y})^+ := \begin{cases} 
  (\delta^n_i)^+ & \text{if } y = x_i, \\
  id_{n+1} & \text{if } y \not\in x,
\end{cases}
\]

provided \(|x| = n \).\(^{14}\) then

\[
x \cdot \varepsilon_{x - y} = x - y, \ (xy) \cdot \varepsilon_{x|\|y} = x|\|y.
\]

The associated jlections are denoted as follows:

\[
\pi_{x - y} := \pi_{\varepsilon_{x - y}}, \ \pi_{x|\|y} := \pi_{\varepsilon_{x|\|y}}.
\]

\(^{13}\)This is always the case if \( r(x) = FV(\exists y B) \).

\(^{14}\)\(^+ \) means the simple extension, see the Introduction.
So we can combine (9), (10) from Definition 5.9.4 in a single clause:

\[ M, a \models \exists y B[x] \iff \exists c \in D_u M, \pi_{x\parallel y}(ac) \models B[x\parallel y], \quad (9 + 10) \]

and similarly for \( \forall \). In fact, if \( y \not\in x \), then \( \pi_{x\parallel y}(ac) = ac \) and \( x\parallel y \equiv xy \). And if \( y = x_i \), then

\[ \pi_{x\parallel y}(ac) = (ac) \cdot (\delta_i^n)^+ = (a - a_i)c, \quad x\parallel y = (x - x_i)x_i. \]

The clause (8) generalises a version of the truth definition for \( \Box B(a) \) in Kripke bundles given in 5.3.4. A similar version for functor semantics was considered in 5.6.15.

In fact, in the case of Kripke bundles

\[ (8) \quad M, a \models \Box B[x] \iff \forall b (aR^n b \Rightarrow M, b \models B[x]) \]

corresponds to

\[ (8.1) \quad M, u \models [a/x] \Box B \iff \forall v \in R(u) \forall b \in D_u^n (aR^n b \Rightarrow M, v \models [b/x]B), \]

i.e. to (*) in 5.3.4.

Let us also recall another reading of the \( \Box \)-clause in Kripke-type semantics considered so far:

\[ (8.2) \quad M, u \models \Box C \iff \text{for any } v \in R(u), \text{for any } v\text{-version } C' \text{ of } C \]

in \( v, M, v \models C' \),

where \( C \) is a \( D_u \)-sentence.

(8.1) transforms to (8.2) if we define \( v \) \-versions of a \( D_u \)-sentence \( [a/x]B \) as \( D_u \)-sentences \( [b/x]B \) for \( b \in R^n(a) \cap D_u^n \) (i.e. for \( b \) that are \( v \)-inheritors of \( a \)).

In a Kripke sheaf \( a \in D_u^n \) has a unique \( v \)-inheritor \( \rho_{uv} \cdot a \). In functor semantics inheritors also result from map actions: \( b = \rho_{uv} \cdot a \), for \( \mu \in C(u, v) \). In Kripke bundles we define the relation \( R^n \) in a special way.

In metaframes we can also call \( R^n \) the 'inheritance relation' between \( n \)-tuples and read (8) as

\[ (8') \quad M, a \models \Box B[x] \iff \text{for any inheritor } b \text{ of } a, \ M, b \models B[x]. \]

But now we cannot always define inheritance relations between \( D_u \)-sentences and transform (8) to (8.2) or (8.1). This happens, because a \( D_u \)-sentence can be presented as \( [a/x]A \) in different ways, as mentioned in 2.4. Different presentations may lead to different sets of inheritors for the same \( D_u \)-sentence. The following example shows that this is crucial.

**Example 5.9.7** Consider a \( D_u \)-sentence \( C := P(a, a) \). Then

\[ C = [aa/xy]B_1 = [a/x]B_2, \]

where \( B_1 := P(x, y) \), \( B_2 := P(x, x) \).
Put

\[ X_1 := R^2(aa) = \{ b \in D^2 \mid (aa)R^2b \}, \]
\[ X_2 := \{(dd) \in D^2 \mid aR^1d \}, \]
\[ B_1 := \{ [b/xy]B_1 \mid b \in X_1 \} = \{ P(b) \mid b \in X_1 \}, \]
\[ B_2 := \{ [d/x]B_1 \mid aR^1d \} = \{ P(b) \mid b \in X_2 \}. \]

Since \( X_1 \) consists of all inheritors of \( aa \), we may call the members of \( B_1 \) 'inheritors of \([aa/xy]B_1\)'. Similarly, the members of \( B_2 \) are 'inheritors of \([a/x]B_2\)'. But it may happen that \( X_1 \neq X_2 \),\(^15\) so \( B_1 \neq B_2 \), and thus we cannot properly define inheritors of \( C \). In this case (8) cannot be rewritten as (8.1). In fact, in a metaframe model \( M = (\mathbb{F}, \xi) \)

\[ M, aa \vDash \Box P(x, y) [xy] \iff X_1 \subseteq \xi^+(P), \]
\[ M, a \vDash \Box P(x, x) [x] \iff X_2 \subseteq \xi^+(P). \]

So there exists a model, in which

\[ M, aa \vDash \Box P(x, y) [xy] \nleftrightarrow M, a \vDash \Box P(x, x) [x] \]

just put

\[ \xi^+(P) := \begin{cases} X_2 \text{ if } X_1 \nsubseteq X_2, \\ X_1 \text{ if } X_2 \nsubseteq X_1. \end{cases} \]

So we see that in arbitrary metaframes there is no reasonable way to define the relation \( M, u \vDash A \) for \( D_u \)-sentences \( A \). That is why we define metaframe forcing in the form \( M, a \vDash A[x] \).

Now let us define the truth in a metaframe model.

**Definition 5.9.8** A modal formula \( A \) is called **true** in a metaframe model \( M = (\mathbb{F}, \xi) \) (notation: \( M \vDash A \)) if \( M, a \vDash A[x] \) for any ordered assignment \( (x, a) \) in \( \mathbb{F} \) such that \( FV(A) \subseteq r(x) \).

**Definition 5.9.9** A modal formula \( A \) is called **valid** in a metaframe \( \mathbb{F} \) (notation: \( \mathbb{F} \vDash A \)) if \( A \) is true in all models over \( \mathbb{F} \).

Let

\[ \text{ML}^{(=)}(\mathbb{F}) := \{ A \in MF^{(=)}_N \mid \mathbb{F} \vDash A \}. \]

**Definition 5.9.10** A formula \( A \in MF^{(=)}_N \) is called **strongly valid** (respectively, strongly valid with equality) in a metaframe \( \mathbb{F} \) if all its \( MF_N^- \) (respectively, \( MF_N^{+-} \)) substitution instances are valid in \( \mathbb{F} \) (notation: \( \mathbb{F} \vdash^+ A \) or respectively, \( \mathbb{F} \vdash^{+-} A \)). Let

\[ \text{ML}(\mathbb{F}) := \{ A \in MF_N \mid \mathbb{F} \vdash^+ A \}. \]
\[ \text{ML}^{=}(\mathbb{F}) := \{ A \in MF^{=}_N \mid \mathbb{F} \vdash^{+-} A \}. \]

**Definition 5.9.11** Let \( \mathcal{F} = (C, D, \rho) \) be a preset over an \( N \)-precategory \( C \). Then its associated metaframe is \( \text{MF}(\mathcal{F}) := ((F_n), D) \), where \( F_n \) is the \( n \)-level of \( \mathcal{F} \) (Definition 5.6.14). In the same way we define \( \text{MF}(\mathcal{F}) \) for a Kripke bundle \( \mathcal{F} \).

\(^{15}\)The reader can easily construct such an example.
So a Kripke bundle or a C-preset model $M = (F, \xi)$ corresponds to a metaframe model $Mf(M) := (Mf(F), \xi)$, and the two definitions of forcing 5.9.4, 5.6.9 are obviously the same:

**Lemma 5.9.12** Let $F$ be a preset over an $N$-precategory (or a Kripke bundle), $M$ a model over $F$. Then for any formula $A \in Mf(N)^{=}$ for any ordered assignment $(x, a)$ in $Mf(F)$ with $FV(A) \subseteq r(x)$

1. $M \models [a/x] A$ iff $Mf(M), a \models A [x]$.  
2. $M \models A$ iff $Mf(M) \models A$.

**Proof** (1) By straightforward induction on the length of $A$, according to 5.9.4, 5.6.9.  
(2) follows from (1), 5.6.9, 5.2.10.  

**Proposition 5.9.13** For a C-preset or a Kripke bundle $F$:

$$ML^{(=)}(F) = ML^{(=)}(Mf(F)),$$  
$$ML^{(=)}(F) = ML^{(=)}(Mf(F)).$$

**Proof** Follows readily from 5.9.12.  

**Remark 5.9.14** One can also define a metaframe $Mf(\Phi)$ for any Kripke quasi-bundle $\Phi$ (cf. 5.5.19) such that

$\Phi \models A \leftrightarrow Mf(\Phi) \models A$

for any modal formula $A$. The counterexample 5.5.19(a) shows that the set $ML(Mf(\Phi))$ is not always a modal predicate logic.

**Definition 5.9.15** A metaframe $F$ is called modally sound (respectively modally sound with equality), in brief, $m^{(=)}$-sound, if $ML^{(=)}(F)$ is an m.p.l.($=$).

Our goal in the next sections (5.10-5.12) will be to describe the class of $m^{(=)}$-sound metaframes; as we shall see, $m^{(=)}$-soundness is equivalent to $m$-soundness (Theorem 5.12.13).

But first, let us point out some peculiarities of forcing in arbitrary metaframes. This notion is still too broad, because $M, a \models A [x]$ may depend on the ordering of $x$ and $a$ and also on the variables $x_i$ that do not occur in $A$.

**Example 5.9.16** Let $F = (W, R)$ be a reflexive singleton $\{u\}$. Consider a metaframe $F$ over $F$ such that $D_u = \{c_1, c_2\}$,  

$R^1 := \{(c_i, c_j) \mid i \leq j\}, \quad R^2 := \{(a, a) \mid a \in D_u^2 \} \cup \{(c_1c_2, c_2c_2), (c_2c_1, c_2c_2)\}$.  

So $R^2$ is the same as the corresponding relation in the Kripke bundle in Fig. 5.9, but without the pair $(c_1c_2, c_2c_2)$.  


We do not specify the frames $F_n$ for $n > 2$; for example, $R^n$ can be universal on $D^n$. Consider a metaframe model $M = (F, \xi)$ such that

$$\xi^+(P) := \{c_2 c_2\}, \quad \xi^+(Q) := \{c_2\}$$

for certain $P \in PL^2$, $Q \in PL^1$.

Let $A := \Diamond P(x_1, x_2)$, then in $M$ we have

$$c_2 c_1 \vDash A[x_2 x_1] \text{ (since } (c_2 c_1) R^2(c_2 c_2)),$$

but

$$c_1 c_2 \vDash A[x_1 x_1] \text{ (since } (c_1 c_2) R^2(c_2 c_2)).$$

So $M, a \vDash A[x]$ depends on the ordering of $x$ and $a$.

Now put $B := \Diamond Q(x)$, then (in $M$)

$$c_1 \vDash B[x] \text{ (since } a_1 R^1 c_2),$$

$$c_1 c_1 \vDash B[xy] \text{ (since } (c_1 c_1) R^2(c_2 c_1)),$$

but

$$c_1 c_2 \not\vDash B[xy] \text{ (since } R^2(c_1 c_2) = \{c_1 c_2\} \text{ and } c_1 c_2 \not\vDash Q(x)[xy]).$$

So $M, a \vDash B[x]$ depends on the variables in $x$ that do not occur in $B$.

**Remark 5.9.17** The above counterexample may cause doubts about the clause (10) in Definition 5.9.4. In fact, as $M, a \vDash A[x]$ may depend on the values of $x_i$ that do not occur in $A$, it seems improper to make $M, a \vDash \exists x_i B[x]$ equivalent to $M, a - a_i \vDash \exists x_i B[x - x_i]$.

Instead we can define an alternative forcing relation $\vDash^*$ by the same clauses (1)–(9) and the following modification of (10).

$$(10^*) \quad M, a \vDash^* \exists x_i B[x] \text{ iff } \exists b \in D^n (b \cdot \delta_i^n = a \cdot \delta_i^n \& M, b \vDash^* B[x])$$

where $[c/a_i]a$ is obtained by replacing $a_i$ with $c$ at position $i$;\(^{16}\) and similarly

$$M, a \vDash^* \forall x_i B[x] \text{ iff } \forall b \in D^n (b \cdot \delta_i^n = a \cdot \delta_i^n \Rightarrow M, b \vDash^* B[x])$$

where $[c/a_i]a$ is obtained by replacing $a_i$ with $c$ at position $i$;

In arbitrary metaframes the relations $\vDash$ and $\vDash^*$ may not be equivalent:

---

\(^{16}\)So $b \cdot \delta_i^n = a \cdot \delta_i^n$ iff $b = [c/a_i]a$ for some $c \in D(a)$.\}
5.10. PERMUTABILITY AND WEAK FUNCTORIALITY

Exercise 5.9.18 Consider a metaframe model $M$ from Example 5.9.16 and formula $C = \neg Q(x_1) \land A$, where $A = \diamond P(x_1, x_2)$. Show that

$$M, c_1c_2 \models \exists x_1 C [x_1x_2] \text{ and } M, c_1c_2 \not\models \exists x_1 C [x_1x_2].$$

Hint: recall that $M, c_1c_2 \models A [x_2x_1]$ and $M, c_1c_2 \not\models A [x_1x_2]$.

On the other hand, later on we will show (see the end of Section 5.12) that $\models$ and $\models^*$ are equivalent in logically sound metaframes.

5.10 Permutability and weak functoriality

Now let us describe a class of metaframes without peculiarities mentioned at the end of Section 5.9. This class will be used in the further description of m-sound metaframes.

First we give the following

Definition 5.10.1 An N-metaframe $\mathbb{F}$ is called permutable if $\pi_\sigma$ is monotonic for every permutation $\sigma$:

$$\forall \sigma \in T_n \forall i \in I_N \forall a, b (aR^n_i b \rightarrow \pi_\sigma(a)R^n_i \pi_\sigma(b)).$$

Note that for $n = 0, 1$ this condition holds trivially, since in these cases $T^n_1 = \{id_n\}$.

Also note that to show permutability, it is sufficient to check monotonicity of jctions corresponding to (simple) transpositions $\sigma_k^n$, where $2 \leq k \leq n$, which are generators of the symmetric group.

Since for any permutation $\sigma$, $\pi_{\sigma^{-1}} = (\pi_\sigma)^{-1}$, we obtain the following equivalent condition:

Lemma 5.10.2 A metaframe $\mathbb{F}$ is permutable iff for any permutation $\sigma \in T_n$, $\pi_\sigma$ is an automorphism of the n-level $F_n$.

Proof (Only if.) $\pi_\sigma$ is a bijection for any $\sigma \in T_n$ by Lemma 0.0.1. $(\pi_\sigma a)R^n_i (\pi_\sigma b) \rightarrow aR^n_i b$ follows from 5.10.1 by applying $\pi_{\sigma^{-1}} = (\pi_\sigma)^{-1}$.  

Exercise 5.10.3 Show that in a 1-modal permutable metaframe, where not all domains are one-element, the relation $R^2$ cannot be a linear ordering.

Definition 5.10.4 A metaframe $\mathbb{F}$ is called weakly functorial (w-functorial) if for any injection $\sigma \in T_{mn}$ (where $m \leq n$), $\pi_\sigma$ is a p-morphism $F_n \Rightarrow F_m$.

From 5.9.6 we already know that $\pi_\sigma$ is surjective for $\sigma \in T_{mn}$.

To show w-functoriality, it suffices to check that $\pi_\sigma$ is a (p-)morphism, whenever $\sigma$ is a simple transposition $\sigma^n_\tau \in T_n$ or a simple embedding $\sigma^n_\tau \in \Sigma_{n,n+1}$, because every injection is a composition of functions of this kind.\(^{17}\)

Remembering that $\pi^n_\tau a = a - a_{n+1}$ for $n \neq 0$, and $\pi^0_\tau a$ is the world of $a$ (for $a \in D^1$), we obtain:

\(^{17}\)See Introduction.
Lemma 5.10.5 A metaframe $\mathbb{F} = (F_n)_{n \in \omega}, D$ is $w$-functorial iff $\mathbb{F}$ is permutable and for any $n > 0$, $i \in I_N$, the following conditions hold:

1. $\forall n > 1 \forall a, b \in D^n (a R^n_i b \Rightarrow (a - a_n) R^{n-1}_i (b - b_n))$ (the monotonicity for $\pi^{n-1}_+$);
2. $\forall n > 0 \forall a, b \in D^n \forall c \in D^1 (a R^n_i b & (ac) \in D^n \Rightarrow \exists d (ac) R^{n+1}_i (bd))$ (the lift property for $\pi^0_+$);
3. $\forall a \in D_u \forall b \in D_v (a R^1_j b = uR_i v)$ (the monotonicity for $\pi^0_+$);
4. $\forall u, v (uR_i v \Rightarrow \forall a \in D_u \exists b \in D_v aR^1_j b)$ (the lift property for $\pi^0_+$).

Lemma 5.10.6 Let $\mathbb{F}$ be a metaframe.

1. $\mathbb{F}$ is permutable iff for any model $M$ over $\mathbb{F}$, for any ordered assignment $(x, a)$, for any formula $A$ with $FV(A) \subseteq x$ and for any permutation $\sigma \in \gamma_n$, where $n = |x|$, $M, a \models A[x] \iff M, a \cdot \sigma \models A[x \cdot \sigma]$.

2. $\mathbb{F}$ is $w$-functorial iff (*) holds for any model $M$ over $\mathbb{F}$, for any ordered assignment $(x, a)$, for any injection $\sigma \in \gamma_{mn}$, where $|x| = n \geq m \geq 0$, and for any formula $A$ with $FV(A) \subseteq r(x \cdot \sigma)$.

Note that if $\sigma$ is injective and $(x, a)$ is an ordered assignment, then the list $x \cdot \sigma$ is distinct, so $(x \cdot \sigma, a \cdot \sigma)$ is also an ordered assignment.

Proof The case $n = 0$ (when $a, x$ are empty and $A$ is a sentence) is trivial.

(I) 'Only if' is proved by induction on $A$, and the proofs of (1) and (2) can be made in parallel. So we assume that $\mathbb{F}$ is respectively permutable or transformable and check the equivalence (*). The model $M$ is fixed, so we drop it from the notation. Let $y := x \cdot \sigma$; then $r(y) \subseteq r(x)$.

- Let $A$ be atomic; then since $FV(A) \subseteq r(y)$, it follows that $A = P^k_j(y \cdot \tau) = P^k_j(x \cdot (\tau \cdot \sigma))$ for some $j, k \geq 0, \tau \in \Sigma_{km}$ (not necessarily injective). Hence by definition, $a \models A[x]$ iff $a \cdot (\tau \cdot \sigma) \in \xi^+(P^k_j)$.

On the other hand,

$$(a \cdot \sigma) \models A[y] \iff (a \cdot \sigma) \cdot \tau \in \xi^+(P^k_j),$$

and since $(a \cdot \sigma) \cdot \tau = a \cdot (\tau \cdot \sigma)$, (*) follows.

- The cases when $A = \bot, B \land C, B \lor C, B \supset C$, are trivial.
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- Let $A = \Box_i B$. Then $a \vdash A [x]$ iff

  $$(3) \quad \forall b \in R^n_i (a) \ b \vdash B [x]$$

  which by induction hypothesis, is equivalent to

  $$(4) \quad \forall b \in R^n_i (a) \ \pi_\sigma (b) \vdash B [\pi_\sigma (x)].$$

  On the other hand, $\pi_\sigma (a) \vdash A [\pi_\sigma (x)]$ iff

  $$(5) \quad \forall b' \in R^n_i (\pi_\sigma (a)) \ b' \vdash B [\pi_\sigma (x)].$$

  But $R^n_i (\pi_\sigma (a)) = \pi_\sigma [R^n_i (a)]$, since $\pi_\sigma$ is a p-morphism. So (4) is equivalent to (5).

- Let $A = \exists y B$.

  Put $\tau := \varepsilon_{x-y}, \ \rho := \varepsilon_{x \cdot \sigma - y}$. Then $\varepsilon_{x||y} = \tau^+, \ \varepsilon_{(x \cdot \sigma)||y} = \rho^+.$

  So according to our definitions and notation (see condition (9+10) in Section 5.9), for any appropriate assignment $(x, a)$ we obtain\(^18\)

  $$(6) \quad a \vdash \exists y B [x] \text{ iff } \exists c \in D_u (ac) \cdot \tau^+ \vdash B [(xy) \cdot \tau^+],$$

  and

  $$(7) \quad a \cdot \sigma \vdash \exists y B [x \cdot \sigma] \text{ iff } \exists c \in D_u ((a \cdot \sigma)c) \cdot \rho^+ \vdash B [((x \cdot \sigma)y) \cdot \rho^+].$$

  In view of (6), (7), to apply the induction hypothesis, it suffices to find an injective $\lambda'$ such that

  $$(8) \quad ((xy) \cdot \tau^+) \cdot \lambda' = ((x \cdot \sigma)y) \cdot \rho^+;$$

  $$(9) \quad ((ac) \cdot \tau^+) \cdot \lambda' = ((a \cdot \sigma)c) \cdot \rho^+;$$

  i.e.

  $$(10) \quad (xy) \cdot (\tau^+ \cdot \lambda') = (xy) \cdot (\sigma^+ \cdot \rho^+);$$

  $$(11) \quad (ac) \cdot (\tau^+ \cdot \lambda') = (ac) \cdot (\sigma^+ \cdot \rho^+).$$

  Note that $x \cdot \sigma - y$ is a distinct list of variables from $x - y$. So by Lemma 0.0.8, $x \cdot \sigma - y = (x - y) \cdot \lambda$ for some injection $\lambda$.

  **Claim** $\tau \cdot \lambda = \sigma \cdot \rho$.

  For the proof note that

  $$x \cdot \tau = x - y, \ (x \cdot \sigma) \cdot \rho = x \cdot \sigma - y,$$

  and thus

  $$x \cdot (\tau \cdot \lambda) = (x \cdot \tau) \cdot \lambda = (x - y) \cdot \lambda = x \cdot \sigma - y = (x \cdot \sigma) \cdot \rho = x \cdot (\sigma \cdot \rho).$$

\(^{18}\)Remember that $(xy) \cdot \tau^+ = x||y, ((x \cdot \sigma)y) \cdot \rho^+ = (x \cdot \sigma)||y.$
Since $x$ is distinct, the claim now follows by Lemma 0.0.5.

Hence by Lemma 0.0.7, $\tau^+ \cdot \lambda^+ = \sigma^+ \cdot \rho^+$, so (10), (11) hold for $\lambda' := \lambda^+$.

(II) Now let us prove the ‘if’ part in both assertions, (1) and (2). We assume (*) and check the $p$-morphism properties of $\pi_\sigma$ (respectively, for $\sigma \in \Upsilon_m$ or $\sigma \in \Upsilon_n$).

To check the monotonicity, we also assume $a R_i^m b$, and show that $\pi_\sigma(a) R_i^m \pi_\sigma(b)$. Let $A = \diamond_i P(\pi_\sigma(x))$, where $x = (x_1, \ldots, x_n)$ is distinct. Consider the model $M = (F, \xi)$ such that $\xi^+(P) = \{\pi_\sigma(b)\}$. Then by Definition 5.9.4

$$M, b \models P(\pi_\sigma(x))[x],$$

and so

$$M, a \models A[x].$$

Hence by (*),

$$M, \pi_\sigma(a) \models A[\pi_\sigma(x)],$$

which means by Definition 5.9.4, $\pi_\sigma(a) R_i^m b'$ for some $b' \in \xi^+(P)$. By the choice of $\xi$, $b' = \pi_\sigma(b)$. Therefore $\pi_\sigma(a) R_i^m \pi_\sigma(b)$.

Note that the same argument is valid in the particular case $m = 0$; then we have $\sigma = \emptyset$, $A = \diamond_i P$ for $P \in PL^0$, $\xi^+(P) = \{\pi_\sigma(b)\}$.

To check the lift property, we assume $x_\sigma(a) R_i b'$ and find $b \in R_i^m(a)$ such that $\pi_\sigma(b) = b'$.

Consider the same formula $A = \diamond_i P(\pi_\sigma(x))$, and the model $M = (F, \xi)$ such that $\xi^+(P) = \{b'\}$. Then

$$M, b' \models P(\pi_\sigma(x))[\pi_\sigma(x)],$$

and thus

$$M, \pi_\sigma(a) \models A[\pi_\sigma(x)].$$

Hence by (*),

$$M, a \models A[x],$$

i.e. by 5.9.4,

$$M, b \models P(\pi_\sigma(x))[x]$$

for some $b \in R_i^m(a)$. But then $\pi_\sigma(b) \in \xi^+(P)$, according to 5.9.4, thus $\pi_\sigma(b) = b'$, by the choice of $\xi$.

Lemma 5.10.6(1) shows that in permutable metaframes forcing $M, a \models A[x]$ is invariant under simultaneous permutations of $x$ and $a$. So we can also define forcing under unordered assignments. Such an assignment (at a world $u$) is nothing but a $D_u$-substitution (Section 2.2), i.e. a function $[a/x]$ sending every $x_i$ to $a_i$. An ordered assignment $(x, a)$ corresponds to the unordered assignment $[a/x] = \{(x_1, a_1), \ldots, (x_n, a_n)\}$, where $n = |x|$. So in a model $M$ over a permutable metaframe we can define

$$M, [a/x] \models A := M, a \models A[x].$$
This definition is sound; in fact, \([a/x] = [b/y]\) iff \((y, b)\) is obtained from \((x, a)\) by some permutation, i.e. iff \(y = x \cdot \sigma,\) \(b = a \cdot \sigma\) for some \(\sigma \in \Sigma_n\). So by Lemma 5.10.6(1),

\[ M, a \models A [x] \text{ iff } M, b \models A [y]. \]

Furthermore, in w-functorial metaframes the truth value of a formula \(A\) depends only on individuals assigned to parameters of \(A\) (or only on a possible world if \(A\) is a sentence). Viz., suppose \(r(y) \supseteq FV(A) = r(x)\). Then \(x = y \cdot \sigma\) for some injection \(\sigma\). So by Lemma 5.10.6(2),

\[ M, a \models A [y] \text{ iff } M, a \cdot \sigma \models A [x]. \]

Thus forcing \(M, a \models A [y]\) reduces to \(M, b \models A [x]\) for an appropriate \(b\). In particular:

**Lemma 5.10.7** If \(r(x) = FV(A)\), \(z\) is a distinct list of other variables and \(c\) is a tuple of individuals (from the world of \(a\)) of the same length, then

\[ M, a \models A [x] \text{ iff } M, ac \models A [xz]. \]

Since in w-functorial metaframes we may use only the forcing relation \(M, a \models A [x]\) for \(r(x) = FV(A)\), in Definition 5.9.4 we need only the clause (9) for \(A = \forall y B\) or \(\exists y B\), because clearly \(y \notin FV(A) = r(x)\).

**Exercise 5.10.8** Show that in a permutable metaframe for any model \(M\)

\[ M, a \models \exists x_i B [x] \text{ iff } \exists c \in D(a) \ M, [c/a_i]a \models B [x], \]

where the tuple \([c/a_i]a\) is obtained from \(a\) by replacing \(a_i\) with \(c\); cf. Remark 5.9.17.

Hint: \((x - x_i)x_i = x \cdot \sigma\) and \((a - a_i)c = ([c/a_i]a) \cdot \sigma\) for \(\sigma \in \Sigma_n\).

So by induction on \(|A|\), one can easily show that

\[ M, a \models A[x] \iff M, a \models^{*} A[x] \]

for a model \(M\) over a permutable metaframe for any \(A \in MF^=\), where \(\models^{*}\) denotes the modified forcing from Remark 5.9.17.

To show that forcing in w-functorial metaframes is congruence independent, we begin with an auxiliary lemma.

**Lemma 5.10.9** Let \(M\) be model over an arbitrary metaframe, \(A\) a modal formula, \((x, a)\) an assignment in \(M\) such that \(FV(A) \subseteq r(x)\). Also let \(y' \notin V(A), y \notin BV(A), x' = [y'/y]x, \) and \(A' = A[y \mapsto y']\). Then

\[ M, a \models A [x] \text{ iff } M, a \models A' [x']. \]
Proof This statement is quite obvious — it means that the truth value of a formula does not depend on the name of a certain free variable. Formally we argue by induction on the complexity of $A$. Let us consider three cases.

(1) If $A = P(x \cdot \sigma)$, then $A' = P(x' \cdot \sigma)$. So

$$M, a \vDash A [x] \iff \pi_\sigma a \in \xi^+(P) \iff M, a \vDash A' [x'].$$

(2) Let $A = \exists z B$. Then $z \neq y, y'$ by the assumption of our lemma, so $A' = \exists z B'$, $(x || z)' = x' || z$, $\pi_{x' || z} = \pi_{x || z}$. Then by the modified clause (9+10) of Definition 5.9.4 and the induction hypothesis,

$$M, a \vDash A [x] \iff \exists c \in D(a) M, \pi_{x' || z}(ac) \vDash B \ [x || z]$$

$$\iff \exists c \in D(a) M, \pi_{x' || z}(ac) \vDash B' \ [x' || z] \iff M, a \vDash \exists z B' (= A') [x'].$$

(3) Let $A = \Box_i B$, $a \in D^m$. By Definition 5.9.4 and the induction hypothesis,

$$M, a \vDash A[x] \iff \forall b \in R_i^n(a) M, b \vDash B \ [x]$$

$$\iff \forall b \in R_i^n(a) M, b \vDash B' \ [x'] \iff M, a \vDash \Box_i B' (= A') [x']$$

All other cases are rather trivial.

Lemma 5.10.10 Let $M$ be model over a $w$-functorial metaframe, and let $A, A'$ be congruent modal formulas. Then for any assignment $[a/x]$ with $FV(A) \subseteq r(x)$

$$M, [a/x] \vDash A \iff M, [a/x] \vDash A'$$

(***)

and thus

$$M \vDash A \iff M \vDash A'.$$

Proof We apply Proposition 2.3.17. Consider the equivalence relation between formulas:

$$A \sim B := \text{for any assignment } [a/x] \text{ such that } FV(A), FV(B) \subseteq r(x),$$

$$M, [a/x] \vDash A \iff M, [a/x] \vDash B.$$ 

It suffices to check that $\sim$ satisfies the conditions (1)–(4) from 2.3.17.

(1) $M, [a/x] \vDash Q_y A \iff M, [a/x] \vDash Q_z (A[y \mapsto z])$ provided $y \notin BV(A)$, $z \notin V(A)$, $Q \in \{\forall, \exists\}$ and $FV(Q y A) \subseteq r(x)$.

(Obviously, $FV(Q z (A[y \mapsto z])) = FV(Q y A)$.)

Consider e.g. the case $Q = \exists$. We may assume $r(x) = FV(Q y A)$, since the metaframe is $w$-functorial. Then $y, z \notin r(x)$, so we have

$$M, [a/x] \vDash \exists y A \iff \exists c \in D(a) M, [ac/xy] \vDash A$$

$$\iff \exists c \in D(a) M, [ac/xz] \vDash A[y \mapsto z] \ (\text{by Lemma 5.10.9})$$

$$\iff M, [a/x] \vDash \exists z (A[y \mapsto z]).$$
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(2) Assuming \( A \sim B \) let us prove \( QyA \sim QyB \) (for \( Q = \exists \)). We again assume \( r(x) = FV(A) \), so \( y \not\in r(x) \), and thus we obtain

\[
M, [a/x] \models \exists yA \iff \exists c \in D(a) \ M, [ac/xy] \models A \iff (\text{since } A \sim B)
\exists c \in D(a) \ M, [ac/xy] \models B \iff M, [a/x] \models \exists yB.
\]

(3) This property holds trivially, by the definition of forcing.

(4) Assuming \( A \sim B \) let us show \( \Box_i A \sim \Box_i B \):

\[
M, [a/x] \models \Box_i A \iff \forall b \in R^n_i(a) \ M, [b/x] \models A
\iff \forall b \in R^n_i(a) \ M, [b/x] \models B \iff M, [a/x] \models \Box_i B.
\]

Hence we readily obtain

**Lemma 5.10.11** Let \( F \) be a w-functorial metaframe. Then for any congruent modal formulas \( A, A' \)

\[
F \models A \iff F \models A'.
\]

**Remark 5.10.12** We cannot extend Lemma 5.10.10 to arbitrary (not w-functorial) metaframes. For example, suppose \( r(xz) = FV(B) \), \( z \not\in BV(B) \) and \( y, y' \) are distinct variables that do not occur in \( B \). Then the formulas \( \exists y(B[z \mapsto y]) \) and \( \exists y'(B[z \mapsto y']) \) are congruent. However by 5.9.4 and 5.10.10

\[
M, ab \models \exists y(B[z \mapsto y]) [xy] \iff \exists c \in D(a) \ M, ac \models B[z \mapsto y] [xy] \iff \exists c \in D(a) \ M, ac \models B [xz],
\]

while

\[
M, ab \models \exists y'(B[z \mapsto y']) [xy] \iff \exists c \in D(a) \ M, abc \models B[z \mapsto y'] [xyy'] \iff \exists c \in D(a) \ M, abc \models B [xyz].
\]

Now, e.g. if \( B = \Diamond P(x, z) \), then (*1) means

\[
\exists c \exists d \exists e ((ac)R^{n+1}(de) \& de \in \xi^+(P)),
\]

and (*2) means

\[
\exists c \exists d \exists e \exists e' ((abc)R^{n+1}(de'e) \& de \in \xi^+(P)).
\]

If a metaframe is permutable, but not w-functorial, these conditions may be not equivalent, cf. Example 5.9.16.

On the other hand, we do not know if w-functoriality is necessary for Lemma 5.10.10. One can try to construct counterexamples explicitly, as in the proof of Lemma 5.10.6.
5.11 Modal metaframes

Definition 5.11.1 An \((N)\)-modal metaframe is a \(w\)-functorial metaframe satisfying

\[(mm_2) \ \forall i \in I_N \ \forall a,b_1,b_2 ((a,a)R_i^2(b_1,b_2) \Rightarrow b_1 = b_2).\]

An equivalent form of \((mm_2)\) is \(R_i^2(\Delta) \subseteq \Delta\), where

\[\Delta := \{(a,a) \mid a \in D^1\}.

For the next lemma recall that

\[a \ \text{sub} \ b \iff \forall j,k (a_j = a_k \Rightarrow b_j = b_k),\]

\[\pi^n a = aa_n \ \text{for} \ a \in D^n.

Lemma 5.11.2 For an \(N\)-modal metaframe for any \(n > 0\), \(i \in I_N\) we have

\[(mm_n) \ \forall a \forall b \ (aR_i^n b \Rightarrow a \ \text{sub} \ b);\]

\[(mm^+_n) \ \forall a \forall b \ (aR_i^n b \Rightarrow (\pi^n a)R_i^{n+1}(\pi^n b)).\]

Proof In fact, \((mm_n)\) follows from \((mm_2)\), since \((a_j, a_k) = a \cdot \lambda^n_{jk}\) (see the Introduction). So for a \(w\)-functorial metaframe \(aR_i^n b\) implies \((a_j, a_k)R_i^2(b_j, b_k)\), and we can apply \((mm_2)\) to \((a_j, a_k)\) and \((b_j, b_k)\).

To check \((mm^+_n)\), note that \(\pi^n a : F_{n+1} \rightarrow F_n\) in a \(w\)-functorial metaframe. Thus by the lift property,

\[aR_i^n b \text{ implies } \exists c \ (aa_n)R_i^{n+1}(bc),\]

and by \((mm_{n+1})\), it follows that \(c = b_n\). Hence

\[\pi^n a = (aa_n)R_i^{n+1}(bb_n) = \pi^n b.\]

The next lemma will be mainly used later on, in the intuitionistic case (section 5.14).

Lemma 5.11.3 If an \(N\)-metaframe satisfies \((mm_m)\), \(i \in I_N\) and \(\sigma \in \Sigma_{mn}\), then \(\pi_\sigma[D^n]\) is \(R_i^m\)-stable.

Proof If \(\pi_\sigma(a)R_i^m b\), then \(\pi_\sigma(a) \ \text{sub} \ b\), and the latter yields \(\sigma \ \text{sub} \ b\), i.e. \(b \in \pi_\sigma[D^n]\), by 5.9.5.

Exercise 5.11.4 Show that for a certain \(i\), \((mm_m)\) is equivalent to \(R_i^m\)-stability of all sets \(\pi_\sigma[D^n]\) for \(\sigma \in \Sigma_{mn}, \ n > 0\).

Hint: \(a \in D^m\) can be presented as \(\pi_\sigma c\) for some tuple \(c\) with different components, where \(\sigma \in \Sigma_{mn}, \ n = |c|\).
5.11. MODAL METAFRAMES

Definition 5.11.5 A metaframe $\mathbb{F}$ is called functorial if every its jection $\pi_\sigma : D^n \to D^m$ is a morphism $F_n \to F_m$.

Lemma 5.11.6 A metaframe is modal iff it is functorial.

Proof (Only if.) Assume that a metaframe $\mathbb{F}$ is modal. Recall that every function $\sigma \in \Sigma_{mn}$ is a composition of simple injections and simple projections.\(^\text{19}\)
Since $\mathbb{F}$ is w-functorial, it is sufficient to show that every jection $\pi^n_\sigma (= \pi^n_{\sigma^n})$ is a p-morphism. Its monotonicity is already stated by (mm). To check the lift property, assume $\pi^n_\sigma (a) = (aa_n)R_i^{n+1}(bc)$. Then $(aa_n) \sub (bc)$ by (mm\(_{n+1}\)), and hence $b_n = c$. Thus $bc = bb_n = \pi^n_\sigma (b)$. We also have $aR_i^n b$ by Lemma 5.10.5(1), since $\mathbb{F}$ is w-functorial.

(If.) We assume that $\mathbb{F}$ is functorial and check (mm\(_2\)). In fact, suppose $(a, a) = \pi^1_\sigma (a)R_i^1(b_1, b_2)$. Since $\pi^1_\sigma$ is a p-morphism, $(b_1, b_2) = \pi^1_\sigma (b)$ for some $b$, i.e. $b_1 = b_2 = b$.

The next lemma shows that forcing in functorial metaframes 'respects variable substitutions'.

Lemma 5.11.7 Let $M = (\mathbb{F}, \xi)$ be a model over a functorial metaframe. Then, for any modal formula $A$
(\(\#\)) for any $\sigma \in \Sigma_{mn}$, for any distinct lists of variables $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ such that $FV(A) \subseteq r(y)$, and for any $a \in D^n$:

$$M, [a/x] \models [x \cdot \sigma / y] A \iff M, [a \cdot \sigma / y] \models A.$$ 

Recall that $[x \cdot \sigma / y] A$ is defined up to congruence. But this does not matter in functorial metaframes, by 5.10.10.

Proof By Lemma 5.10.10, the claim (\(\#\)) does not change if we replace $A$ (in both sides) with any $B \equiv A$. So we may replace $A$ with its clean version $A^\circ$ such that $BV(A^\circ) \cap r(xy) = \emptyset$ (and $FV(A^\circ) = FV(A) \subseteq r(y)$); then we have

$$[x \cdot \sigma / y] A \equiv A^\circ [y \mapsto x \cdot \sigma].$$ 

To simplify notation, we now assume $A^\circ = A$ and proceed by induction on $A$.

- Let $A = P_j^k (y \cdot \tau), \tau \in \Sigma_{km}$. Then $[x \cdot \sigma / y] A = P_j^k ((x \cdot \sigma) \cdot \tau) = P_j^k (x \cdot (\sigma \cdot \tau))$.

So by Definition 5.9.4, we have:

$$M, [a/x] \models [x \cdot \sigma / y] A \iff a \cdot (\sigma \cdot \tau) = ((a \cdot \sigma) \cdot \tau) \in \xi(P_j^k) \iff M, [a \cdot \sigma / y] \models A.$$ 

- Let $A = (y_j = y_k)$, then $[x \cdot \sigma / y] A = (x_{\sigma(j)} = x_{\sigma(k)})$. So we have

$$M, [a/x] \models [x \cdot \sigma / y] A \iff a_{\sigma(j)} = a_{\sigma(k)} \iff M, [a \cdot \sigma / y] \models (y_j = y_k).$$

---

\(^{19}\)See Introduction.
Let $A = \exists z B$, then $z \not\in r(xy)$ by the choice of $A$, and so

$$[x \cdot \sigma/y] A = \exists z [x \cdot \sigma/y] B = \exists z [(xz) \cdot \sigma^+/yz] B.$$  

Hence

$$M, [a/x] \models [x \cdot \sigma/y] A \iff \exists c M, [ac/xz] \models [(xz) \cdot \sigma^+/yz] B \iff \exists c M, [(ac) \cdot \sigma^+/yz] \models B \text{ (by the induction hypothesis)} \iff \exists c M, [(a \cdot \sigma)c/yz] \models B \iff M, [a \cdot \sigma/y] \models A.$$  

Let $A = \Box_i B$. By Definition 5.9.4

$$M, [a/x] \models [\pi_\sigma(x)/y] A \iff \forall b \in R_i^n(a) M, [b/x] \models [\pi_\sigma(x)/y] B \iff \forall b \in R_i^n(a) M, [\pi_\sigma(b)/y] \models B \text{ (by the induction hypothesis)} \iff \forall b' \in \pi_\sigma[R_i^n(a)] M, [b'/y] \models B. \quad (1)$$

On the other hand, $M, [\pi_\sigma(a)/y] \models A$ is equivalent to

$$\forall b' \in R_i^m(\pi_\sigma(a)) M, [b'/y] \models B. \quad (2)$$

Now, since $\pi_\sigma$ is a $p$-morphism, we have $R_i^m(\pi_\sigma(a)) = \pi_\sigma[R_i^n(a)]$. Thus $(1) \iff (2)$.

All other cases are obvious.

Hence it follows that variable substitutions preserve validity.

Lemma 5.11.8 Let $F$ be a functorial metaframe, $A$ a modal formula such that $F \models A$. Then for any variable substitution $[y/x]$, $F \models [y/x] A$.

Proof Since $[y/x] A$ does not depend on the variables beyond $FV(A)$, we may assume that $r(x) = FV(A)$; then $r(y) = FV([y/x] A)$. Let $z$ be a distinct list such that $r(z) = r(y)$; then $y = z \cdot \sigma$ for some transformation $\sigma$. By Definitions 5.9.8, 5.9.9 and Lemma 5.10.7, $F \models [y/x] A$ iff for any model $M$ over $F$, for any assignment $[a/z]$ in $F$

$$M, [a/z] \models [y/x] A.$$ 

By 5.11.7, the latter is equivalent to

$$M, [a \cdot \sigma/x] \models A,$$

which follows from $F \models A$. Therefore, $[y/x] A$ is valid in $F$.

The next lemma can be considered as a converse to 5.11.7 in a stronger form.

Lemma 5.11.9 Let $F$ be a metaframe such that for any model $M$ over $F$, for any quantifier-free formula $A$
(‡) for any \( \sigma \in \Sigma_{mn} \), for any distinct lists \( x \) of length \( n \) and \( y \) of length \( m \) such that \( \text{FV}(A) \subseteq r(y) \), for any \( a \in F_n \)

\[
M, a \models (\{ x \cdot \sigma/y \} A) [x] \iff M, a \cdot \sigma \models A [y].
\]

Then \( F \) is functorial.

**Proof** Given an arbitrary \( \sigma \in \Sigma_{mn} \), let us check the p-morphism properties for \( \pi_\sigma \).

1. \( aR_i^n b \Rightarrow (\pi_\sigma a)R_i^m (\pi_\sigma b) \).
   In fact, assume \( aR_i^n b \). Consider the formula \( A := \Diamond_i P(y), \ P \in PL^m \) and the model \( M = (\mathcal{F}, \xi) \) such that \( \xi^+(P) = \{ \pi_\sigma b \} \). Then \( [\pi_\sigma x/y] A = \Diamond_i P(\pi_\sigma x) \),

\[
M, b \models P(\pi_\sigma x) [x],
\]

and so

\[
M, a \models [\pi_\sigma x/y] A [x].
\]

Hence by assumption (‡) we have \( M, \pi_\sigma a \models A [y] \), and thus

\[
M, b' \models P(y) [y]
\]

for some \( b' \in R_i^m (\pi_\sigma a) \). Hence \( b' \in \xi^+(P) \), i.e. \( b' = \pi_\sigma b \), by the choice of \( \xi \), so we obtain \( (\pi_\sigma a)R_i^m (\pi_\sigma b) \).

2. \( (\pi_\sigma a)R_i^m b' \Rightarrow \exists b \in R_i^n (a) \pi_\sigma b = b' \).
   Assume \( (\pi_\sigma a)R_i^m b' \). Consider the same formula \( A = \Diamond_i P(y) \) and the model \( M = (\mathcal{F}, \theta) \) such that \( \theta^+(P) = \{ b' \} \). Then

\[
M, b' \models P(y) [y],
\]

and so

\[
M, \pi_\sigma a \models A [y].
\]

Thus by assumption (‡),

\[
M, a \models [\pi_\sigma x/y] A (= \Diamond_i P(\pi_\sigma x)) [x],
\]

i.e. \( aR_i^n b \) for some \( b \) such that

\[
M, b \models P(\pi_\sigma x) [x].
\]

Hence \( \pi_\sigma b \in \theta^+(P) \), i.e. \( \pi_\sigma b = b' \).

Obviously, if \( m = 0 \), \( \sigma = \emptyset_n \), then we can use the formula \( A = \Diamond_i p \) (with \( p \in PL^0 \)) both in (1) and (2).  

**Remark 5.11.10** Note that in the above proof we use the condition (‡) only for formulas of the form \( \Diamond_i P(y) \) or \( \Diamond_i p \).
Hence we obtain

**Proposition 5.11.11** For a metaframe $\mathcal{F}$ the following properties are equivalent:

1. $\mathcal{F}$ is functorial;
2. 5.11.9 (‡) holds for any formula $A$;
3. 5.11.9 (‡) holds for any quantifier-free $A$.

Actually (‡) in (2) should be formulated as the equivalence

$$M, a \vDash ([x \cdot \sigma/y]A) [x] \iff M, a \cdot \sigma \vDash A [y]$$

for any congruent version of $[x \cdot \sigma/y]A$.

Note that variable substitutions are defined up to congruence, while in non-functorial metaframes forcing may be sensitive to congruence. But for a quantifier-free $A$, the formula $[x \cdot \sigma/y]A$ is unique, so there is no ambiguity in (3).

**Proposition 5.11.12** Let $M$ be a model over a functorial metaframe, $u \in M$, and let $A$, $A^*$ be modal formulas, $[a/x]$, $[a^*/x^*]$ assignments giving rise to equal $D_u$-sentences: $[a/x]A = [a^*/x^*]A^*$. Then

$$M, a \vDash A [x] \iff M, a^* \vDash A [x^*].$$

**Proof** Recall that (Lemma 2.4.2) a $D_u$-sentence $[a/x]A$ can be presented in the form $[b/y]B$, where $b$ is a list of distinct individuals (from $D_u$) and $r(y) = FV(B)$; this presentation is unique up to the choice of a distinct list $y$. So let us show that

1. $M, [a/x] \vDash A$ iff $M, [b/y] \vDash B$

and

1*). $M, [a^*/x^*] \vDash A$ iff $M, [b/y] \vDash B$.

Since both these assertions are similar, it is sufficient to check (1). Let $x'$ be a sublist of $x$ such that $FV(A) = r(x')$, $a'$ the corresponding sublist of $a$ (i.e. if $x' = x \cdot \tau$ for an injection $\tau$, then $a' = a \cdot \tau$). Thus

2. $M, [a/x] \vDash A$ iff $M, [a'/x'] \vDash A$

by Lemma 5.10.6, since $\mathcal{F}$ is w-functorial. Here $b$ is a list of distinct individuals from $a'$, and $B$ is obtained from $A'$ by identifying those variables $x_j$ in $x'$, which correspond to equal individuals $a_j$ in $a'$ (cf. the proof of Lemma 2.4.2). Thus if $m = |x'| = |a'|$ and $n = |y| = |b|$ is the number of distinct individuals in $a'$, then for some function $\sigma \in \Sigma_{mn}$, we have $a' = b \cdot \sigma$, $B = [y \cdot \sigma/x']A$, Now by 5.11.7(‡),

3. $M, [b/y] \vDash B (= [y \cdot \sigma/x']A) \iff M, [b \cdot \sigma/x'] \vDash A$.

Since $b \cdot \sigma = a'$, (1) follows from (2) and (3).
Again we have the converse to the above proposition in the following stronger form.

**Corollary 5.11.13** Let $F$ be a metaframe such that for any model $M$ over $F$, for any $u \in M$, quantifier-free formulas $A, A^*$ and assignments $[a/x], [a^*/x^*]$ in $F$ such that $[a/x]A = [a^*/x^*]A^*$.

$(\#\#)$ \ $M, a \vDash A [x] \iff M, a^* \vDash A^* [x^*].$

Then $F$ is functorial.

**Proof** Note that $(\#\#)$ implies 5.11.9 $(\#)$ for a quantifier-free $A$, since $[a/x][(x \cdot \sigma/y)A] = [a \cdot \sigma/y]A$ by 2.4.2(4). So we can apply Lemma 5.11.9.  

Therefore modal (i.e. functorial) metaframes are exactly those in which forcing can be defined for $D_n$-sentences, cf. Section 5.9.

### 5.12 Modal soundness

In this section we show that modal metaframes are exactly $m(=)$-sound metaframes.

Recall that by Definition 5.9.8, for a metaframe model $M = (F, \xi)$, $M \vDash A$ if $M, a \vDash A [x]$ for any ordered assignment $(x, a)$ in $F$ with $FV(A) \subseteq r(x)$. For a w-functorial metaframe we can fix the list $x$ as the following simple lemma shows.

**Lemma 5.12.1** Let $M = (F, \xi)$ be a model over a w-functorial $N$-metaframe $F$, $A$ an $N$-modal formula. Then the following conditions are equivalent.

1. $M \vDash A$;
2. $M, a \vDash A [x]$ for any ordered assignment $(x, a)$ in $F$ such that $FV(A) = r(x)$;
3. there exists a list of distinct variables $y$ containing $FV(A)$ such that $M, b \vDash A [y]$ for any ordered assignment $(y, b)$ in $F$.

**Proof** Obviously, (1) implies (2), and (2) implies (3).

The other way round, assuming (2), let us show (1), i.e. $M, [b/y] \vDash A$ for any assignment $[b/y]$ with $r(y) \supseteq FV(A)$. By 0.0.7, we have $x = y \cdot \sigma$ for some injection $\sigma$, hence by Lemma 5.10.6, $M, [b/y] \vDash A$ is equivalent to $M, [b \cdot \sigma/x] \vDash A$, which holds by our assumption.

Finally, assuming (3), let us check (2). In fact, again we have $x = \pi_{\sigma}(y)$ for an injection $\sigma$, and $a = \pi_{\sigma}(b)$ for some $b$, due to the surjectivity of $\pi_{\sigma}$ (see Lemma 5.9.6); thus 5.10.6 can be applied.

Recall that $ML(=)(F)$ denotes the set of all formulas valid in $F$ (Definition 5.9.9).

---

20 This equality holds for a quantifier-free $A$; for an arbitrary $A$ it should be replaced by congruence.
Lemma 5.12.2 For any metaframe F:

(1) $\text{ML}^{(\pi)}(F)$ is closed under necessitation and generalisation.

(2) If F is w-functorial, then $\text{ML}^{(\pi)}(F)$ is closed under modus ponens.

(3) If F is w-functorial, then for any $B \in M F^w$

$$F \vdash B \iff F = \forall yB \iff F \vdash \neg B.$$  

(4) If F is functorial, then $\text{ML}^{(\pi)}(F)$ is closed under strict substitutions.

Proof
(1) Rather trivial. Note that by definition (cf. (8) and (9+10) in Section 5.9)

$$M, a \models \Box, B[x] \iff \forall b \in R^a_{\pi}(a) \ M, b \models B[x];$$

and

$$M, a \models \forall yB[x] \iff \forall c \in D_u \ M, \pi_{x\|y}(ac) \models B[x\|y].$$

(if $a \in D_u^a$).

(2) If $M, [a/x] \models B \supset C$ and $M, [a/x] \models B$, for any assignment $(x, a)$ with a fixed $x$ containing $FV(B \supset C) \supseteq FV(C)$, then we have $M, [a/x] \models C$. Hence $M \models C$ by Lemma 5.12.1.

(3) Obviously, it suffices to check the first equivalence. $(\Rightarrow)$ follows readily from (1).

$(\Leftarrow)$ Suppose $F \models \forall yB$ and consider a model $M$ over $F$. Let $r(x) = FV(\forall yB)$, then for any assignment $(x, a)$ we have $M, a \models \forall yB[x]$. Since $y \notin x$, this means that $M, ac \models B[xy]$ for any assignment $(xy, ac)$. Hence $M \models B$ by Lemma 5.12.1.

(4) Suppose $F \models A$. Consider a strict substitution $S = [C/P(y)]$, where P is an m-ary predicate letter, $y = (y_1, \ldots, y_m)$ is a distinct list containing $FV(C)$, and let us show that $F \models SA$.

Recall that a substitution instance $[C/P(y)]A$ is defined up to congruence and can be obtained from a clean version $A^c$ of $A$ by replacing every subformula of the form $P(y')$ with $[y'/y]C$ ($y'$ may be not distinct). By Lemma 5.10.10, for any model $M$ over $F$ we have

$$M \models A \iff M \models A^c,$$

and thus

$$F \models A \iff F \models A^c.$$

So we may assume that $A$ is clean (i.e. $A = A^c$).

For a model $M = (F, \xi)$ and a list of distinct variables x such that $r(x) = FV(A) \supseteq FV([C/P(y)]A)$, we have to show that $M \models SA$, i.e. that

$$(4.1) \ M, [a/x] \models [C/P(y)]A \text{ for any assignment } [a/x].$$
Since \( A \) is clean, \( BV(A) \cap r(x) = \varnothing \).

Consider the model \( N = (\mathcal{F}, \eta) \) such that
\[
\eta^+(P) = \{ b \mid M, [b/y] \models C \},
\eta^+(Q) = \xi^+(Q) \text{ for any other predicate letter } Q.
\]

We claim that
\[(4.2) \quad N, [a/x] \models A \text{ iff } M, [a/x] \models SA.\]

By our assumption, \( \mathcal{F} \models A \), thus \( N, [a/x] \models A \), and so (4.2) implies (4.1). To show (4.2), let us prove
\[(4.3) \quad N, [a/z] \models B \text{ iff } M, [a/z] \models SB\]
for any subformula \( B \) of \( A \) and for any assignment \( [a/z] \) with \( r(z) \supseteq FV(B) \), \( r(z) \cap BV(B) = \varnothing \).

The proof is by induction on the complexity of \( B \).

If \( B = P(z \cdot \sigma) \) for some map \( \sigma \), then by Definition 5.9.4 and the choice of \( N \),
\[
N, [a/z] \models B \text{ iff } (a \cdot \sigma) \in \eta^+(P) \text{ iff } M, [a \cdot \sigma/y] \models C.
\]

By Lemma 5.11.7 in a functorial metaframe we have
\[
M, [a \cdot \sigma/y] \models C \text{ iff } M, [a/z] \models [z \cdot \sigma/y]C (= SB),
\]
and thus \( B \) satisfies (4.3).

If \( B = B_1 \land B_2 \) and (4.3) holds for \( B_1, B_2 \), we obtain it for \( B \):
\[
N, [a/z] \models B \text{ iff } N, [a/z] \models B_1 \land N, [a/z] \models B_2 \text{ iff } M, [a/z] \models SB_1 \land M, [a/z] \models SB_2 \text{ iff } M, [a/z] \models SB_1 \land SB_2 (= SB).
\]

The cases \( B = B_1 \lor B_2 \), \( B = B_1 \lor B_2 \) are similar to the above, and the cases \( B = \perp, B = Q(z \cdot \sigma) (Q \neq P) \) are trivial.

If \( B = \Box_i B_1 \), (4.3) holds for \( B_1 \), then for an \( n \)-tuple \( a \) we have:
\[
N, [a/z] \models B \text{ iff } \forall b \in R^n_i(a) \quad N, [b/z] \models B_1 \text{ iff } \forall b \in R^n_i(a) \quad M, [b/z] \models SB_1 \text{ iff } M, [a/z] \models \Box_i(SB_1) (= SB),
\]
i.e. (4.3) holds for \( B \).

If \( B = \exists v B_1 \), then by our assumption, \( v \notin r(z) \). Also \( r(zv) \cap BV(B_1) = \varnothing \), since \( B \) is clean. So by Definition 5.9.4 and the induction hypothesis we have
\[
N, [a/z] \models B \text{ iff } \exists c (ac \in D^{n+1} \land N, [ac/zv] \models B_1) \text{ iff } \exists c (ac \in D^{n+1} \land M, [ac/zv] \models SB_1) \text{ iff } M, [a/z] \models \exists v(SB_1) (= SB).^{21}
\]

So (4.3) holds for \( B \).

The case \( B = \forall v B_1 \) is left to the reader.

\[\text{\[21\text{Note that in our case } S\exists v B_1 = \exists v SB_1.\]}\]
Lemma 5.12.2 (3), (4) yields the following convenient description of the set of strongly valued formulas $\text{ML}^{(=)}(\mathbb{F})$ for a functorial metaframe $\mathbb{F}$.

**Proposition 5.12.3** Let $\mathbb{F}$ be a functorial $N$-metaframe. Then

\[
\text{ML}^{(=)}(\mathbb{F}) = \{ A \in MF_N^{(=)} \mid \mathbb{F} \models A^n \text{ for any } n \in \omega \}
\]

\[
= \{ A \in MF_N^{(=)} \mid \mathbb{F} \models \overline{A^n} \text{ for any } n \in \omega \}.
\]

**Proof** Recall that $\overline{A^n} = \forall A^n$, so the second equality follows from 5.12.2(3). Since $A^n$ is a substitution instance of $A$, we have

\[
\{ A \in MF_N^{(=)} \mid \forall n \mathbb{F} \models A^n \} \supseteq \text{ML}^{(=)}(\mathbb{F}).
\]

The other way round, suppose $\forall n \mathbb{F} \models A^n$. By Lemma 2.5.30, every substitution instance $S_A$ of $A$ is congruent to a formula of the form $[y/x]S_1A^n$, where $S_1$ is a strict substitution. By 5.12.2(4) and 5.11.8, the latter formula is valid in $\mathbb{F}$. Thus $S_A$ is valid, by 5.10.11.

**Corollary 5.12.4** If $\mathbb{F}$ is a functorial metaframe, then $\text{ML}^{(=)}(\mathbb{F})$ is conservative over $\text{ML}(\mathbb{F})$.

**Remark 5.12.5** For an arbitrary $N$-metaframe $\mathbb{F}$, we can only state that $\text{ML}^{(=)}(\mathbb{F}) \subseteq \text{ML}(\mathbb{F})$. In fact, if all $MF_N^{(=)}$-instances of a formula $A \in MF_N$ are valid in $\mathbb{F}$, then all its $MF_N$-instances are also valid, but not the other way round. So we cannot claim that $\text{ML}(\mathbb{F}) \subseteq \text{ML}^{(=)}(\mathbb{F})$.

The next two lemmas are full analogues of 5.3.5 and 5.3.6, so we leave their proofs to the reader.

**Lemma 5.12.6** Let $\mathbb{F}$ be an arbitrary $N$-metaframe, $M = (\mathbb{F}, \xi)$ a model over $\mathbb{F}$, and let $\xi^n$ be a propositional valuation in its $n$-th level $F_n$, $n \geq 0$ such that $\xi^n(p_k) = \xi^+(P^n_k)$ for any $k \geq 0$. Let $M_n = (F_n, \xi^n)$ be the corresponding propositional Kripke model. Then for any $N$-modal propositional formula $A$, for any $a \in F_n$ and for any assignment $(x, a)$

\[
M_n, a \models A \text{ iff } M, a \models A^n[x].
\]

**Lemma 5.12.7** Let $\mathbb{F}$ be a w-functorial $N$-metaframe, $A$ an $N$-modal propositional formula. Then

\[
\mathbb{F} \models A^n \text{ iff } F_n \models A.
\]

**Remark 5.12.8** The w-functoriality of $\mathbb{F}$ is essential in Lemma 5.12.2, which is used in the proof of 5.12.7.

By 5.12.3 and 5.12.7 we obtain:

**Proposition 5.12.9** For a modal metaframe $\mathbb{F}$

\[
\text{ML}^{(=)}(\mathbb{F}) = \bigcap_{n \in \omega} \text{ML}(F_n).
\]
Corollary 5.12.10 All theorems of $K_N$ are strongly valid in every $N$-modal metaframe $\mathcal{F}$.

Proof Obvious, since they are valid in all propositional frames $F_n$.

Exercise 5.12.11 Show that Proposition 5.12.9 (and Corollary 5.12.10) holds for any N-metaframe $\mathcal{F}$ (not necessarily functorial). Hint: for a substitution instance $A'$ of a propositional formula $A$ and for an assignment $(s,a)$ (where $FV(A') \subseteq r(x)$ and $a \in D^n$) the truth of $M,a \models A'[x]$ can be checked in the propositional frame $F_n$.

However the claim of the above exercise is not so important, because as we will show in Theorem 5.12.13, only modal (i.e. functorial) $N$-metaframes validate $QK^=(N)$. Therefore only modal metaframes are interesting from the logical point of view.

Lemma 5.12.12 If a metaframe $\mathcal{F}$ is functorial, then the set $ML^=(\mathcal{F})$ is closed under necessitation, generalisation, modus ponens, and arbitrary substitutions.

Proof For a functorial $\mathcal{F}$, $ML^=(\mathcal{F})$ is substitution closed, by Lemma 2.5.27. (Note that $ML^=(\mathcal{F})$ is the largest substitution closed subset, the 'substitution interior' of $ML^=(\mathcal{F})$.)

By Lemma 5.12.2 and Proposition 5.12.3, it follows that $ML^=(\mathcal{F})$ is closed under necessitation and generalisation (for a functorial $\mathcal{F}$) as well as under modus ponens (for an arbitrary metaframe $\mathcal{F}$).

For example, consider generalisation. Suppose $A \in ML^=(\mathcal{F})$. Then $A^n \in ML^=(\mathcal{F})$ for any $n$, and hence $(\forall xA)^n = \forall xA^n \in ML^=(\mathcal{F})$ by 5.12.2. Hence $\forall xA \in ML^=(\mathcal{F})$ by 5.12.3.

Theorem 5.12.13 (Soundness theorem) Let $\mathcal{F}$ be an N-metaframe. Then the following properties are equivalent:

1. $\mathcal{F}$ is modal;
2. $ML(\mathcal{F})$ is a predicate $N$-modal logic without equality;
3. $ML=(\mathcal{F})$ is a predicate $N$-modal logic with equality;
4. $QK^N \subseteq ML(\mathcal{F})$;
5. $QK^=(N) \subseteq ML=(\mathcal{F})$.

Proof $(1) \Rightarrow (3), (1) \Rightarrow (2)$. For a modal metaframe $\mathcal{F}$, let us check the strong validity of the predicate axioms and the axioms of equality, i.e. the validity of their $n$-shifts (for $n \geq 0$). We fix a model $M = (\mathcal{F}, \xi)$ and do not indicate it in the notation of forcing.
• $A_1 := \forall y P(x, y) \supset P(x, z)$, for $y, z \notin r(x), y \neq z$.

Suppose $[ab/x z] \models \forall y P(x, y), ab \in D_{u}^{n+1}$; then by 5.9.4, for any $c \in D_{u}$, $[abc/x z y] \models P(x, y)$, which is equivalent to $\forall c \in D_{u} ac \in \xi^+(P)$. Hence $ab \in \xi^+(P)$, and thus $[ab/x z] \models P(x, z)$. Therefore $[ab/x z] \models A_1$.

• $A_2 := P(x, z) \supset \exists y P(x, y)$, where $y, z \notin r(x), y \neq z$.

Similarly to the previous case, assume $[ab/x z] \models P(x, z)$, i.e. $ab \in \xi^+(P)$. Then for some $c \in D_{u}$ (viz. for $c = b$), $[abc/x z y] \models P(x, y)$, and thus $[ab/x z] \models \exists y P(x, y)$. So $[ab/x z] \models A_2$.

• $A_3 := \forall y (Q(x) \supset P(x, y)) \supset (Q(x) \supset \forall y P(x, y))$, where $y \notin r(x)$.

Assume $[a/x] \models \forall y (Q(x) \supset P(x, y))$, $a \in D_{u}^{n}$. Then for any $b \in D_{u}$, we have $[ab/xy] \models Q(x) \supset P(x, y)$.

If also $[a/x] \models Q(x)$, i.e. $a \in \xi^+(Q)$, then for any $b \in D_{u}$, $[ab/xy] \models Q(x)$. Hence $[ab/xy] \models P(x, y)$ for any $b$, and thus we obtain $[a/x] \models \forall y P(x, y)$.

Therefore $M \models A_3$.

• $A_4 := \forall y (P(x, y) \supset Q(x)) \supset (\exists y P(x, y) \supset Q(x))$, $y \notin x$.

Assume $[a/x] \models \forall y (P(x, y) \supset Q(x))$

and $[a/x] \models \exists y P(x, y)$.

Then $[ab/xy] \models P(x, y)$ for some $b \in D_{u}$, and also $[ab/xy] \models P(x, y)$.

Thus $[ab/xy] \models Q(x)$, i.e. $a \in \xi^+(Q)$, which implies $[a/x] \models Q(x)$. Therefore $M \models A_4$.

• $A_5 := (x = x)$.

$[a/x] \models x = x$ holds trivially, since $a = a$.

• $A_6 := (y = z \supset (P(x, y) \supset P(x, z)), y, z \notin r(x)$.

We may assume that $y, z$ are distinct, since otherwise $M \models A_6$ is trivial.

If $[abc/xyz] \models y = z$, and $[abc/xyz] \models P(x, y)$, then by Definition 5.9.4, $b = c$ and $ab \in \xi^+(P)$, which implies $[abc/xyz] \models P(x, z)$ by the same definition. Therefore $M \models A_6$. 

5.12. MODAL SOUNDNESS

The implications (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (5) are obvious. (5) $\Rightarrow$ (4) follows by Remark 5.12.5.

(4) $\Rightarrow$ (1). We assume $\mathbf{QK}_N \subseteq \mathbf{ML}(\mathbb{F})$ and show that $\mathbb{F}$ is functorial.

(I) $\mathbb{F}$ is permutative, i.e.

\[ aR^n_i b \Rightarrow \pi_\sigma(a) R^n_i \pi_\sigma(b) \text{ for any } \sigma \in \Upsilon_n, \ n > 0. \]

Suppose $aR^n_i b$, $a \in D^n_u$. Consider the $\mathbf{QK}_N$-theorem

\[ B_1 := \exists x(P(\pi_\sigma(x)) \land \bigcirc_i Q(\pi_\sigma(x))) \supset \exists x(P(x) \land \bigcirc_i Q(x)), \ |x| = n, \]

and a model $M = (\mathbb{F}, \xi)$ such that $\xi^+(P) = \{\pi_\sigma(a)\}$, $\xi^+(Q) = \{\pi_\sigma(b)\}$. Then $M, b \vDash Q(\pi_\sigma(x))[x]$ and $M, a \vDash (P(\pi_\sigma(x)) \land \bigcirc_i Q(\pi_\sigma(x))[x]$. Hence

\[ M, u \vDash \exists x(P(\pi_\sigma(x)) \land \bigcirc_i Q(\pi_\sigma(x))]. \]

Since $M, u \vDash B_1$, we have

\[ M, u \vDash \exists x(P(x) \land \bigcirc_i Q(x)), \]

i.e.

\[ M, c \vDash (P(x) \land \bigcirc_i Q(x))[x] \]

for some $c \in D^n_u$. Thus $c \in \xi^+(P)$, i.e. $c = \pi_\sigma(a)$ and $cR^n_i d$ for some $d \in \xi^+(Q)$, i.e. for $d = \pi_\sigma(b)$. Therefore $\pi_\sigma(a)R^n_i \pi_\sigma(b)$.

(II) $\mathbb{F}$ is w-functorial.

We apply Lemma 5.10.6.

(IIa) $aR^1_i b \Rightarrow u R^1_i \nu$, whenever $a \in D_u$, $b \in D_v$.

In fact, assume $aR^1_i b$. Consider the $\mathbf{QK}_N$-theorem

\[ B_2 := \exists y \bigcirc_i p \supset \bigcirc_i p, \]

where $p \in PL^0$, and a model $M = (\mathbb{F}, \xi)$ such that $\xi^+(p) = \{v\}$. Then (by Definition 5.9.4) $M, b \vDash p[y]$, (since $\pi_\sigma(b) = v$), and so

\[ M, a \vDash \bigcirc_i p[y], \]

since $aR^1_i b$. Thus

\[ M, u \vDash \exists y \bigcirc_i p[y]. \]

Since $M, u \vDash B_2$, we have $M, u \vDash \bigcirc_i p$, which implies $uR^1_i \nu$.

(IIb) $(ac)R^i_{n+1}(bd) \Rightarrow aR^n_i b$ for $n > 0$.

Suppose $(ac)R^i_{n+1}(bd)$. Consider the $\mathbf{QK}_N$-theorem

\[ B_3 := \exists y \bigcirc_i P(x) \supset \bigcirc_i P(x), \]

where $|x| = n$, $y \not\in r(x)$, and a model $M = (\mathbb{F}, \xi)$ such that $\xi^+(P) = \{b\}$. Then

\[ M, bd \vDash P(x)[xy]. \]
Since \((ac)R^{n+1}_i(bd)\), we have
\[ M, ac \vDash \Diamond_i P(x) [xy], \]
and so
\[ M, a \vDash \exists y \Diamond_i P^n(x) [x]. \]
Now from \(M, a \vDash B_3 [x]\), we obtain \(M, a \vDash \Diamond_i P(x) [x]\), and thus \(aR^n_i b\) by the choice of \(M\).

(Ilc) \(ac \in D^{n+1} \& aR^n_i b \Rightarrow \exists d (ac)R^{n+1}_i(bd)\) for \(n > 0\).

In fact, assume \(ac \in D^{n+1}_u\), \(aR^n_i b\), and consider the \(QK_N\)-theorem
\[ B_4 := \Diamond_i P(x) \supset \forall y \Diamond_i P(x), \]
where \([x] = n\), \(y \not\in x\), and a model \(M = (F, \xi)\) such that \(\xi^+(P) = \{b\}\). Then \(M, b \vDash P(x) [x]\), and so
\[ M, a \vDash \Diamond_i P(x) [x]. \]
Since \(M, a \vDash B_4 [x]\), we also have
\[ M, a \vDash \forall y \Diamond_i P(x) [x], \]
i.e. \(M, ae \vDash \Diamond_i P(x) [xy]\) for any \(e \in D_u\). In particular, \(M, ac \vDash \Diamond_i P(x) [xy]\).
So there exists \(g = (g_1, \ldots, g_{n+1}) \in D^{n+1}\) such that
\[ (ac)R^{n+1}_i g \& M, g \vDash P(x) [xy]. \]
Hence \((g_1, \ldots, g_n) = b\), i.e. \((ac)R^{n+1}_i(bg_{n+1})\), and so we can take \(d = g_{n+1}\).

(IId) \(uR_i v \Rightarrow \forall a \in D_u \exists b \in D_v aR^1_i b\).

Consider the \(QK_N\)-theorem
\[ B_5 := \Diamond_i p \supset \forall y \Diamond_i p, \]
for \(p \in PL^0\) and a model \(M = (F, \xi)\) such that \(\xi^+(p) = \{v\}\). Then we obviously have \(M, v \vDash p\), and \(M, u \vDash \Diamond_i p\). Since \(M, u \vDash B_5\), it follows that \(M, u \vDash \forall y \Diamond_i p\). Thus
\[ \forall a \in D_u M, a \vDash \Diamond_i p [y], \]
which yields \(M, b \vDash p [y]\) for some \(b \in R^1_i(a)\). By definition, the latter means that \(\pi_\varnothing(b) \in \xi^+(p)\), i.e. \(\pi_\varnothing(b) = v\), or \(b \in D_v\).

(III) \((a, a)R^2_i(b_1, b_2) \Rightarrow b_1 = b_2, \) for \(a \in D_u, b_1, b_2 \in D_v\).

This property \((mm_2)\) easily follows from the validity of the \(QK^-N\)-theorem \(x = y \supset \Box_i (x = y)\). In fact, obviously, \(aa \vDash x = y [xy]\). Thus \(aa \vDash x = y \supset \Box_i (x = y)\).
y ⊨ □_i(x = y)[xy] implies aa ⊨ □_i(x = y)[xy], which is equivalent to (III).

So we have proved (5) ⇒ (1).

To check (III) for the case without equality (i.e. for the proof of (4) ⇒ (1)) we need a longer argument.

Let (a, a)R^a_i(b_1, b_2), a ∈ D_u. Consider the QK_N-theorem

\[ B_6 := \exists x(P(x) ∧ □_iQ(x, x)) ⊃ \exists x_1\exists x_2(P(x_1) ∧ P(x_2) ∧ □_iQ(x_1, x_2)) \]

Take a model M = (F, ξ) such that

\[ ξ^+(P) = \{a\}, \quad ξ^+(Q) = \{(b, b) \mid b ∈ D^1\}. \]

Then M, a ⊨ □_iQ(x, x)[x], since M, b ⊨ Q(x, x)[x] for any b. We also have M, u ⊨ P(x)[x]; hence M, u ⊨ \exists x(P(x) ∧ □_iQ(x, x)). Thus from M, u ⊨ B_6 we obtain:

\[ M, u ⊨ \exists x_1\exists x_2(P(x_1) ∧ P(x_2) ∧ □_iQ(x_1, x_2)), \]

i.e.

\[ M, a_1a_2 ⊨ P(x_1) ∧ P(x_2) ∧ □_iQ(x_1, x_2)[x_1x_2] \]

for some a_1, a_2 ∈ D_u. Then M, a_1a_2 ⊨ P(x_1) ∧ P(x_2)[x_1x_2], which implies a_1 = a_2 = a, since ξ^+(P) = \{a\}. Now we have

\[ M, aa ⊨ □_iQ(x_1, x_2)[x_1x_2] \]

and so

\[ M, b_1b_2 ⊨ Q(x_1, x_2)[x_1x_2], \]

since (a, a)R^a_i(b_1, b_2). Thus (b_1, b_2) ∈ ξ^+(Q), i.e. b_1 = b_2.

Therefore F is modal.

Theorem 5.12.13 means that modal metaframes are exactly m-sound (and also m^-sound) metaframes. This allows us to introduce the metaframe semantics \( M_N^a \) and \( M_N \) for modal predicate logics (with or without equality) generated by the class of N-modal (or functorial) metaframes. This is the largest sound modal Kripke-type semantics generated by metaframes. Therefore we obtain a precise criterion of logical soundness in metaframes. In the next section this criterion will be applied to Kripke bundles and C-sets.

Now let us make some remarks on terminology. According to general definitions, a metaframe validating a modal logic L (predicate or propositional), should be called an ‘L-metaframe’. If L is a propositional logic, ‘L-metaframes’ are those, for which every \( F_n \) is a propositional L-frame (Proposition 5.12.9); such metaframes are not necessarily modal. But if L is a predicate logic, then F is an L-metaframe iff L ⊆ ML(=)(F); so every L-metaframe is modal (Theorem 5.12.13).
In particular, for a propositional logic $\mathbf{A}$, $\mathbf{QA}$-metaframes are just modal $\mathbf{A}$-metaframes. So e.g. $F$ is an $S4$-metaframe iff every $F_n$ is reflexive and transitive. An $S4$-metaframe is also called propositionally intuitionistic. On the other hand, a $QS4$-metaframe is a 1-modal $S4$-metaframe.

Finally let us explain why the modified forcing $\models^*$ described in 5.9.17 does not really change the notion of logical soundness. To see this we consider an arbitrary modification $\models'$ of forcing satisfying the clauses (1)-(9) from Definition 5.9.4. Then we define the notions of truth, validity and strong validity for $\models'$ in the natural way. Let $\text{ML}'(\models)(F)$ be the corresponding set of strongly valid formulas in a metaframe $F$. If this set is an m.p.l. $(\models)$, we say that $F$ is modally sound for $\models'$ (respectively, without or with equality). This notion is described as follows.

**Proposition 5.12.14** Consider a modified forcing relation $\models'$ satisfying 5.9.4(1)-(9).

1. If $\text{QK}_N \subseteq \text{ML}'(\models)(F)$, then $F$ is a modal metaframe;
2. If $\models'$ is equivalent to $\models$ in modal metaframes, i.e.

   \[
   M, a \models' A [x] \iff M, a \models A [x]
   \]

   for any model $M$ over a modal metaframe, for any assignment $(x, a)$ in $M$ and formula $A$ with $\text{FV}(A) \subseteq \text{r}(x)$, then for any metaframe $F$ the following conditions are equivalent:

   (i) $F$ is modally sound for $\models'$ (with or without equality);
   (ii) $F$ is modally sound;
   (iii) $F$ is modal.

Moreover, $\text{ML}'(\models)(F) = \text{ML}(\models)(F)$ for modal (i.e. modally sound) $F$.

**Proof**

1. We can repeat the part (4) $\Rightarrow$ (1) from the proof of 5.12.13. The argument does not use the clause (10), because it does not involve forcing $M, a \models \exists y B[x]$ with $y \in \text{r}(x)$.\(^{22}\)
2. Obvious.

Thus a 'reasonable' modification of 5.9.4(10) does not affect logical soundness. This applies to the forcing $\models^*$ from 5.9.17, because it is equivalent to $\models$ in modal metaframes. In fact, by 5.9.4 we have

\[
M, a \models \exists x_i B[x] \iff M, a - a_i \models \exists x_i B [x - x_i] \\
\iff \exists c \in D(a) M, (a - a_i)c \models B [(x - x_i)c_i].
\]

\(^{22}\)One can easily rewrite the proof using only clean formulas and forcing $M, a \models A [x]$ with $\text{r}(x) = \text{FV}(A)$.  

Now let $|x| = n$, and let $\sigma \in \Upsilon_n$ be a permutation

$$
\begin{pmatrix}
1 \ldots i-1 & i & \ldots & n-1 & n \\
1 \ldots i-1 & i+1 & \ldots & n & i
\end{pmatrix};
$$

then $((x - x_i)x_i) \cdot \sigma = x$. By 5.10.6 we can further write

$$M, a \vdash \exists x_i B [x] \iff \exists c \in D(a) M, ((a - a_i)c) \cdot \sigma \vdash B [x].$$

But $((a - a_i)c) \cdot \sigma = (a_1, \ldots, a_i-1, c, a_{i+1}, \ldots, a_n)$, so the existence of $c$ is equivalent to the existence of $b \in D^n$ such that $b - b_i = a - a_i$ and

$$M, b \vdash B [x].$$

Thus $\vdash$ satisfies the inductive clause (10*) from 5.9.17 for $\exists$ (and similarly for $\forall$). Hence the equivalence of $M, a \vdash A [x]$ and $M, a \vdash^* A [x]$ easily follows by induction on $|A|$.

### 5.13 Representation theorem for modal metaframes

Recall that a preset $F = (C, D, \rho)$ over an $N$-precategory $C$ gives rise to the metaframe $Mf(F) = (F_n)_{n \in \omega}$ with the relations

$$aR^m \rho b \iff \exists \gamma \in Mor C \rho \cdot a = b,$$

cf. Definition 5.6.14. By 5.9.13, $ML(=) (Mf(F)) = ML(=) (F)$.

We also know that every Kripke bundle $F = (F, D, \rho)$ is associated with a metaframe $Mf(F) = (F_n)_{n \in \omega}$ (cf. Definition 5.3.2) as well as with a preset $F'$ (over some precategory), cf. 5.8.5. By Proposition 5.8.5 these constructions are coherent, i.e. $Mf(F) = Mf(F')$. Recall (Definition 5.3.2) that the relations in $Mf(F)$ are

$$aR^m \rho b \iff \forall j a\rho_j b_j \ & & a \ sub \ b.$$

By 5.9.13, $ML(=) (F) = ML(=) (Mf(F))$; hence $ML(=) (F) = ML(=) (F')$.

**Lemma 5.13.1**

1. If $F$ is a preset over an $N$-precategory $C$ (or an $N$-modal Kripke bundle), then $Mf(F)$ is an $N$-modal metaframe.

2. If $F$ is a $C$-set over a category $C$ (or an intuitionistic Kripke bundle), then $Mf(F)$ is a QS4-metaframe.

**Proof**

1. By the above remark, it is sufficient to consider only presets. To show the w-functoriality, consider $\sigma \in \Upsilon_{mn}$, and show that $\pi_\sigma : F_n \rightarrow F_m$. To check the
monotonicity, suppose $aR_i^\gamma b$, $u = \pi_\sigma(a)$, $v = \pi_\sigma(b)$. Then $\rho_\gamma \cdot a = b$ for some $\gamma \in C_i(u, v)$.

Now note that

$$(3) \quad \rho_\gamma \cdot (a \cdot \sigma) = (\rho_\gamma \cdot a) \cdot \sigma.$$ 

In fact,

$$(\rho_\gamma \cdot (a \cdot \sigma))_j = \rho_\gamma((a \cdot \sigma)_j) = \rho_\gamma(a_{\sigma(j)}) = (\rho_\gamma \cdot a)_{\sigma(j)} = ((\rho_\gamma \cdot a) \cdot \sigma)_j.$$ 

So by (3) we have:

$$\rho_\gamma \cdot (a \cdot \sigma) = (\rho_\gamma \cdot a) \cdot \sigma = b \cdot \sigma.$$ 

Thus $(a \cdot \sigma)R_i^n(b \cdot \sigma)$.

Next, let us check the lift property for $\pi_\sigma$. If $(a \cdot \sigma)R_i^n b'$, $u = \pi_\sigma(a)$, $v = \pi_\sigma(b')$, then $\rho_\gamma \cdot (a \cdot \sigma) = b'$ for some $\gamma \in C_i(u, v)$. By (1) it follows that $(\rho_\gamma \cdot a) \cdot \sigma = b'$, i.e. $b' = \pi_\sigma b$, where $b = \rho_\gamma \cdot a$, and thus $aR_i^n b$.

Therefore $\pi_\sigma : F_n \rightarrow F_m$.

To check the property 5.11.1 (mm2), suppose $(a, a)R_i^n (b_1, b_2)$, $a \in D_u$, $b_1$, $b_2 \in D_v$. Then for some $\gamma \in C_i(u, v)$

$$\rho_\gamma \cdot (a, a) = (b_1, b_2),$$

i.e. $b_1 = \rho_\gamma(a) = b_2$. Thus (mm2) holds.

(2) If $F$ is a $C$-set, then every $F_n$ is an S4-frame by 5.6.16. The same holds for intuitionistic Kripke bundles by 5.5.1 or by the observation that if $F$ is an intuitionistic Kripke bundle, then the corresponding $C$-preset $F'$ is a $C$-set (Proposition 5.8.5), and $Mf(F) = Mf(F')$.

Therefore by Proposition 5.9.13, Theorem 5.12.13, Proposition 5.12.3, and Corollary 5.12.4 we obtain

**Proposition 5.13.2** Let $F$ be a preset over an $N$-precategory $C$ (or an $N$-modal Kripke bundle). Then

1. $ML(=)(F) = ML(=)(Mf(F))$ is an $N$-m.p.l. $(=)$;
2. $ML(=)(F) = \{ A \in MF_N^{(=)} : \forall m F \vdash A^m \} = \{ A \in MF_N^{(=)} : \forall m F \vdash \overline{A}^m \}$;
3. $ML(=)(F)$ is conservative over $ML(F)$.
4. If $F$ is $C$-set or an intuitionistic Kripke bundle, then $ML(=)(F) \supset QS4(=)$.

Let us now show that all countable modal metaframes can be represented by $C$-sets.

**Definition 5.13.3** We say that $D = (D_u : u \in W)$ is a system of countable domains if every $D_u$ is countable. In this case we also say that a metaframe with a system of domains $D$ has countable domains.

---

23 Recall that according to our terminology, a countable set may be finite.
Theorem 5.13.4 (Representation theorem)

(1) Let $F$ be an $N$-modal metaframe with countable domains. Then $F = Mf(F')$ for some preset $F'$ over an $N$-precategory $C$. Moreover, if $F$ is a $\text{QS4}$-metaframe, then $C$ is a category and $F'$ is a $C$-set.

(2) There exists a modal ($\text{QS4}$-) metaframe $F$, for which (1) does not hold, and all but one domains are countable.

Proof

(1) Let $F = ((F_n)_{n \in \omega}, D)$. Consider the $N$-precategory $C$ with $\text{Ob}C = W$ and

(1.1) $C_i(u, v) = \{f : D_u \to D_v \mid \forall n > 0 \forall a \in D_u^n \ a R^n_i (f \cdot a)\}.$

Note that if $F$ is a $\text{QS4}$-metaframe ($N = 1$), then $C$ is a concrete category, i.e. the composition of morphisms is the composition of functions and identity morphisms are identity functions. Recall that $R^n$ are reflexive and transitive in this case; thus $f \circ g \in C(u, w)$ whenever $f \in C(u, v), \ g \in C(v, w), \ uRvRw$, and also $\text{id}_D \in C(u, u)$ for $u \in W$.

Consider the $C$-set $F' = (F, D, \rho)$, in which $\rho_f = f$ for $f \in C_i(u, v)$ and $F = FR(C)$.

Let us ensure that $F = Mf(F')$ according to Definition 5.6.14, i.e.

(1.2) $a R^n_i b \iff \exists f \in C_i(u, v) \ f \cdot a = b$

for $a \in D_u^n, \ b \in D_v^n$, (recall that $\rho_f = f$).

In fact, if $f \cdot a = b$, $f \in C_i(u, v)$, then $a R^n_i b$ by (1.1).

The other way round, suppose $a R^n_i b$. Let $a_1, \ldots, a_n, a_{n+1} \ldots$ be an enumeration of $D_u$ starting at our $a = (a_1, \ldots, a_n)$. Using Lemma 5.10.5 (2), by induction we construct a sequence $b_1, \ldots, b_n, b_{n+1} \ldots \in D_v$ such that $b = (b_1, \ldots, b_n)$ and $(a_1, \ldots, a_k) R^k_i (b_1, \ldots, b_k)$ for all $k \geq n$. Then the function $f$ sending each $a_k$ to $b_k$ belongs to $C_i(u, v)$. In fact, $(a_1, \ldots, a_k) R^k_i (b_1, \ldots, b_k)$ for $k \geq n$, by our construction, and this is also true for $k < n$, by monotonicity of $\pi_\sigma$, where $\sigma : I_k \to I_n$ is the inclusion map. Thus (1.2) holds.

Finally note that (1.2) implies $F(= FR(C)) = F_0:\$

$u R_i v \iff C_i(u, v) \neq \emptyset.$

In fact, suppose $u R_i v, a \in D_u$. By 5.10.5(4), $a R^k_i b$ for some $b \in D_v$, thus by (1.2), $f(a) = b$ for some $f \in C_i(u, v)$ and so $C_i(u, v) \neq \emptyset$.

The other way round, if $f \in C_i(u, v)$, then $a R^k_i f(a)$ and $f(a) \in D_v$. Hence $u R_i v$ by 5.10.5(3).

(2) Consider a $\text{QS4}$-metaframe $F = ((F_n)_{n \in \omega}, D)$, such that

- $F_0 = (W, R)$ is a 2-element chain:
  $W = \{u_0, u_1\}, \ R = W \times W - \{(u_1, u_0)\};$
- $|D_{u_0}| > |D_{u_1}| = K_0;$
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- \( aR^n b \iff a \text{ sub } b \ & \ & b \text{ sub } a \iff \forall i,j (a_i = a_j \iff b_i = b_j) \) for \( a \in D^n_{u_i}, \ b \in D^n_{u_k}, \ j \leq k. \)

To check that \( F \) is a modal metaframe, we apply 5.10.2, 5.10.5.

First, for a permutation \( \sigma \in \Gamma_n, \pi_\sigma \) is an automorphism of \( F_n \). In fact, \( aR^n b \) iff \( \forall i, j (a_i = a_j \iff b_i = b_j) \) iff \( \forall i, j (a_{\sigma(i)} = a_{\sigma(j)} \iff b_{\sigma(i)} = b_{\sigma(j)}) \) iff \( (a \cdot \sigma)R^n (b \cdot \sigma) \).

A similar argument shows that \( \pi_\sigma \) is monotonic for any \( \sigma \in \Gamma_n \).

Second, \( \pi_n^+ \) has the lift property. In fact, suppose \( aR^{n-1} b \), i.e. \( a \text{ sub } b \) and \( b \text{ sub } a \). Let \( u, v \) be the worlds of \( a, b \), respectively. Then for any \( c \in D_u \) there is \( d \in D_v \) such that \( (ac)R^n (bd) \): if \( c = a_i \) for some \( i \), take \( d = b_i \); otherwise take \( d \neq b_i \) for any \( i \) (since \( D_v \) is infinite, such an individual \( d \) always exists).

The case \( n = 1 \) described in 5.10.5(4) is trivial. By definition, the property \((mm_2)\) also holds: \( (a, a)R^2 (b_1, b_2) \Rightarrow b_1 = b_2 \).

All the \( F_n \) are S4-frames, since the relation \( \text{sub} \) is reflexive and transitive.

Now suppose \( F \) corresponds to a \( C \)-set \( F' = (F, D, \rho) \). Let \( \mu \in C(u_0, u_1) \); then \( \rho_\mu \) is a function from \( D_{u_0} \) to \( D_{u_1} \). But \( |D_{u_0}| > |D_{u_1}| \), so there exist \( a_1, a_2 \in D_{u_0}, \ b \in D_{u_1} \) such that \( a_1 \neq a_2 \) and \( \rho_\mu(a_1) = \rho_\mu(a_2) = b \). Then \( (a_1, a_2)R^2 (b, b) \) in \( Mf(F') \), but not in \( F \).

One can easily construct similar examples for any \( N > 1 \).

Therefore modal metaframes with countable domains are nothing but \( C \)-sets. However we do not know if the semantics of modal metaframes is stronger than functor semantics (either in the case of \( C \)-presets or \( C \)-sets). For the intuitionistic case (discussed below) this question is also open.

5.14 Intuitionistic forcing and monotonicity

5.14.1 Intuitionistic forcing

Now let us consider the intuitionistic case. We shall begin with a simple observation that QS4-metaframes generate a sound semantics for superintuitionistic logics via Gödel–Tarski translation (Proposition 5.14.7). But this semantics is probably not maximal in the intuitionistic case, so our goal will be to identify a larger class of 'intuitionistic sound metaframes'.

Let us first give a definition of intuitionistic forcing in S4-metaframes.

Definition 5.14.1 A valuation \( \xi \) in an S4-metaframe \( F \) is called intuitionistic if it is intuitionistic in every \( F_n \), i.e. for any predicate letter \( P_j^n, \ n \geq 0 \)

\[ R^n(\xi^+(P_j^n)) \subseteq \xi^+(P_j^n). \]

The pair \( (F, \xi) \) is called an intuitionistic metaframe model.

Henceforth until Section 5.19, we shall consider mainly S4-metaframes.
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Definition 5.14.2 An intuitionistic model \( M = (\mathbb{F}, \xi) \) gives rise to the forcing relation \( M, a \vDash A[x] \) (where \( a \in D^n, \; |x| = n, \; A \in IF^n, \; FV(A) \subseteq r(x) \)) defined by induction:

1. \( M, a \vDash \bot[x] \);
2. \( M, a \vDash P^n_k(x \cdot \sigma)[x] \) iff \( (a \cdot \sigma) \in \xi^+(P^n_k) \);
3. \( M, a \vDash (x_j = x_k)[x] \) iff \( a_j = a_k \);
4. \( M, a \vDash (B \land C)[x] \) iff \( M, a \vDash B[x] \) and \( M, a \vDash C[x] \);
5. \( M, a \vDash (B \lor C)[x] \) iff \( M, a \vDash B[x] \) or \( M, a \vDash C[x] \);
6. \( M, a \vDash (B \supset C)[x] \) iff \( \forall b \in R^n(a) \; (M, b \vDash B[x] \Rightarrow M, b \vDash C[x]) \);
7. \( M, a \vDash \forall yB[x] \) iff \( \forall b \in R^n(a) \; \forall c \in D(b) \; M, \pi_{x \mid y}(bc) \vDash B[x|y] \)
   \( \vDash \exists yB[x] \) iff \( \exists b \in (R^n)^{-1}(a) \; \exists c \in D(b) \; M, \pi_{x \mid y}(bc) \vDash B[x|y] \).

This definition of forcing is motivated by Gödel–Tarski translation. E.g. the intuitionistic forcing for \( \forall yB \) resembles the modal forcing for \( \square \forall yB \). In more detail the connection between modal and intuitionistic forcing will be discussed later on. However note that for the \( \exists \)-case the intuitionistic and the modal definition differ, unlike Kripke frame or Kripke sheaf semantics. This situation was already discussed for quasi-bundles in section 5.5.

Definition 5.14.3 Let \( \mathbb{F} \) be an S4-metaframe. An intuitionistic formula \( A \) is called

- **true** in an intuitionistic model \( M \) over \( \mathbb{F} \) (notation: \( M \vDash A \)) if \( M, a \vDash A[x] \) for any assignment \( (x, a) \) with \( r(x) \supseteq FV(A) \);
- **valid** in \( \mathbb{F} \) (notation: \( \mathbb{F} \vDash A \)) if it is true in all intuitionistic models over \( \mathbb{F} \);
- **strongly valid** (respectively, strongly valid with equality) in \( \mathbb{F} \) if all its IF- (respectively, IF\(^\equiv\))-substitution instances are valid in \( \mathbb{F} \). Strong validity is denoted by \( \vDash^+ \) (respectively, \( \vDash^{+\equiv} \)).

Similarly to the modal case, we use the notation

\[
\text{IL}_{(\approx)}^+(\mathbb{F}) := \{ A \in IF(\approx) \mid \mathbb{F} \vDash^+ A \},
\]
\[
\text{IL}_{(\equiv)}^+(\mathbb{F}) := \{ A \in IF(\equiv) \mid \mathbb{F} \vDash^{+\equiv} A \}.
\]

Note that \( \text{IL}_{\approx}^-(\mathbb{F}) \) is the largest substitution closed subset of \( \text{IL}_{\approx}^+(\mathbb{F}) \).

As we have pointed out for Kripke bundles, it may be the case that \( \mathbb{F} \vDash SA \) for any IF-substitution \( S \), but not for any IF\(^\equiv\)-substitution; so \( \text{IL}_{\approx}^-(\mathbb{F}) \) may be not conservative over \( \text{IL}(\mathbb{F}) \), cf. Remark 5.12.5 for the modal case. Similarly to the modal case we have

\[
\text{IL}_{\approx}^-(\mathbb{F}) \cap IF \subseteq \text{IL}(\mathbb{F}).
\]
Definition 5.14.4 A metaframe $\mathcal{F}$ is called intuitionistic sound ($i(\rightarrow)$-sound, for short) if $\mathcal{F}$ is an s.p.l. ($\rightarrow$).

As we shall see in Section 5.16, $i^{\rightarrow}$-soundness implies $i$-soundness.

Now let us show that all $\mathcal{Q}S4$-metaframes are $i^{\rightarrow}$-sound. This happens because intuitionistic forcing for $\mathcal{Q}S4$-metaframes corresponds to modal forcing via Gödel–Tarski translation.

Definition 5.14.5 Let $\mathcal{F}$ be a $\mathcal{Q}S4$-metaframe; $M = (\mathcal{F}, \xi)$ a metaframe model. The pattern of $M$ is the model $M_0 = (\mathcal{F}, \xi_0)$ such that

$$\xi_0^+(P^m_j) = \{ a \in D^m | R^n(a) \subseteq \xi^+(P^m_j) \}$$

for any $j, n \geq 0$.

It is clear that $M_0$ is an intuitionistic model and $M_0 = M$ if $M$ itself is intuitionistic.

Lemma 5.14.6 Let $M$ be the same as in Definition 5.14.5. Then for any $A \in IF^{\rightarrow}$ and any assignment $(x, a)$ with $FV(A) \subseteq x$

$$M_0, a \vdash A[x] \iff M, a \vdash A^T[x].$$

Proof By induction, cf. Lemmas 3.2.15, 5.5.7.

If $A = P(\pi_\sigma x), \sigma \in \Sigma_{mn}, |x| = n$, then

$$M_0, a \vdash A[x] \iff \pi_\sigma a \in \xi_0^+(P) \iff R^m(\pi_\sigma a) \subseteq \xi^+(P),$$

$$M, a \vdash \Box A[x] \iff \forall b \in R^n(a) \pi_\sigma(b) \in \xi^+(P) \iff \pi_\sigma(R_n(a)) \subseteq \xi^+(P).$$

But $R_m(\pi_\sigma a) = \pi_\sigma(R_n(a))$ for a morphism $\pi_\sigma : F_n \rightarrow F_m$ (see the remark after Definition 1.3.30), so the statement is true in this case.

For the induction step let us only check the quantifier cases.

Suppose $A = \forall y B$ and the statement holds for $B$. Then

$$M_0, a \vdash A \iff \forall b \in R^n(a) \forall c \in D(b) M_0, \pi x_{\|y}(bc) \vdash B[x\|y] \iff$$

$$\forall b \in R^n(a) \forall c \in D(b) M, \pi x_{\|y}(bc) \vdash B^T[x\|y] \iff M, a \vdash \Box \forall y B^T(= A^T)[x].$$

The argument for the $\exists$-case is slightly longer as the intuitionistic and the modal definitions differ in this case.

Suppose $A = \exists y B$ and the statement holds for $B$. We have

$$M_0, a \vdash A \iff \exists b \in (R^n)^{-1}(a) \exists c \in D(b) M_0, \pi x_{\|y}(bc) \vdash B[x\|y] \iff$$

$$\exists b \in (R^n)^{-1}(a) \exists c \in D(b) M, \pi x_{\|y}(bc) \vdash B^T[x\|y] \iff$$

$$\exists b \in (R^n)^{-1}(a) M, b \vdash \exists y B^T(= A^T)[x].$$

The latter holds if $M, a \vdash A^T$—just take $b = a$. The other way round, suppose $M, b \vdash A^T$ as. As we know, $\mathcal{Q}S4^{(=} \vdash A^T \equiv \Box A^T$ (Lemma 2.11.2), so $M \vdash A^T \equiv \Box A^T$ by soundness (5.12.13), and thus $M, b \vdash \Box A^T[x]$, which implies $M, a \vdash A^T$. 
The case \( A = \exists x_i B \) is reduced to the above:

\[
M_0, a \models A \iff M_0, a - a_i \models \exists x_i B [x - x_i] \iff M, a - a_i \models \exists x_i B^T [x - x_i] \iff M, a \models \exists x_i B^T (= A^T) [x].
\]

All other cases are left to the reader.

**Proposition 5.14.7** Let \( \mathcal{F} \) be an QS4-metaframe, \( A \in \mathcal{IF}^= \). Then

1. \( \mathcal{F} \models A \) iff \( \mathcal{F} \models A^T \).

2. The following three assertions are equivalent:
   
   (a) \( \mathcal{F} \models + A \);
   
   (b) \( \forall \mathbf{m} \mathcal{F} \models A^m \);
   
   (c) \( \mathcal{F} \models + A^T \).

3. \( \mathcal{IL}^{(=)}(\mathcal{F}) = \mathcal{TML}^{(=)}(\mathcal{F}) \) and therefore \( \mathcal{IL}^{(=)}(\mathcal{F}) \) is an s.p.l.(=).

**Proof** Similar to 5.5.12, 5.5.13.

(1) (Only if.) Assume \( \mathcal{F} \models A \). For a metaframe model \( M \) over \( \mathcal{F} \) let us show \( M \models A^T \), i.e. \( M, a \models A^T [x] \) for any appropriate assignment \((x, a)\). By Lemma 5.14.6, the latter is equivalent to \( M_0, a \models A [x] \), which follows from our assumption.

(If.) Assume \( \mathcal{F} \models A^T \). Let \( M \) be an intuitionistic metaframe model over \( \mathcal{F} \), and let us show \( M \models A \), i.e. \( M, a \models A [x] \) for any appropriate assignment \((x, a)\). Since \( M_0 = M \), by Lemma 5.14.6, \( M, a \models A [x] \) iff \( M, a \models A^T [x] \), and the latter follows from \( \mathcal{F} \models A^T \).

(2) The proof is completely analogous to 5.5.12. Use 5.12.3 for (b)\( \Rightarrow \) (c) and soundness (5.12.13) for (c)\( \Rightarrow \) (a).

(3) follows readily from the equivalence (a)\( \Leftrightarrow \) (c) in (2).

Proposition 5.14.7 shows that QS4-metaframes are \( i^{(=)} \)-sound. As we shall see later on, the class of \( i \)-sound metaframes is larger, but the question, whether \( i \)-sound metaframes generate a stronger semantics, is open. The same happens to \( i^{=} \)-soundness.

**Corollary 5.14.8** If \( \mathcal{F} \) is a C-set (or an intuitionistic Kripke bundle), then \( \mathcal{IL}^{(=)}(\mathcal{F}) \) is an s.p.l.(=); moreover, \( \mathcal{IL}^{(=)}(\mathcal{F}) = \mathcal{TML}^{(=)}(\mathcal{F}) \).

**Exercise 5.14.9** Using 5.14.6, check the following properties of intuitionistic forcing in QS4-metaframes analogous to the properties of modal forcing (cf. Lemma 5.10.6(2), Lemma 5.10.10, Lemma 5.11.7, Proposition 5.11.12).

Let \( \mathcal{F} \) be a QS4-metaframe, \( M \) an intuitionistic model over \( \mathcal{F} \). Then

1. for any ordered assignment \((x, a)\), for any \( \sigma \in \Upsilon_{mn} \), where \( |x| = n \geq m \) and for any formula \( A \in \mathcal{IF}^= \) with \( FV(A) \subseteq r(x \cdot \sigma) \)

\[
M, a \models A [x] \iff M, a \cdot \sigma \models A [x \cdot \sigma].
\]
(2) for any congruent intuitionistic formulas \( A, B \) and any ordered assignment \((x, a)\) with \( \text{FV}(A) \subseteq r(x) \)

\[
M, a \models A[x] \iff M, a \models B[x];
\]

(3) for any \( \sigma \in \Sigma_{mn} \), for any distinct lists of variables \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_m) \), for any intuitionistic formula \( A \) with \( \text{FV}(A) \subseteq r(y) \), for any \( a \in D^n \)

\[
M, a \models ([x \cdot \sigma / y] A)[x] \iff M, a \cdot \sigma \models A[y];
\]

(4) if \( A, A^* \) are intuitionistic formulas, \([a/x], [a^*/x^*]\) assignments giving rise to equal \( D_u\)-sentences: \([a/x] A = [a^*/x^*] A^*\), then

\[
M, a \models A[x] \iff M, a^* \models A[x^*].
\]

As we shall see later on, all the claims in 5.14.9 extend to arbitrary \( \bar{\mathfrak{q}}\)-sound metaframes.

Now let us consider arbitrary \( \mathbf{S}4\)-metaframes. Let us first note that the quantifier clauses in Definition 5.14.2 are analogues of the combined clause (9\+10) for modal forcing from Section 5.9. Now if we consider two options in the intuitionistic case, we obtain

(VII.1) if \( y \notin x \), then

\[
M, a \models \forall y B[x] \text{ iff } \forall b \in R^n(a) \forall c \in D(b) M, bc \models B[xy],
\]

\[
M, a \models \exists y B[x] \text{ iff } \exists b \in (R^n)^{-1}(a) \exists c \in D(b) M, bc \models B[xy];
\]

(VII.2) for \( 1 \leq i \leq n \)

\[
M, a \models \forall x_i B[x] \text{ iff } \forall b \in R^n(a) \forall c \in D(b) M, (b-b_i)c \models B[(x-x_i)x_i]
\]

\[
\text{iff } \forall d \in \pi^n_i (R^n(a)) \forall c \in D(d) M, dc \models B[(x-x_i)x_i].
\]

\[
M, a \models \exists x_i B[x] \text{ iff } \exists b \in (R^n)^{-1}(a) \exists c \in D(b) M, dc \models B[(x-x_i)x_i].
\]

Note that (VII.1) corresponds to the clause (9) from Definition 5.9.4. Let us show that in \( \mathbf{QS}4\)-frames (VII.2) corresponds to the clause 5.9.4(10), i.e.

\[
M, a \models \forall x_i B[x] \text{ iff } M, \hat{a}_i \models \forall x_i B[\hat{x}_i],
\]

\[
M, a \models \exists x_i B[x] \text{ iff } M, \hat{a}_i \models \exists x_i B[\hat{x}_i].
\]

In fact, by (VII.1)

\[
M, a - a_i \models \forall x_i B[x-x_i] \text{ iff } \forall d \in R^{n-1}(a-ai) \forall c \in D(d) M, dc \models B[(x-x_i)x_i]
\]

\[
\text{iff } \forall d \in R^n(\pi^n_i a) \forall c \in D(d) M, dc \models B[(x-x_i)x_i].
\]

Now in \( \mathbf{QS}4\)-metaframes \( \pi^n_i \) is a morphism, so the latter is equivalent to (VII.2). But in arbitrary \( \mathbf{S}4\)-metaframes (VII.2) may not be true, because it may happen that \( \pi^n_i (R^n(a)) \neq R^n(\pi^n_i (a)) \), cf. Section 5.9.24 The reader can construct a counterexample as an exercise, cf. 5.9.16, 5.9.18.

\[24\] Or note that

\[
M, a \models \forall x_i B[x] \iff M, a - a_i \models \forall x_i B[x-x_i]
\]

may not imply

\[
M, a \models \Box \forall x_i B[x] \iff M, a - a_i \models \Box \forall x_i B[x-x_i].
\]
Later on we shall see that (VII.2) and its analogue for the $\exists$-case hold in all i($\equiv$)-sound metaframes.

**Remark 5.14.10** These comments motivate an alternative definition of forcing ($\vdash^*$) for the quantifier case instead of (VII):

This can be rewritten as follows:

There also exists an alternative version ($\vdash^*$) resembling 5.9.17:

All these definitions are equivalent in i($\equiv$)-sound metaframes.

Now we obtain an analogue to Lemma 5.12.2 (1), (2) and Corollary 5.12.12.

**Proposition 5.14.11** Let $\mathcal{F}$ be an S4-metaframe. Then

(1) $\text{IL}^{(\equiv)}(\mathcal{F})$ is closed under generalisation and modus ponens;

(2) $\text{IL}^{(\equiv)}(\mathcal{F})$ is closed under formula substitutions;

(3) the following conditions are equivalent:

(a) $\text{IL}^{(\equiv)}(\mathcal{F})$ is an s.p.l. ($\equiv$);

(b) $\text{QH}^{(\equiv)} \subseteq \text{IL}^{(\equiv)}(\mathcal{F})$;

(c) $\text{QH}^{(\equiv)} \subseteq \text{IL}^{(\equiv)}(\mathcal{F})$.

**Proof** (1) (I) Let us first consider generalisation. Let $M$ be an intuitionistic model over $\mathcal{F}$, $B \in IF^{(\equiv)}$. Assuming $M \models B$, let us prove $M \models \forall y B$, i.e. $M, a \models \forall y B [x]$ for any assignment $(x, a)$ with $r(x) \supseteq FV(\forall y B)$. Suppose $|x| = n$.

If $y \not\in x$, then

$$M, a \models \forall y B [x] \iff \forall b \in R^n(a) \forall c \in D(b) \ M, \pi_{x,y}(bc) \models B [x][y].$$

But $M, \pi_{x,y}(bc) \models B [x][y]$ holds, since $M \models B$.

(II) Now let us consider modus ponens.

Let $M$ be an intuitionistic model $M$ over $\mathcal{F}$. Assuming $M \models B$, $B \supseteq C$, let us check that $M \models C$.

Let $x \subseteq FV(C)$, $y = FV(B) - x$. By our assumption, for any assignment $(xy, ab)$,

$$M, ab \models B[xy]; \ M, ab \models (B \supseteq C)[xy].$$

Since every relation $R^n$ is reflexive, we readily obtain (for any assignment $(xy, ab)$)

(3.1) $M, ab \models C[xy]$.

Now let us show that

(3.2) $M, a \models C [x]$. 
In fact, (3.1) implies

(3.3) \( M, a \vdash \forall y C[x] \),

as one can easily check. By (c), we also have

(3.4) \( M, a \vdash (\forall y C \supset C)[x] \),

since \( \forall y C \supset C \) is an intuitionistic theorem. Now (3.2) follows from (3.3), (3.4), and the reflexivity of \( R^n \).

(2) Note that the composition of substitutions is a substitution, cf. the argument in the modal case in 5.12.12.

(3) The implications (a) \( \Rightarrow \) (b), (b) \( \Rightarrow \) (c) are obvious.

(c) \( \Rightarrow \) (b) is also obvious, \( \text{IL}(=)(\mathcal{F}) \) is substitution closed.

So assuming (c), let us check (a). Due to (2), it remains to consider modus ponens and generalisation.

First note that \( \text{IL}(=)(\mathcal{F}) \) is closed under congruence. In fact, suppose \( A \in \text{IL}(=)(\mathcal{F}) \) and \( A \cong B \). Then \( (A \supset B) \in \mathcal{QH}(=) \) by 2.6.9, and hence \( (A \supset B) \in \mathcal{QH}(=) \) (by (Ax3), MP and substitution). So by the assumption (c),

\( (A \supset B) \in \text{IL}(=)(\mathcal{F}) \).

Since by (1) \( \text{IL}(=)(\mathcal{F}) \) is closed under MP, it follows that \( B \in \text{IL}(=)(\mathcal{F}) \).

Now the closedness of \( \text{IL}(=)(\mathcal{F}) \) under modus ponens follows easily.

In fact, suppose

\[ A, A \supset B \in \text{IL}(=)(\mathcal{F}). \]

Then \( S, SA, S(A \supset B) \in \text{IL}(=)(\mathcal{F}) \) for any formula substitution. By Lemma 2.5.13, \( S(A \supset B) \cong (SA \supset SB) \). Since \( \text{IL}(=)(\mathcal{F}) \) is congruence closed and MP-closed (by (1)), this implies \( SB \in \text{IL}(=)(\mathcal{F}) \). Eventually \( B \in \text{IL}(=)(\mathcal{F}) \).

Finally, let us consider generalisation. We suppose \( A \in \text{IL}(=)(\mathcal{F}) \) and show that \( \forall x A \in \text{IL}(=)(\mathcal{F}) \). This does not follow directly from (1), because a substitution instance of \( \forall x A \) is not always of the form \( \forall x A' \) for a substitution instance \( A' \) of \( A \) (cf. Lemma 2.5.13), and so we need a little detour.

So for a substitution \( S \), let us prove that \( \mathcal{F} \vdash S \forall x A \).

Let \( y \) be a new variable, \( y \notin V(\forall x A) \cup FV(S) \); then by 2.3.25(13)

\[ \forall x A \cong \forall y[y/x]A. \]

Hence by 2.5.12 and 2.5.13(3),

\[ S \forall x A \cong S \forall y[y/x]A \cong \forall y S[y/x]A. \]

As we already know, the validity in \( \mathcal{F} \) is closed under congruence and generalisation. So it is sufficient to show that \( \mathcal{F} \vdash S[y/x]A \).

By Lemma 2.5.14 (for complex substitutions) we can rename the bound variables of \( S \); so

(11) \[ S[y/x]A \cong S_1[y/x]A, \]
with $x \not\in BV(S_1)$. Now we introduce yet another new variable $x'$. Then we obviously have

(\#2) \quad S_1[y/x]A \equiv [x/x'][x'/x]S_1[y/x]A.

By 2.5.17, there exists a formula substitution $S_0$ such that $x \not\in FV(S_0)$ and

(\#3) \quad [x'/x]S_1[y/x]A \equiv S_0[y/x]A.

But then by 2.5.15 we obtain

(\#4) \quad S_0[y/x]A \equiv [y/x]S_0A

Eventually from (\#1)--(\#4) and 2.3.24 it follows that

(\#5) \quad S[y/x]A \equiv [x/x'][y/x]S_0A.

It remains to notice that validity in $\mathbb{F}$ respects variable substitutions. In fact, if $\mathbb{F} \vdash B$, then $\mathbb{F} \vdash \forall x B$ by (1).

By 2.6.15(xxv), $\forall x B \supset [y/x]B$ is an intuitionistic theorem, so it is valid in $\mathbb{F}$ by our assumption (c). By (1), we can apply modus ponens, thus $\mathbb{F} \vdash [y/x]B$.

Therefore, since $A \in \mathbb{IL}^{(=)}(\mathbb{F})$, we obtain $\mathbb{F} \vdash [x/x'][y/x]S_0A$; thus $\mathbb{F} \vdash S[y/x]A$ by (\#5), since $\mathbb{IL}^{(=)}(\mathbb{F})$ is closed under congruence.

Remark 5.14.12 The above proof cannot be simplified by concluding $\mathbb{F} \vdash S_0[y/x]A$ directly from $A \in \mathbb{IL}^{(=)}(\mathbb{F})$, before we know that $[y/x]A \in \mathbb{IL}^{(=)}(\mathbb{F})$. Since $S_0$ does not always commute with $[y/x]$ for $x \in FV(S_0)$, we had to replace $x$ with $x'$ and $S_0$ with $S_1$.

Corollary 5.14.13 $i^{=}-$soundness implies $i$-soundness.

Proof \quad $QH^{=} \subseteq \mathbb{IL}^{=}($F$)$ implies $QH = (QH^{=})^o \subseteq (\mathbb{IL}^{=}($F$))^o \subseteq \mathbb{IL}($F$)$, which implies $i$-soundness by 5.14.11(3).

5.14.2 Monotonic metaframes

Now let us consider an important property of intuitionistic sound metaframes.

Definition 5.14.14 An S4-metaframe $\mathbb{F}$ is called monotonic(=) if it satisfies the condition

(\text{im}) \quad M, a \models A[x] \& a R^n b \Rightarrow M, b \models A[x]

for any $A \in IF^{(=)}$, for any intuitionistic model $M = (\mathbb{F}, \xi)$, a distinct list $x$ of length $n$ containing $FV(A)$ and any $a, b \in D^n$.

We already mentioned this property in Section 5.5 when quasi-bundles were discussed. The next lemma proves that monotonicity is really necessary.
Lemma 5.14.15 If the formula \( B := p \supset (T \supset p) \) is strongly valid in an S4-metaframe \( \mathcal{F} \) (in the language with or without equality, respectively), then \( \mathcal{F} \) is monotonic\(^{=}\).

Proof Let us check \((\text{im})\) for an arbitrary formula \( A \). Consider the following substitution instance \( B' := A \supset (T \supset A) \) of \( B \).

Assume \( aR^nb \) and \( M, a \models A[x] \). Then \( M, a \models T \supset A[x] \) since \( M, a \models B'[x] \) and \( R^n \) is reflexive. Obviously, \( M, b \models T(= \bot \supset \bot)[x] \), and thus \( M, b \models A[x] \).

The idea of the previous proof is quite clear; \( B' \) expresses the property \((\text{im})\) for \( A \) — it states that if \( A \) is true now, then \( T \supset A \) is also true, and thus \( A \) will always be true in the future, by the definition of forcing for implication.

Hence we obtain

Proposition 5.14.16 Let \( \mathcal{F} \) be an S4-metaframe such that \( H \subseteq \text{IL}(\supset)(\mathcal{F}) \). Then \( \mathcal{F} \) is monotonic\(^{=}\).

Let us now describe monotonic metaframes in terms of accessibility relations.

Definition 5.14.17 An S4-metaframe \( \mathcal{F} \) is called semi-functorial (or \( s\)-functorial, for short) if for any \( \sigma \in \Sigma_{mn} \); \( m, n \geq 0 \)

\[(0') \forall a, b \in D^n (aR^n b \Rightarrow (\pi_{\sigma} a)R^n (\pi_{\sigma} b)),\]

i.e. if every \( \pi_{\sigma} \) is monotonic.

Lemma 5.14.18 An S4-metaframe \( \mathcal{F} \) is semi-functorial iff the following holds.

\((11)\) \( \mathcal{F} \) is permutative (see Definition 5.10.1), i.e. \((0')\) holds for all (simple) permutations \( \sigma \).\(^{25}\)

\[(12.1) \forall n > 0 \forall u, v \in W \forall a \in D^u \forall b \in D^v \ (aR^n b \Rightarrow uRv).\]

\[(12.2) \forall n > 0 \forall u, v \in W \forall a \in D^u \forall b \in D^v \forall c \in D_u \forall d \in D_v \ ((ac)R^{n+1}(bd) \Rightarrow aR^nb).\]

\[(13) \forall n > 0 \forall a, b \in D^n (aR^n b \Rightarrow (aa_n)R^{n+1}(bb_n)).\]

(Also note that \((12.1)\) can be replaced by its particular case with \( n = 1 \).)

Proof Every \( \sigma \in \Sigma_{mn} \) is a composition of permutations, simple embeddings and simple projections. So it suffices to check the monotonicity of \( \pi_{\sigma} \) for \( \sigma = \sigma^u_n \ (n \geq 0) \), \( \sigma^n_n \ (n > 0) \) and for all permutations. Now recall that \( \sigma^0_n = 1, \pi_{\sigma_n}(a) = u \) for \( a \in D^u \), \( \pi_{\sigma_n}(a) = a \).

Definition 5.14.19 A semi-functorial metaframe is called semi-functorial with equality (briefly, \( s^=\)-functorial) if it satisfies the condition (cf. 5.11.1)

\(^{25}\)Cf. Definition 5.10.1, Lemma 5.10.5.
Lemma 5.14.20 In an s-functorial metaframe all properties

\( (mm_n) \forall a, b \in M (aR^nb \Rightarrow a \subseteq b) \)

are equivalent for \( n \geq 2 \). So \((mm_n)\) holds in every \( s^n\)-functorial metaframe.

Proof \((mm_2)\) implies \((mm_n)\), since \( aR^nb \) implies \((a_i, a_j)R^2(b_i, b_j)\) by (I1), (I2.2); cf. Lemma 5.11.2.

The other way round, suppose \((mm_n)\). Let \( a = (a, a)R^2(b_1, b_2) = b \), and consider the projection \( \sigma \in \Sigma_{n^2} \) such that

\[
\sigma(i) = \begin{cases} 
1 & \text{for } i = 1, \\
2 & \text{for } i > 1. 
\end{cases}
\]

Then \( \pi_a a = a^n \), \( \pi_a b = b_1b_2^{n-1} \).

By s-functoriality \( (\pi_a a)R^n(\pi_a b) \), hence by \((mm_n)\), \((\pi_a a) \subseteq (\pi_a b)\), and thus \( b_1 = b_2 \).

An inductive argument shows that \( s^n\)-functoriality implies monotonicity. The steps of the proof are almost trivial, due to the definition of forcing 5.14.2, so the only problem is the atomic case.

Recalling that

\[(\mathcal{F}, \xi), a \vDash P(x \cdot \sigma)[x] \iff \pi_a a \in \xi^+(P)\]

and \( \xi^+(P) \) is \( R^n \)-stable, we see that monotonicity exactly corresponds to \( (0, \sigma) \).

Now let us give more details.

Lemma 5.14.21 A metaframe is \( s^{(\vDash)}\)-functorial iff it is monotonic\(^{(\vDash)}\)

Proof (Only if.) By induction on the complexity of \( A \).

If \( A = P^m_j(\pi_a x) \), then \((im)\) holds, since \( a \vDash P^m_j(\pi_a x)[x] \iff \pi_a a \in \xi^+(P^m_j) \).

By monotonicity of \( \mathcal{F} \), \( aR^nb \) implies \((\pi_a a)R^m(\pi_a b) \), and since \( M \) is intuitionistic, we obtain \( \pi_a b \in \xi^+(P^m_j) \), i.e. \( b \vDash A[x] \).

If \( A = (x_i = x_j) \), \( aR^nb \), we have

\[
a \vDash x_i = x_j [x] \iff a_i = a_j, \\
b \vDash x_i = x_j [x] \iff b_i = b_j.
\]

By (I4), \( a \subseteq b \), so \( a_i = a_j \) implies \( b_i = b_j \).

The induction step for \( \land, \lor \) is obvious.

Let \( A = B \supset C \). Suppose \( aR^nb \) and \( M, b \not\vDash A[x] \), i.e. \( M, c \vDash B[x] \), \( M, c \not\vDash C[x] \) for some \( c \in R^n(b) \). By transitivity, \( aR^nc \) and thus \( M, a \not\vDash A[x] \).

Let \( A = \forall y B \). Suppose \( y \notin x \), \( aR^nb \) and \( M, b \not\vDash A[x] \). Then by Definition 5.14.2, \( M, cd \not\vDash B[x_y] \) for some \( c \in R^n(b) \) and some \( d \in D(c) \). By transitivity, \( aR^nc \), and thus \( M, a \not\vDash A[x] \).
Now suppose \( A = \forall x_i B \), \( \mathbf{a} R^m b \), \( M, b \not\models A[x] \). Then by Definition 5.14.2, \( M, \widehat{b}_i, \not\models A[\widehat{x}_i] \). Recall that \( \widehat{a}_i = \pi_\sigma a \) for \( \sigma = \delta^n \). So since \( \pi_\sigma \) is monotonic, \( \mathbf{a} R^m b \) implies \( \widehat{a}_i R^m \widehat{b}_i \), and thus \( M, \widehat{a}_i, \not\models A[\widehat{x}_i] \), as we have already proved. Therefore \( M, a \not\models A[x] \), by Definition 5.14.2.

Let \( A = \exists y B \), \( y \notin x \), and suppose \( \mathbf{a} R^m b \) and \( M, a \models A[x] \). Then \( M, c d \models B[xy] \) for some \( c \in (R^n)^{-1}(a) \) and some \( d \in D(c) \). By transitivity, we have \( \mathbf{c} R^m b \), and hence we obtain \( M, b \models A[x] \).

The case \( A = \exists x_i B \) is similar to \( A = \forall x_i B \) and is left to the reader.

(If.) Assuming (im), let us show that \( \mathcal{F} \) is monotonic\((=)\). Note that (mm)\(_2\) readily follows from (im) for \( A = (x_1 = x_2) \).

If \( \mathbf{a} R^m b \), then \( M, a \models P^m(\pi_\sigma x)[x] \) in the intuitionistic model \( M = (\mathcal{F}, \xi) \) such that \( \xi^+(P^m) = R^m(\pi_\sigma a) \). Thus by (im), \( M, a \models P^m(\pi_\sigma x)[x] \), i.e. \( \pi_\sigma b \in \xi^+(P^m) \), and so \( (\pi_\sigma a) R^m(\pi_\sigma b) \).

Now we obtain

**Proposition 5.14.22** Every \( \mathcal{F}(=)\)-sound S4-metaframe is \( s(=)\)-functional.

**Remark 5.14.23** Monotonicity is an intrinsic property of intuitionistic forcing, thus Lemma 5.14.8 explains why we confine ourselves to \( s \)-functorial metaframes in our further considerations.

### 5.15 Intuitionistic soundness

In this section we prove intuitionistic soundness for a certain class of S4-metaframes. Its description is more complicated than in the modal case. As explained in the previous section, we certainly need \( s \)-functoriality corresponding to monotonicity. We also need the 'quasi-lift property', an intuitionistic analogue of the lift property for jections \( \pi_\sigma \). But unlike the modal case, this is yet insufficient, and we need two extra conditions related to interpretation of quantifiers.

In S4-metaframes we use the notation \( \approx^n \) for the corresponding equivalence relation \( \approx_{R^n} \) on \( D^n \); it is called \( n \)-equivalence.

**Definition 5.15.1** An S4-metaframe \( \mathcal{F} \) is called quasi-functorial (\( i \)-functorial, for short) if all its jections \( \pi_\sigma \) are quasi-morphisms.

So we have

**Lemma 5.15.2** A metaframe \( \mathcal{F} = ((F_n)_{n \in \omega}, D) \) is \( i \)-functorial iff it is \( s \)-functorial (i.e. all \( \pi_\sigma \) are monotonic) and all \( \pi_\sigma \) have the following quasi-lift property:

\[
(1') \forall a \in D^n \forall b' \in D^m ((\pi_\sigma a) R^m b' \Rightarrow \exists b \in D^n (a R^n b \& \pi_\sigma b \approx^m b')).
\]
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Recall that in the modal case we defined functorial metaframes, where all $\pi_\sigma$ are morphisms, and also weakly functorial metaframes, where $\pi_\sigma$ are morphisms for injective $\sigma$. The intuitionistic analogue of functoriality is quasi-functoriality, but as we shall see now, there is no need in the intuitionistic analogue of weak functoriality.

**Lemma 5.15.3** Let $M$ be an intuitionistic model over an $i$-functorial wi-metaframe $F$. Then we can replace $R^n$ with $\approx^n$ in the truth definition for $\exists$:

$$M, a \models \exists y B[x] \iff \exists b \approx^n a \exists c \in D(b) M, \pi_{x|y}(bc) \models B[x|y].$$

**Proof** In fact, if there exists $bc \in D^{n+1}$ such that $bR^na$ and $M, \pi_{x|y}(bc) \models B[x|y]$, then, by the quasi-lift property $(\lambda a:)$, there exists $b'c' \in D^{n+1}$ such that $(bc)R^{n+1}(b'c')$, $b' \approx^n a$, (recall that $\pi_{\sigma^+_n}(b'c') = b'$). Now by $(0\delta^n_i)$,

$$\pi_{x|y}(bc)R^n\pi_{x|y}(b'c'),$$

and thus by monotonicity 5.14.14 (im),

$$M, \pi_{x|y}(b'c') \models B[x|y].$$

The modified definition of forcing in the $\exists$-case resembles the familiar definition for intuitionistic Kripke models, but with $\approx^n$ replacing $\approx$. So in our semantics, a 'witness' for $\exists y B$ does not always exist in the present world, but should exist in an equivalent world (cf. section 5.5). This reflects the basic idea that intuitionistic models26 'distinguish' worlds up to $\approx^0$, individuals up to $\approx^1$, $n$-tuples of individuals up to $\approx^n$.

**Remark 5.15.4** One can see that in the above proof monotonicity and quasi-lift property are used explicitly only for injections $\sigma^+_n$ and $\varepsilon_{x|y}$. But we also use the monotonicity of forcing relying on the monotonicity of $\pi_\sigma$ for arbitrary (perhaps, non-injective) $\sigma$, cf. Lemma 5.14.21.

Therefore in further definitions we suppose monotonicity of all $\pi_\sigma$.

The next lemma shows that in functorial metaframes the intuitionistic truth definition for the $\exists$-case is quite analogous to the modal one. This simplifies the proof of Lemma 5.14.6.

\[^{26}\text{Due to the monotonicity of forcing, cf. 5.14.14.}\]
Lemma 5.15.5 Let $M$ be an intuitionistic model over a functorial metaframe. Then for any $B \in IF =, y \notin x$

$$M, a \Vdash \exists y B[x] \iff \exists c \in D(a) \ M, ac \Vdash B[xy].$$

Proof The same as in 5.15.3, but now $\pi_\sigma^n$ is a morphism, so we can take $b' = a$. Note that forcing is monotonic, because in a functorial metaframe every $\pi_\sigma$ is monotonic. 

Remark 5.15.6 In a similar way one can show that the equivalence

$$M, a \Vdash \exists x_i B \ [x] \iff \exists c \in D(a) \ M, \hat{a} \Vdash B \ [x_i x_i].$$

holds in functorial metaframes. Then we readily have

$$M, a \Vdash \exists x_i B \ [x] \iff M, \hat{a} \Vdash \exists x_i B \ [x_i].$$

The same equivalence for $\forall$ was already proved in 5.14, cf. the condition (VII.2).

Intuitionistic metaframes

As we already know, functoriality is equivalent to modal soundness (Lemma 5.11.6 and Theorem 5.12.13). But i-functoriality does not yet imply intuitionistic soundness as we shall see in section 5.16. Now let us consider two other soundness conditions specific for the intuitionistic case.

Definition 5.15.7 An intuitionistic metaframe (\textit{\textit{i}-metaframe, for short}) is an i-functorial metaframe satisfying

\begin{align*}
(2\sigma) \text{ (forward 2-lift property)} & \quad \forall a \in D^n \forall b \in D^{m+1}((\pi_\sigma a)R^m(\pi_\sigma b) \Rightarrow \exists c \in D^{n+1}(\pi^m aR^n(\pi^m c) & \land \pi_{\sigma+c}R^{m+1}b)). \\
(3\sigma) \text{ (backward 2-lift property)} & \quad \forall a \in D^n \forall b \in D^{m+1}((\pi^m b)R^m(\pi_\sigma a) \Rightarrow \exists c \in D^{n+1}(\pi^m cR^n a & \land bR^{m+1}(\pi_{\sigma+c}))). 
\end{align*}

Recall that $\sigma^m_+ \in \Upsilon_{m,m+1}$, $\sigma^n_+ \in \Upsilon_{n,n+1}$ are simple embeddings and $\sigma^+ \in \Sigma_{m+1,n+1}$ is the simple extension of $\sigma \in \Sigma_{mn}$, see the Introduction.

So a metaframe is intuitionistic iff it satisfies (0$\sigma$), (1$\sigma$), (2$\sigma$), (3$\sigma$) for any $\sigma$.

Definition 5.15.8 A weak intuitionistic (\textit{\textit{wi-}}) metaframe is an s-functorial metaframe satisfying (1$\sigma$), (2$\sigma$), (3$\sigma$), for all injections $\sigma$.

Note that in this case (0$\sigma$) holds for arbitrary $\sigma$.

Definition 5.15.9 An intuitionistic metaframe with equality (an \textit{\textit{i=}-metaframe, for short}) and respectively, a weak intuitionistic metaframe with equality (a \textit{\textit{wi=}-metaframe}) is an s\textsuperscript{=}-functorial i-metaframe (respectively, an s\textsuperscript{=}-functorial \textit{\textit{wi=}-metaframe}).
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Figure 5.10. Forward 2-lift property

Figure 5.11. Backward 2-lift property.
Lemma 5.15.10 \( i^e \)-metaframes (respectively \( wi^e \)-metaframes) are exactly \( i \)-metaframes (respectively, \( wi \)-metaframes) satisfying (mm\(_2\)) from 5.11.1.

Our next goal is to show that \( i^e \)-metaframes are exactly \( i^- \)-sound metaframes (Theorem 5.16.13) and at the same time — \( wi^e \)-metaframes (Theorem 5.16.10(3)).

Remark 5.15.11 Let us give a motivation for 2-lift properties. The starting point is the following simple observation on tuples (called the 'lift property' in [Skvortsov and Shehtman, 1993]). Let

\[
\sigma \in \Sigma_{mn}, \ a = (a_1, \ldots, a_n) \in D^n, \ b \in D^{m+1}, \ \pi_\sigma a = \pi_{+}^n b,
\]
i.e. \( b = a_{\sigma(1)} \ldots a_{\sigma(m)} b_{m+1} \); then there exists \( c \in D^{n+1} \) such that \( a = \pi_\sigma^n c \) and \( \pi_\sigma + c = b \); in fact, put \( c = ab_{m+1} \). This shows (2\( \sigma \)), (3\( \sigma \)) for the case when \( R^k \) are the equality relations. See Fig. 5.12 below.

\[
\begin{array}{c}
\pi_\sigma ^n c \\
\pi_\sigma \\
\pi_\sigma a
\end{array} \quad \begin{array}{c}
\pi_\sigma ^m b \\
\pi_\sigma \\
\pi_\sigma a
\end{array}
\]

Figure 5.12. \( \pi_\sigma a \) is a 'common part' of \( b \) and \( a \); \( c \) is their 'join'.

We can also check these properties for a metaframe \( \mathcal{M}(\mathcal{F}) \) associated with a \( C \)-set \( \mathcal{F} = (C, D, \rho) \) over a category \( C \). In fact, if for \( a \in D_u^n \) and \( b \in D_v^{m+1} \) we have

\[
\pi_\sigma a = (a_{\sigma(1)}, \ldots, a_{\sigma(m)}) R^m(b_1, \ldots, b_m) = \pi_\sigma ^m b,
\]
then there exists a morphism \( \mu \in C(u, v) \) such that \( \pi_\sigma ^m b = \rho_\mu \cdot (\pi_\sigma a) \), i.e. \( \forall i \ b_i = \rho_\mu (a_{\sigma(i)}) \). Then consider

\[
c' := \rho_\mu \cdot a = (\rho_\mu (a_1), \ldots, \rho_\mu (a_n)) \in D_u^n
\]
and extend it to

\[
c := c' b_{m+1} \in D_v^{n+1}.
\]

It follows that

\[
a R^n c' = \pi_\sigma ^n c, \ \pi_\sigma + c = (\pi_\sigma c') b_{m+1} = (\rho_\mu \cdot (\pi_\sigma a)) b_{m+1} = b
\]
and thus obviously

\[
(\pi_\sigma + c) R^{m+1} b.
\]
So (2σ) holds; (3σ) is checked in a dual way.

We see that in the first example the condition (2σ) expresses the 'co-amalgamation property' of tuples (Fig. 5.12).

The second example corresponds to Fig. 5.14. Here \( \rho^m_\mu \) sends a to \( \rho_\mu \cdot a \); similarly for \( \rho^n_\mu \). Again c is the 'join' of \( c' \) and b; it exists, because they have a 'common part' \( \pi^m_+ b \).

**Lemma 5.15.12** In i-functorial metaframes (2σ) is equivalent to
\[
\forall a \in D^n \forall b \in D^{m+1} (\pi_\sigma a \equiv^m \pi^m_+ b \Rightarrow \exists c \in D^{n+1} (a R^n(\pi^m_+ c) \& \pi_\sigma + c \equiv^{m+1} b)).
\]
(2σ)

Similarly (3σ) is equivalent to
\[
\forall a \in D^n \forall b \in D^{m+1} (\pi^m_+ b \equiv^m \pi_\sigma a \Rightarrow \exists c \in D^{n+1} ((\pi^m_+ c) R^m a \& b \equiv^{m+1} \pi_\sigma + c)).
\]
(3σ)

As we can see, these conditions are obtained from (2σ) and (3σ) by changing \( R, R^{m+1} \) to \( \equiv^m, \equiv^{m+1} \).

**Proof** (2σ) ⇒ (2σ). We assume (2σ) and prove a stronger claim than (2σ). Namely, for \( a \in D^{n+1}, b \in D^{m+1} \) such that (\( \pi_\sigma a \)) \( R^m(\pi^m_+ b) \) we find \( c' \in D^{n+1} \) such that
\[
a R^n(\pi^m_+ c) \& \pi_\sigma + c' \equiv^m b.
\]

First we apply (2σ) to \( a, b \) and find \( c \) shown in Fig. 5.13.

Then using the quasi-lift property (1σ+) we find \( c' \). Finally we note that
\[
a R^n(\pi^m_+ c) R^m(\pi^m_+ c') \text{ by monotonicity } (0\pi^n_+).
\]

(2σ) ⇒ (2σ). Assume (2σ). Given \( a, b \) such that \( (\pi_\sigma a) R^m(\pi^m_+ b) \), by the quasi-lift property (1σ), we can find \( a' \in R^n(a) \) with \( \pi_\sigma a' \equiv^m \pi^m_+ b \), see Fig. 5.15. Then by applying (2σ) to \( a' \) and \( b \), we obtain \( c \) such that \( a' \equiv^n \pi^m_+ c, b \equiv^{m+1} \pi_\sigma + c \). Hence by transitivity \( a R^n(\pi^m_+ c \cdot \pi_\sigma + c R^{m+1} b \) is obvious.

The proof of (3σ) ⇔ (3σ) is dual, so we skip it.

Figure 5.13.
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Figure 5.14.

Figure 5.15.
Lemma 5.15.13 In i-functorial metaframes (2σ) is equivalent to
\[ \forall a \in D^n \forall b \in D^{m+1}(\pi_\sigma a \approx^m \pi^m_+ b \Rightarrow \exists c \in D^{n+1}(aR^n(\pi^n_+ c) \& (\pi_\sigma + c)R^{m+1}b)) \]
and dually for (3σ).

Proof Similar to Lemma 5.15.12 (an exercise).

Now we are going to prove intuitionistic analogues of the results from Sections 5.11 and 5.12.

The property (1σ) together with s-functoriality means that \( \pi_\sigma \) is a quasi-p-morphism. In modal logic we have a stronger requirement that \( \pi_\sigma \) is a p-morphism. In fact, in intuitionistic models, due to monotonicity (im), \( \approx^n \)-equivalent worlds are indistinguishable. Thus equality transforms into \( \approx^n \) in the intuitionistic case.

The next assertion is an intuitionistic analogue of Lemma 5.10.6.

Lemma 5.15.14 Let \( \mathcal{F} \) be an s\( (\approx) \)-functorial metaframe. Then \( \mathcal{F} \) is weakly functorial iff for any \( A \in \mathcal{F}(\approx) \), for any assignment \((x,a)\) of length \( n \), for any injection \( \sigma \in \Upsilon_{mn} \) such that \( \text{FV}(A) \subseteq x \cdot \sigma \), for any intuitionistic model \( M = (\mathcal{F}, \xi) \)

\[ M, a \models A[x] \Leftrightarrow M, a \cdot \sigma \models A[x \cdot \sigma]. \]  

(∗i)

So (∗i) means that \( M, a \models A[x] \) does not depend on the choice of a list \( x \supseteq \text{FV}(A) \) and possible renumbering of variables.

Proof \((\Rightarrow)\) is proved by induction on the complexity of \( A \) (cf. Lemma 5.10.6). The atomic cases are obvious. The cases when the main connective of \( A \) is \( \lor \) or \( \land \), are also easy, and we leave them to the reader.

Let \( A = B \supset C \). Suppose \( M, a \nvdash A[x] \), i.e.

\[ M, b \models B[x]; M, b \nvdash C[x] \]

for some \( b \in R^n(a) \). Then \((a \cdot \sigma)R^m(b \cdot \sigma)\) by s-functoriality, and so

\[ M, b \cdot \sigma \models B[x \cdot \sigma], M, b \cdot \sigma \nvdash C[x \cdot \sigma] \]

by the induction hypothesis, thus \( M, a \cdot \sigma \nvdash A[x \cdot \sigma] \).

The other way round, suppose \( M, a \cdot \sigma \nvdash A[x \cdot \sigma] \), i.e. \( M, b' \models B[x \cdot \sigma], M, b' \nvdash C[x \cdot \sigma] \). for some \( b' \in R^m(a \cdot \sigma) \). Then by (1σ), \( b \cdot \sigma \approx^m b' \)

for some \( b \in R^n(a) \). Now by monotonicity,

\[ M, b \cdot \sigma \models B[x \cdot \sigma]; M, b \cdot \sigma \nvdash C[x \cdot \sigma], \]

and thus by the induction hypothesis,

\[ M, b \models B[x]; M, b \nvdash C[x]. \]

Hence \( M, a \nvdash A[x] \).
Now let us consider the case $A = \forall y B$. Supposing $M, a \vdash \neg A[x \cdot \sigma]$, and let us prove $M, a \vdash \neg A[x]$. By definition, we have $c \in R^m(a \cdot \sigma)$, $d \in D(c)$ such that

$$M, \pi_{(x \cdot \sigma)\upharpoonright y}(cd) \vdash B[(x \cdot \sigma)\upharpoonright y].$$

Since $(\pi_\sigma a)^R^m c = \pi_\sigma^m (cd)$, by applying $(2\sigma)$, we obtain $b \in D^n$, $e \in D^1$ such that

$$a^{R^n} \pi_\sigma^m (be) = b & (b \cdot \sigma)e = \pi_\sigma^m (be)R^{m+1}(cd).$$

Hence by $s$-functoriality,

$$\pi_{(x \cdot \sigma)\upharpoonright y}((b \cdot \sigma)e)R^l \pi_{(x \cdot \sigma)\upharpoonright y}(cd),$$

where $l$ is the length of $(x \cdot \sigma)\upharpoonright y$.

Now note that

$$\pi_{(x \cdot \sigma)\upharpoonright y}((x \cdot \sigma)y) = (x \cdot \sigma)\upharpoonright y = (x\upharpoonright y) \cdot \tau = \pi_{x\upharpoonright y} (xy) \cdot \tau$$

for some $\tau$, as we saw in the proof of 5.10.6. Hence by a standard argument we have the same for arbitrary tuples:

$$\pi_{(x \cdot \sigma)\upharpoonright y}((b \cdot \sigma)e) = \pi_{x\upharpoonright y} (be) \cdot \tau.$$ 

Then by monotonicity we obtain

$$\pi_{x\upharpoonright y} (be) \cdot \tau \vdash B[(x \cdot \sigma)\upharpoonright y],$$

and since $(x \cdot \sigma)\upharpoonright y = (x\upharpoonright y) \cdot \tau$, by the induction hypothesis, it follows that

$$\pi_{x\upharpoonright y} (be) \vdash B[x\upharpoonright y].$$

But $a^{R^n} b$, thus $a \vdash \forall y B[x]$, as required.

The other way round, suppose $a \vdash \neg A[x]$. Then

$$\pi_{x\upharpoonright y} (be) \vdash B[x\upharpoonright y]$$

for some $b \in R^n(a)$, $e \in D(b)$. Hence by the induction hypothesis,

$$\pi_{x\upharpoonright y} (be) \cdot \tau \vdash B[(x\upharpoonright y) \cdot \tau],$$

where $\tau$ is the same as above, i.e. we obtain

$$\pi_{(x \cdot \sigma)\upharpoonright y}((b \cdot \sigma)e) \vdash B[(x \cdot \sigma)\upharpoonright y].$$

But $(a \cdot \sigma)^R^n (b \cdot \sigma)$, by $s$-functoriality; thus eventually,

$$a \cdot \sigma \vdash \neg A[x \cdot \sigma],$$

by Definition 5.14.2.
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The case $A = \exists y B$ is dual to $A = \forall y B$. The proof for this case is obtained by inverting the accessibility relations $R^n$, changing $\not\vdash$ to $\vdash$ and applying $\langle 3\sigma \rangle$ instead of $\langle 2\sigma \rangle$.

In more detail, suppose $M, a \cdot \sigma \vdash A \langle x \cdot \sigma \rangle$. By definition, there exists $c \in (R^m)^{-1}(a \cdot \sigma)$, $d \in D(c)$ such that

$$\pi_{(x \cdot \sigma)}|_y(cd) \vdash B[(x \cdot \sigma)]|_y.$$  

From $cR^m(\pi_{\sigma}a)$, by $\langle 3\sigma \rangle$, we obtain $b \in D^n$, $e \in D^1$ such that

$$\pi^+_1(be) = bR^na \& (cd)R^{m+1}(b \cdot \sigma)e.$$  

Hence by s-functoriality,

$$\pi_{(x \cdot \sigma)}|_y(cd)R^l\pi_{(x \cdot \sigma)}|_y((b \cdot \sigma)e) = \pi_{x|_y}(be) \cdot \tau,$$

where $l$ is the length of $(x \cdot \sigma)|_y$. Then by monotonicity

$$\pi_{x|_y}(be) \cdot \tau \vdash B[(x \cdot \sigma)]|_y,$$

and since $(x \cdot \sigma)|_y = (x|_y) \cdot \tau$ by the induction hypothesis, it follows that

$$\pi_{x|_y}(be) \vdash B|x|_y.$$  

Since $aR^nb$, we obtain $a \not\vdash \forall yB [x]$.

To show the converse, suppose $a \vdash A [x]$. Then

$$\pi_{x|_y}(be) \vdash B|x|_y$$

for some $b \in (R^n)^{-1}(a)$, $e \in D(b)$. Hence by the induction hypothesis,

$$\pi_{x|_y}(be) \cdot \tau \vdash B[(x|_y) \cdot \tau].$$  

This is the same as

$$\pi_{(x \cdot \sigma)}|_y((b \cdot \sigma)e) \vdash B[(x \cdot \sigma)]|_y.$$  

Since by s-functoriality, $(b \cdot \sigma)R^a(a \cdot \sigma)$, it follows that

$$a \cdot \sigma \vdash A [x \cdot \sigma].$$

($\Leftarrow$) Assuming the equivalence $\langle *i \rangle$ for suitable formulas $A$, let us prove the properties $\langle 1\sigma \rangle$, $\langle 2\sigma \rangle$, $\langle 3\sigma \rangle$ for an injection $\sigma \in \Upsilon_{mn}$.

$\langle 1\sigma \rangle$ Let $a \in D^n$, $b' \in D^m$, and assume $(\pi_{\sigma}a)R^m b'$. Consider the formula

$$A := P(\pi_{\sigma}x) \supset Q(\pi_{\sigma}x)$$

and the intuitionistic model $M = (F, \xi)$ such that

$$\xi^+(P) := R^m(b'), \xi^+(Q) := \{c \in D^m \mid \not\vdash(cR^m b')\}.$$  

(recall that $F$ is an S4-frame). Then $b' \in \xi^+(P) - \xi^+(Q)$, thus $M, \pi_{\sigma}a \not\vdash A[\pi_{\sigma}x]$. Therefore by $\langle *i \rangle$, $M, a \not\vdash A [x]$, i.e. there exists $b \in R^m(a)$ such that $M, b \vdash P(\pi_{\sigma}x) [x]$ and $M, b \not\vdash Q(\pi_{\sigma}x) [x]$. Thus $\pi_{\sigma}b \in (\xi^+(P) - \xi^+(Q)) = \approx^m (b')$, i.e. $b' \approx^m \pi_{\sigma}b$.  


(2σ) Let \( a \in D^n \), \( c = (\pi^m_{\sigma}c) \in D^{m+1} \), and assume that \((\pi_{\sigma}a)R^m(\pi^m_{\sigma}c)\). Consider the formula \( A := \forall y P(\pi_{\sigma}x, y) \) with \( x = (x_1, \ldots, x_n) \), \( y \not\in x \), and the intuitionistic model \( M = (F, \xi) \) such that 

\[
\xi^+(P) = \{ c' \in D^{m+1} | \neg(c'R^{m+1}c) \}.
\]

Since \( c \not\in \xi^+(P) \), we have

\[
M, c \not\models P(\pi_{\sigma}x, y)[(\pi_{\sigma}x)y],
\]

and so \( M, \pi_{\sigma}a \not\models A[\pi_{\sigma}x] \), since \((\pi_{\sigma}a)R^m(\pi^m_{\sigma}c)\). Thus by (⋆), \( M, a \not\models A[x] \), i.e. there exists \( b = (\pi^m_{\sigma}b) \in D^{n+1} \) such that 

\[
aR^n(\pi^m_{\sigma}b) \land M, b \not\models P(\pi_{\sigma}x, y)[xy].
\]

But \((\pi_{\sigma}y) = \pi_{\sigma}(xy)\), thus \( \pi_{\sigma}+(b) \not\in \xi^+(P) \), so by definition of \( M \), 

\[
\pi_{\sigma}+(b)R^{m+1}c,
\]

as required.

(3σ) The proof is quite similar to the previous case. Now let \( a \in D^n \), \( c = (\pi^m_{\sigma}c) \in D^{m+1}, (\pi^m_{\sigma}c)R^m(\pi_{\sigma}a)\).

Consider the formula \( A := \exists y P(\pi_{\sigma}x, y) \), where \( x = (x_1, \ldots, x_n) \), \( y \not\in x \), and the intuitionistic model \( M = (F, \xi) \) such that \( \xi^+(P) = R^{m+1}(c) \).

Since \( c \in \xi^+(P) \), we have \( M, c \models P(\pi_{\sigma}x, y)[(\pi_{\sigma}x)y] \), and so \( M, \pi_{\sigma}a \models A[\pi_{\sigma}x] \), since \((\pi^m_{\sigma}c)R_m(\pi_{\sigma}a)\). Thus by (⋆) \( M, a \models A[x] \), i.e. there exists \( b = (\pi^m_{\sigma}b) \in D^{n+1} \) such that \((\pi^m_{\sigma}b)R^n a \) and \( M, b \models P(\pi_{\sigma}x, y)[xy] \), which is equivalent to

\[
M, b \models P(\pi_{\sigma}+(xy))[xy],
\]

i.e. \( \pi_{\sigma}b \in \xi^+(P) \), and hence \( cR^{m+1}(\pi_{\sigma}b) \).

---

As in the modal case, Lemma 5.15.14 allows us to arbitrarily change a list \( x \supseteq FV(A) \) in \( M, a \models A[x] \); e.g. we can take \( x = FV(A) \). So in the truth conditions for quantifiers we may always assume \( y \not\in x \), and thus we have

\[
a \models \forall y B[x] \iff \forall b \in R^n(a) \forall c \in D(b) bc \models B[xy];
\]

\[
a \models \exists y B[x] \iff \exists b R^n a \exists c \in D(b) bc \models B[xy].
\]

(or equivalently, \( bR^n a \) may be replaced with \( b \approx^n a \) in the clause for \( \exists \), cf. Lemma 5.15.3).

Now let us prove an intuitionistic analogue of Lemma 5.10.9.

**Lemma 5.15.15** Let \( M = (F, \xi) \) be an intuitionistic model over an \( S_4 \)-metaframe, \( A \in IF^{(=)}, FV(A) \subseteq r(x) \). Also let \( y \not\in BV(A), y \not\in V(A), x' := [y'][y] x, y' \not\in r(x), \) and \( A' := A[y \mapsto y'] \). Then

\[
M, a \models A[x] \iff M, a \models A'[x'].
\]
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Proof Since $F$ is a $w^{(=)}$-metaframe, due to the previous remarks, we prove the equivalence only for $x = FV(A)$. We argue by induction and consider only the atomic and the quantifier cases; other cases can be easily checked by the reader.

Let $A = P(x \cdot \sigma)$. Then $A' = P((x \cdot \sigma)' = P(x' \cdot \sigma)$, so

\[
\text{a} \vdash A \ [x] \iff (a \cdot \sigma) \in \xi^+(P) \iff \text{a} \vdash A' [x'].
\]

Let $A = \forall z B$ and suppose the assertion holds for $B$. Then $z \neq y, y'$ by the assumption of the lemma, and $z \notin x$, by our additional assumption; so $z \notin x'$. Thus

\[
\text{a} \vdash A [x] \iff \forall b \in R^n(a) \forall c \in D(b) \ bc \vdash B [xz];
\]

\[
\text{a} \vdash A' [x'] \iff \forall b \in R^n(a) \forall c \in D(b) \ bc \vdash B' [x'z].
\]

Since $(xz)' = x'z$, the right parts of these equivalences, are also equivalent by the induction hypothesis. So the assertion holds for $A$.

If $A = \exists z B$, the argument is similar:

\[
\text{a} \vdash A [x] \iff \exists b \approx^n a \exists c \in D(b) \ bc \vdash B [xz];
\]

\[
\text{a} \vdash A' [x'] \iff \exists b \approx^n a \exists c \in D(b) \ bc \vdash B' [x'z],
\]

and again the right parts of these equivalences are equivalent. □

Let us prove an intuitionistic analogue of 5.10.10.

Lemma 5.15.16 Let $F$ be a $w^{(=)}$-metaframe, $M$ an intuitionistic model over $F$ Then:

(c) $M, a \vdash A [x] \iff M, a \vdash A^1 [x]

for any congruent formulas $A$, $A^1$ and any appropriate assignment $(x, a)$.

Proof (Cf. Lemma 5.10.10.) We assume $x = FV(A)$, $|x| = n$.

It is sufficient to consider the case when $A^1$ is obtained from $A$ by replacing its subformula $QyB$ with $Qy'B'$, where $Q$ is a quantifier, $y' \not\in V(A)$, $y \not\in BV(B)$, $B' = B[y \mapsto y']$. Now the proof is by induction.

(1) If $A = \forall y B$, then $A^1 = \forall y'B'$. By our assumption, $x = FV(A)$, so $y \not\in x$, and thus

\[
\text{a} \vdash \forall y B [x] \iff \forall b \in R^n(a) \forall d \in D(b) \ ad \vdash B [xy] \iff
\]

\[
\forall b \in R^n(a) \forall d \in D(b) \ ad \vdash B' [xy'] \quad \text{(by Lemma 5.15.15 )}
\]

\[
\iff \text{a} \vdash \forall y'B' [x].
\]

(2) If $A = \exists y B$, the proof is almost the same. We have $A^1 = \exists y'B'$. Suppose $y', y \not\in x$, and thus

\[
\text{a} \vdash \exists y B [x] \iff \exists b \approx^n a \exists d \in D(b) \ ad \vdash B [xy] \iff
\]

\[
\exists b \approx^n a \exists d \in D(b) \ ad \vdash B' [xy'] \quad \text{(by Lemma 5.15.15 )}
\]

\[
\iff \text{a} \vdash \exists y'B' [x].
\]
(1), (2) prove the base of the induction. The step is rather routine, so we only consider the case $A = \forall z C$, $A^1 = \forall z C^1$. We have $x = FV(A)$, $z \notin x$, and thus

$$a \vdash A[x] \iff \forall b \in R^n(a) \forall d \in D(b) \mathrel{a} \vdash C[xz] \iff \forall b \in R^n(a) \forall d \in D(b) \mathrel{a} \vdash B^1[xz] \quad \text{(by induction hypothesis)}$$

The remaining cases are left to the reader. □

Let us prove the intuitionistic analogue of 5.11.7.

**Lemma 5.15.17** Let $\mathbb{F}$ be an $s(=)$-functorial metaframe. Then $\mathbb{F}$ is an $i$-metaframe iff

$$(\#)\text{ for any } \sigma \in \Sigma_{mn}, \text{ distinct lists of variables } x = (x_1, \ldots, x_n), \text{ y } = (y_1, \ldots, y_m), \text{ and formula } A \in IF(=) \text{ with } FV(A) \subseteq r(y), \text{ for any model } M \text{ over } \mathbb{F}, \text{ a } \in D^n:\n
$$

$$M, a \vdash [(x \cdot \sigma)/y]A[x] \iff M, a \cdot \sigma \vdash A[y].$$

Similarly to 5.11.7, this equivalence is stated for all congruent versions of $[x \cdot \sigma/y]A$.

**Proof** (‘Only if.’) Recall that $[(x \cdot \sigma)/y]A$ is defined up to congruence and can be obtained from a clean version $A^\circ$ of $A$ such that $BV(A^\circ) \cap r(xy) = \emptyset$ (and $FV(A^\circ) = FV(A) \subseteq y$) by replacing $[y \mapsto x \cdot \sigma]$. By Lemma 5.15.16 we may assume that $A^\circ = A$. Now we argue similarly to 5.11.7 by induction.

(1) If $A$ is $y_i = y_j$ or $P^k_j(y \cdot \tau)$, $\tau \in \Sigma_{km}$ or $A$ is, the proof is the same as in 5.11.7.

(2) Suppose $A = B \supset C$. If $a \models [(x \cdot \sigma)/y]A[x]$, i.e. for some $b \in R^n(a)$

$$b \vdash [(x \cdot \sigma)/y]B[x]$$

and

$$b \not\models [(x \cdot \sigma)/y]C[x],$$

then by the induction hypothesis

$$b \cdot \sigma \vdash B[y] \text{ and } b \cdot \sigma \not\models C[y].$$

Since $aR^n b$ implies $(a \cdot \sigma)R^n (b \cdot \sigma)$ by (0$\sigma$), we have $a \cdot \sigma \not\models (B \supset C)[y]$.

The other way round, if

$$a \cdot \sigma \not\models (B \supset C)[y],$$

then for some $c \in R^m(a \cdot \sigma)$

$$c \vdash B[y] \text{ and } c \not\models C[y].$$
Now by (1σ), there exists \( b \in R^n(a) \) such that \( b \cdot \sigma \approx^m c \). Then by monotonicity

\[
b \cdot \sigma \vdash B[y] \text{ and } b \cdot \sigma \not\vdash C[y].
\]

So by the induction hypothesis,

\[
b \vdash [(x \cdot \sigma)/y]B \ [x] \text{ and } b \not\vdash [(x \cdot \sigma)/y]C \ [x].
\]

Since \( aR^n b \), this implies

\[
a \not\vdash [(x \cdot \sigma)/y]A \ [x].
\]

(3) Let \( A = \forall z B \), then \( z \not\in r(xy) \) by the choice of \( A \) so

\[
[(x \cdot \sigma)/y] A = \forall z [(xz) \cdot \sigma^+/yz] B.
\]

If

\[
a \not\vdash [(x \cdot \sigma)/y]A \ [x],
\]

then there exists \( b \in R^n(a), d \in D(b) \) such that

\[
b d \not\vdash [(xz) \cdot \sigma^+/yz] B \ [xz].
\]

So by the induction hypothesis

\[
(b \cdot \sigma)d = (bd) \cdot \sigma^+ \not\vdash B[yz].
\]

Since \((a \cdot \sigma)R^n(b \cdot \sigma)\), it follows that

\[
a \cdot \sigma \not\vdash A[y].
\]

The other way round, suppose \( a \cdot \sigma \not\vdash A[y] \). Then for some \( c' \in R^n(a \cdot \sigma), d \in D(c') \)

\[
c'd \not\vdash B[yz].
\]

So for \( c := c'd \) we have \((\pi_\sigma a)R^m c' = \pi^m_+ c\). Hence by (2σ) there exists \( b \in D^{n+1} \) such that

\[
aR^n(\pi^m_+ b) \text{ and } (\pi_\sigma b)R^{m+1} c.
\]

Put \( b' := \pi^m_+ b \), then

\[
b = b'n_{n+1}, \quad \pi_\sigma b = (\pi_\sigma b')b_{n+1}.
\]

Since

\[
(\pi_\sigma b)R^{m+1} c \not\vdash B[yz],
\]

by monotonicity we have

\[
\pi_\sigma b \not\vdash B[yz].
\]

Then by the induction hypothesis,

\[
b \not\vdash [(xz) \cdot \sigma^+/yz] B \ [xz].
\]
But $b = b'b_{n+1}$, while $aR^nb'$. Hence by the definition of forcing

$$a \not\models \forall z[(xz) \cdot \sigma^+/yz]B [x]$$

as required.

(4) The case $A = \exists zB$ is dual to the previous one. The proof makes use of (3σ), cf. the $\exists$-case in the proof of 5.15.14.

(5) The cases of $\land, \lor$ are almost trivial.

('If') (Cf. the proof of 5.15.14.) For $\sigma \in \Sigma_{mn}$ let us check the properties (1σ), (2σ), (3σ).

(1σ) Let $a \in D^n, \ b' \in D^m, (\pi_\sigma a)R^m b'$. Consider $A := P(y) \supset Q(y)$, $|y| = m$ and the intuitionistic model $M = (\mathbb{F}, \xi)$ such that

$$\xi^+(P) = R^m(b'), \ \xi^+(Q) = R^m(b') \approx^m (b').$$

Then $b' \in \xi^+(P) - \xi^+(Q)$, thus $M, \pi_\sigma a \not\models A[y]$. Therefore by (#),

$$M, a \not\models [\pi_\sigma x/y] A(= P(\pi_\sigma x) \supset Q(\pi_\sigma x)) [x],$$

i.e. there exists $b \in R^m(a)$ such that $M, b \models P(\pi_\sigma x)[x]$ and $M, b \not\models Q(\pi_\sigma x)[x]$. Thus $\pi_\sigma b \in (\xi^+(P), \xi^+(Q)) = \approx^m (b')$, i.e. $b' \approx^m \pi_\sigma b$.

(2σ) Let $a \in D^n, \ c = (\pi^+\sigma c)d \in D^{m+1}$, and assume that $(\pi_\sigma a)R^m(\pi^+\sigma c)$. Consider the formula $A := \forall zP(y, z)$ with $|y| = m, z \not\in r(yx)$, and the intuitionistic model $M = (\mathbb{F}, \xi)$ such that

$$\xi^+(P) = \{c' \in D^{m+1} \mid \neg(c'R^{m+1}c)\}.$$

Since $c \not\in \xi^+(P)$, we have

$$M, c \not\models P(y, z)[yz],$$

and so $M, \pi_\sigma a \not\models A[y]$, since $(\pi_\sigma a)R^m(\pi^+\sigma c)$. Thus by (#), $M, a \not\models \forall zP(\pi_\sigma x, z)[x], 27$ i.e. there exists $b = (\pi^+\sigma b)d \in D^{n+1}$ such that

$$aR^n(\pi^+\sigma b) \& M, b \not\models P(\pi_\sigma x, z)[xz].$$

But $(\pi_\sigma x)z = \pi_\sigma +(xz)$, thus $\pi_\sigma +(b) \not\in \xi^+(P)$, so by definition of $M, \pi_\sigma +(b)R^{m+1}c$, as required.

(3σ) The proof is similar to (2σ). We assume $a \in D^n, \ c = (\pi^+\sigma c)d \in D^{m+1}, (\pi^+\sigma c)R^m(\pi_\sigma a)$ and consider the formula $A := \exists zP(y, z)$, where $|y| = m, z \not\in r(yx)$, and the intuitionistic model $M = (\mathbb{F}, \xi)$ such that $\xi^+(P) = R^{m+1}(c)$. Checking the details is left to the reader.

---

27It is essential that here we use $\forall zP(\pi_\sigma x, z)$, a congruent version of $[[x \cdot \sigma]/y]A$ with a new variable $z$ (not $x \in r(x) - r(\pi_\sigma x)$).
Note that unlike the modal case (Lemma 5.11.9, Proposition 5.11.11), we cannot consider only quantifier free formulas \( A \) in the proof of ‘if’ in the above lemma, because \((2\sigma),(3\sigma)\) are related to quantifiers.

Now we obtain an intuitionistic analogue of 5.11.12:

**Proposition 5.15.18** Let \( M \) be a model over a \( w(=) \)-metaframe, \( u \in M \), \( A, A^* \in IF(=) \), and let \([a/x], [a^*/x^*]\) be assignments in \( M \) giving rise to equal \( D_u \)-sentences: \([a/x] A = [a^*/x^*] A^*\). Then

\[
(\#) \quad M, a \models A[x] \iff M, a^* \models A[x^*].
\]

**Proof** Similar to 5.11.12, but now we apply 5.15.14, 5.15.17 instead of 5.10.6, 5.11.7.

Proposition 5.15.18 allows us to define forcing for \( D_u \)-sentences \([a/x] A\) in \( i \)-metaframes in the same manner as we did in Kripke sheaves, Kripke bundles and modal metaframes. In the modal case this definition is successful, due to the property

\[
(mm) \quad \forall a \forall b \ (a R^n b \Rightarrow a \text{ sub } b),
\]

cf. Lemma 5.11.2, Proposition 5.11.12. This property also holds in \( i^n \)-metaframes (5.14.20), but not always in \( i \)-metaframes (cf. Example 2* in 5.16). But \( i \)-metaframes still enjoy the property \((\#\#)\), perhaps thanks to the following weaker version of \((mm)\).

**Exercise 5.15.19** Show that

\[
(mm \sim) \quad \forall a \forall b \ (a R^n b \Rightarrow \exists b' \approx^n b \ a \text{ sub } b')
\]

holds in \( i \)-functorial metaframes. Hint: choose \( \sigma \) such that \( a \text{ sub } \sigma \& \sigma \text{ sub } a \) and apply \((1\sigma)\) to \( c \) such that \( a = \pi \circ c \).

Now we can prove intuitionistic analogues of results from Section 5.12.

The following lemma is an analogue of 5.12.1.

**Lemma 5.15.20** Let \( M = (F, \xi) \) be a model over an \( w(=) \)-metaframe \( F \), and let \( A \in IF(=) \). Then the following conditions are equivalent.

1. \( M \models A \);
2. \( M, a \models A[x] \) for any ordered assignment \((x, a)\) in \( F \) such that \( FV(A) = r(x) \);
3. there exists a list of distinct variables \( r(y) \supset FV(A) \) such that \( M, b \models A[y] \) for any ordered assignment \((y, b)\) in \( F \).

Thus \((b)\) does not depend on the choice of \( y \).

**Proof** Along the same lines as 5.12.1, but now we apply 5.15.14 instead of 5.10.6.
The next lemma is an analogue of 5.12.2.

**Lemma 5.15.21** Let $\mathcal{F}$ be an $i(\equiv)$-metaframe. Then

1. $\mathcal{IL}(\equiv)(\mathcal{F})$ is closed under modus ponens;
2. for any $B \in IF(\equiv)$
   
   \[ \mathcal{F} \vdash B \iff \mathcal{F} \vdash \forall yB \iff \mathcal{F} \vdash \forall B; \]

3. $\mathcal{IL}(\equiv)(\mathcal{F})$ is closed under generalisation;
4. $\mathcal{IL}(\equiv)(\mathcal{F})$ is closed under strict substitutions.

Note that (1) for $B \in IF$ holds in every wi-metaframe $\mathcal{F}$.

**Proof**

1. Similar to the proof of 5.12.2(2), but now we use Lemma 5.15.14 (and the reflexivity of $R^\equiv 5.15.20$).
2. Cf. the proof of Lemma 5.12.2(3); again we use 5.15.14 and the reflexivity of $R^\equiv$.
3. Follows from (2).
4. The argument is essentially the same as in the proof of 5.12.2(4). Assuming that $S = [C/P(y)]$ is a strict substitution ($FV(C) \subseteq y$) and $\mathcal{F} \vdash A$, we show that $\mathcal{F} \vdash SA$, i.e. $M, a \vdash SA[x]$ for any intuitionistic model $M$ and any assignment $(x, a)$ with $r(x) = FV(A) \supseteq FV(SA)$.

   By Lemma 5.15.16, we may assume that $A$ is clean, $BV(A) \cap r(x) = \emptyset$.

   Consider the model $N = (\mathcal{F}, \eta)$ such that
   
   \[
   \eta^+(P) = \{ b \mid M, b \vdash C[y] \}, \\
   \eta^+(Q) = \xi^+(Q) \text{ for any other predicate letter } Q.
   \]

   This model is intuitionistic, by monotonicity of $\mathcal{F}$. Now it suffices to prove by induction that

   \[
   (\#) \quad N, a \vdash B[z] \text{ iff } M, a \vdash SB[z]
   \]

   for any subformula $B$ of $A$ and for any assignment $(z, a)$ with $r(z) \supseteq FV(B)$, $r(z) \cap BV(B) = \emptyset$.

   If $B = P(z \cdot \sigma)$ for a map $\sigma$, then $SB = [z \cdot \sigma/y] C$. By Definition 5.14.2, the choice of $N$, and Lemma 5.15.17 we have

   \[
   N, a \vdash B[z] \text{ iff } (a \cdot \sigma) \in \eta^+(P) \text{ iff } M, a \cdot \sigma \vdash C[y] \text{ iff } M, a \vdash [z \cdot \sigma/y] C[z],
   \]

   and thus (\#) holds.
If $B = B_1 \supset B_2$ and $(\dagger)$ holds for $B_1$, $B_2$, we obtain it for $B$:

\[
N, a \models B [z] \iff \forall b \in R^n(a) (N, b \models B_1 [z] \Rightarrow N, b \models B_2 [z]) \iff \\
\forall b \in R^n(a) (M, b \models SB_1 [z] \Rightarrow M, b \models SB_2 [z]) \iff \\
M, a \models SB_1 \supset SB_2 (= SB) [z]
\]

(where $n = |a|$).

Suppose $B = \forall v B_1$ and $(\sharp)$ holds for $B_1$. By our assumption, $v \notin r(z)$, so we have

\[
N, a \models B [z] \iff \forall b \in R^n(a) \forall c \in D(b) N, bc \models B_1 [zv] \\
\iff \forall b \in R^n(a) \forall c \in D(b) M, bc \models SB_1 [zv] \\
\iff M, a \models \forall v(SB_1) (= SB) [z].
\]

Thus $(\sharp)$ holds for $B$.

The cases $B = B_1 \land B_2$, $B_1 \lor B_2$, $\exists v B_1$ are left to the reader, and the case $B = \bot$ is trivial.

Now similarly to 5.12.3 we obtain the following

**Proposition 5.15.22** Let $\mathcal{F}$ be an $i(=)$-metaframe. Then

\[
\text{IL}(=)(\mathcal{F}) = \{ A \in \text{IF}(=) \mid \mathcal{F} \models A^n \text{ for any } n \in \omega \}, \\
= \{ A \in \text{IF}(=) \mid \mathcal{F} \models A^n \text{ for any } n \in \omega \}.
\]

Therefore $\text{IL}^= (\mathcal{F})$ is conservative over $\text{IL}(\mathcal{F})$.

We also obtain the following analogues of 5.12.9, 5.12.10.

**Proposition 5.15.23** For an $i(=)$-metaframe $\mathcal{F}$

\[
\text{IL}_n(=)(\mathcal{F}) = \bigcap_{n \in \omega} \text{IL}(F_n).
\]

**Corollary 5.15.24** All theorems of $\mathcal{H}$ are strongly valid in every $i$-metaframe.

**Proof** $\mathcal{H} \subseteq \text{IL}(F_n)$ if $F_n$ is an $\text{S}4$-frame.

**Remark 5.15.25** Note that we cannot extend Lemma 5.15.21 and Proposition 5.15.22 to $\text{w}i$-metaframes, but Proposition 5.15.23 still holds (cf. Exercise 5.12.11 for the modal case). However in the next section we will show that only $i(=)$-metaframes can validate predicate all $\text{QH}$-axioms (and theorems).

**Theorem 5.15.26** (Soundness theorem) For any $i(=)$-metaframe $\mathcal{F}$, $\text{IL}(=)(\mathcal{F})$ is an $s.p.l.(=)$. 

Proof  Soundness of the inference rules is proved by the same argument as in 5.12.12. Now similarly to 5.12.13, we prove the strong validity of the predicate axioms (and the axioms of equality — for i"-metaframes); all these formulas can be supposed clean. So consider an intuitionistic model $M = (F, \xi)$.

- $A_1 := \forall y P(x, y) \supset P(x, z)$, for $y, z \not\equiv x, y \neq z, |x| = n$.
  It is sufficient to show that
  \[ ab \models \forall y P(x, y) \ [xz] \Rightarrow ab \models P(x, y) \ [xz]. \]
  So assume $ab \models \forall y P(x, y) \ [xz]$; then by 5.15.14 $a \models \forall y P(x, y) \ [x]$. Hence (by the reflexivity of $R^n$) for any $c \in D(a)$, $ac \models P(x, y) \ [xy]$, which is equivalent to $ac \models \xi^+(P)$, and thus to $ac \models P(x, z) \ [xz]$. Therefore $ab \models P(x, z) \ [xz]$.

- $A_2 := P(x, z) \supset \exists y P(x, y)$, where $y, z \not\equiv x, y \neq z, |x| = n$.
  Assume $ab \models P(x, z) \ [xz]$, i.e. $ab \in \xi^+(P)$, which is equivalent to $ab \models P(x, y) \ [xy]$. Since $a \approx_n a$, by the definition of forcing it follows that $a \models \exists y P(x, y) \ [x]$; hence $ab \models \exists y P(x, y) \ [xz]$ by 5.15.14.

- $A_3 := \forall y (Q(x) \supset P(x, y)) \supset (Q(x) \supset \forall y P(x, y))$, where $y \not\equiv x, |x| = n$.
  Assuming
  \[ (3.1) \quad a \models \forall y (Q(x) \supset P(x, y)) \ [x], \]
  let us show
  \[ (3.2) \quad a \models (Q(x) \supset \forall y P(x, y)) \ [x]. \]
  In fact, (3.1) means
  \[ (3.3) \quad \forall b \in R^n(a) \forall c \in D(b) \quad bc \models (Q(x) \supset P(x, y)) \ [xy], \]
  and we have to show for any $b \in R^n(a)$
  \[ (3.4) \quad b \models Q(x) \ [x] \Rightarrow b \models \forall y P(x, y) \ [x]. \]
  So we assume
  \[ (3.5) \quad b \models Q(x) \ [x] \]
  and prove that
  \[ (3.6) \quad b \models \forall y P(x, y) \ [x], \]
  which means
  \[ (3.7) \quad \forall d \in R^n(b) \forall c \in D(d) \quad dc \models P(x, y) \ [xy]. \]
Let $b R^n d$; by monotonicity, (3.5) implies

(3.8) $d \vdash Q(x) [x]$;

then $d \in \xi^+(Q)$, and thus

(3.9) $\forall c \in D(d) \ dc \vdash Q(x) [xy]$.

On the other hand, $R^n$ is transitive, so $d \in R^n(a)$, and (3.3) implies

(3.10) $\forall c \in D(d) \ dc \vdash (Q(x) \supset P(x, y)) [xy]$.

Eventually from (3.9) and (3.10) we obtain

$\forall c \in D(d) \ dc \vdash P(x, y) [xy]$,

which yields (3.6).

Therefore (3.4) holds.

$A_4 := \forall y(P(x, y) \supset Q(x)) \supset (\exists y P(x, y) \supset Q(x))$, for $y \not\in x$, $|x| = n$.

Assuming

(4.1) $a \vdash \forall y(P(x, y) \supset Q(x)) [x]$ let us show

(4.2) $a \vdash (\exists y P(x, y) \supset Q(x)) [x]$.

Consider an arbitrary $b \in R^n(a)$ such that

(4.3) $b \vdash \exists y P(x, y) [x]$.

We have to show that

(4.4) $b \vdash Q(x) [x]$.

By (4.3) we have for some $b' \approx^n b$, $c \in D(b')$,

(4.5) $b'c \vdash P(x, y) [xy]$.

Now $a R^n b'$, so by (4.1)

(4.6) $b'c \vdash (P(x, y) \supset Q(x)) [xy]$.

From (4.5), (4.6) we obtain

(4.7) $b'c \vdash Q(x) [xy]$,

which implies

(4.8) $b' \vdash Q(x) [x]$,

and the latter implies (4.4) by monotonicity, since $b' \approx^n b$. 
CHAPTER 5. METAFRAME SEMANTICS

- A_5 := (x = x). The proof is trivial.
- A_6 := (y = z ∩ P(x, y) ∩ P(x, z)), y, z ≠ x, y, z are distinct, |x| = n.
  In fact, assume

(6.1) abc ⊩ y = z \[xyz]\]

and

(6.2) (abc)R^{n+2}d.

From (6.1) we readily obtain b = c, and (6.2) implies (abc) sub d by 5.14.20 (I4). Hence d_{n+1} = d_{n+2}. Now we have to check that

(6.3) d ⊩ P(x, y) [xyz] ⇒ d ⊩ P(x, z) [xyz].

So suppose

(6.3) d ⊩ P(x, y) [xyz].

By 5.15.18, this implies

(6.4) (d_1, \ldots, d_{n+1}) ⊩ P(x, z) [xz].

Since d_{n+1} = d_{n+2}, (6.4) is the same as

(6.5) (d_1, \ldots, d_n, d_{n+2}) ⊩ P(x, z) [xz].

which (again by 5.15.18) implies

\[d ⊩ P(x, z) [xyz].\]

Thus (6.3) holds.

\[\blacksquare\]

Recall that QS4-metaframe is i^-functorial. So we can conclude that IL^{(=)}(F) is a superintuitionistic predicate logic for any QS4-metaframe F. Thus soundness theorem 5.15.26 implies Proposition 5.14.7.

5.16 Maximality theorem

In this section we prove necessary conditions for intuitionistic soundness. To this end, we present Definitions 5.15.7, 5.15.8, 5.15.9 in a more convenient form. But first let us recall basic properties of i-metaframes and introduce some their versions.

(0σ) ∀a, b ∈ D^n(aR^m b ⇒ (π_σ a)R^m(π_σ b)),

\[\blacksquare\]
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(1σ) \( \forall a \in D^n \forall b' \in D^m((\pi_\sigma a)R^m b' \Rightarrow \exists b \in D^n(aR^n b \land \pi_\sigma b \approx^m b')). \)

(2σ) (forward 2-lift property)

\[
\forall a \in D^n \forall b \in D^{m+1}((\pi_\sigma a)R^m(\pi_+^m b) \Rightarrow \exists c \in D^{n+1}(aR^n(\pi_+^n c) \land (\pi_\sigma + c)R^{m+1}b));
\]

(3σ) (backward 2-lift property)

\[
\forall a \in D^n \forall b \in D^{m+1}((\pi_\sigma + b)R^m(\pi_\sigma a) \Rightarrow \exists c \in D^{n+1}((\pi_+^n c)R^n a \land bR^{m+1}(\pi_\sigma + c)));
\]

\((mm_2) \) \( \forall a, b_1, b_2 ((a, a)R^2(b_1, b_2) \Rightarrow b_1 = b_2). \)

Recall that i-metaframes are the S4-metaframes satisfying (0σ), (1σ), (2σ), (3σ) for all \( \sigma \); wi-metaframes satisfy the same conditions for all injective \( \sigma \); (w)i\( ^{-}\)-metaframes also satisfy (mm\( _2 \)). The correlation between different kinds of metaframes is shown in the diagram below.

\[
\text{i-functorial metaframes} \quad \rightarrow \quad \text{i-metaframes} \quad \rightarrow \quad \text{wi-metaframes} \quad \rightarrow \quad \text{s-functorial metaframes}
\]

\[
\text{i\( ^{-}\)-metaframes} \quad \rightarrow \quad \text{wi\( ^{-}\)-metaframes} \quad \rightarrow \quad \text{s\( ^{-}\)-functorial metaframes}
\]

By Lemma 5.14.14 the condition \( \forall \sigma (0\sigma) \) (i.e. s-functorality) is equivalent to the conjunction of the following properties:

(I1) (0σ) holds for all permutations \( \sigma \);

(I2.1) \( \forall n > 0 \forall u, v \in W \forall a \in D_u^n \forall b \in D_v^n (aR^n b \Rightarrow uRv); \)

(I2.2) \( \forall n > 0 \forall u, v \in W \forall a \in D_u^n \forall b \in D_v^n \forall c \in D_u \forall d \in D_v ((ac)R^{n+1}(bd) \Rightarrow aR^n b); \)

(I3) \( \forall n > 0 \forall a, b \in D^n (aR^n b \Rightarrow (aa_n)R^{n+1}(bb_n)); \)

We also need the following conditions:

(I2.3) \( \forall n > 0 \forall u, v (uRv \Rightarrow \forall a \in D_u^n \exists t \approx^0 v \exists b \in D_t^n aR^n b') \)

(I2.4) \( \forall n, a, b (\pi_+^n(a)R^n b \Rightarrow \exists c \in D^{n+1} (aR^{n+1} c \land (\pi_+^n c) \approx^n b)); \)

(14.1) for any \( a, b \in D^1 \)

\[
\pi_\sigma(a)R^0 \pi_\sigma(b) \Rightarrow \exists c_1, c_2 (\pi_\sigma(c_1) = \pi_\sigma(c_2) \land aR^1 c_1 \land c_2 R^1 b);
\]
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(I4.2) for any \( n > 0, \ a, b \in D^{n+1} \)
\[
(\pi_+^n a) R^n (\pi_+^n b) \Rightarrow \exists c \in D^n \ \exists d, e \in D(c) (a R^{n+1} (cd) \land (ce) R^{n+1} b);
\]

(I5.1) \( (aa_n) R^{n+1} b \land b_n \neq b_{n+1} \Rightarrow \exists c \in R^n (a \land (cc_n) \approx^{n+1} b) \);

(I5.2) for any \( a \in D^n, \ b \in D^{n+2}, \ n > 0 \)
\[
(aa_n) R^{n+1} (\pi_+^{n+1} b) \land b_n \neq b_{n+1} \Rightarrow \exists c \in D^n \ \exists d \in D(c) (a R^n c \land (cc_n d) R^{n+2} b);
\]

(I5.3) for any \( a \in D^n, \ b \in D^{n+2}, \ n > 0 \)
\[
(\pi_+^{n+1} b) R^{n+1} (aa_n) \land b_n \neq b_{n+1} \Rightarrow \exists c \in D^n \ \exists d \in D(c) (c R^n a \land b R^{n+2} (cc_n d));
\]

and also the following:\footnote{Recall that \( \bar{c}_1 \) is obtained by eliminating \( c_1 \) from \( c \).}

(I5) for any \( a \in D^n, \ b \in D^{n+2}, \ n > 0 \)
\[
(aa_n) R^{n+1} b_1 \land b_{n+1} \neq b_{n+2} \Rightarrow \exists c \in D^{n+1} \ b \approx^{n+2} cc_{n+1};
\]

(I5') for any \( a \in D^1, \ b \in D^{n+1}, \ n > 0 \)
\[
(a, a) R^2 (b_n, b_{n+1}) \land b_n \neq b_{n+1} \Rightarrow \exists c \in D^n \ b \approx^{n+1} cc_n.
\]

Later on we will show (Lemma 5.16.9) that each of (I5), (I5') is equivalent to (I5.1) & (I5.2) & (I5.3).

Lemma 5.16.1 In i-functorial metaframes (I4.1), (I4.2) are respectively equivalent to

(I4.1') for any \( a, b \in D^1 \)
\[
\pi_0 (a) \approx^0 \pi_0 (b) \Rightarrow \exists c_1, c_2 (\pi_0 (c_2) = \pi_0 (c_1) \land a R^1 c_1 \land c_2 R^1 b),
\]

(I4.2') for any \( n > 0, \ a, b \in D^{n+1} \)
\[
\pi_+^n a \approx^n \pi_+^n b \Rightarrow \exists c \in D^n \ \exists d, e \in D(c) (a R^{n+1} (cd) \land (ce) R^{n+1} b).
\]

Proof Note that (I4.1) is \( (2 \varnothing_1) \). In fact \( \varnothing_1 = \sigma_0^0 \) (and \( \pi_0 (a) = \pi_0^0 (a) \) is the world of \( a \in D^1 \)), so \( (2 \varnothing_1) \) is
\[
\forall a, b \in D^1 (\pi_0 (a) R^0 \pi_0 (b) \Rightarrow \exists c \in D^2 (a R^1 \pi_+^1 c \land \pi_0^1 (c) R^1 b)).
\]

Now if \( c = c_1 c_2 \), then \( \pi_+^1 (c) = c_1, \ \pi_0^1 (c) = c_2 \). By the same reason, (I4.1') is \( (2^\sim \varnothing_1) \) from 5.15.12. So by 5.15.12, (I4.1) \( \Leftrightarrow \) (I4.1').

Analogously, (I4.2) is \( (2 \sigma_+^n) \) and (I4.2') is \( (2^\sim \sigma_+^n) \), so (I4.1) \( \Leftrightarrow \) (I4.2').
The next lemma is similar to 5.11.3 and 5.11.6.

**Lemma 5.16.2** Let $\mathcal{F}$ be an $s$-functorial metaframe, and let $\sigma \in (\Sigma_{mn} - \Phi_{mn})$, $m \geq 2$. Then $\mathcal{F}$ is $s'$-functorial (i.e. satisfies $(mm_2)$) iff $\pi_\sigma[D^n]$ is $R^m$-stable.

Recall that $\pi_\sigma$ is surjective iff $\sigma$ is injective.

**Proof** (Only if.) Assume that $\pi_\sigma[D^n]$ is $R^m$-stable, but $\mathcal{F}$ does not satisfy $(mm_2)$. Suppose $(aa)R_2(b_1b_2)$, $b_1 \neq b_2$, and $i,j \in I_m$, $i \neq j$, $\sigma(i) = \sigma(j)$.

Now let $a := a^m$; then $a = a^n \cdot \sigma \in \pi_\sigma[D^n]$. Next, let $c := b_1^{-1}b_2b_1^{m-i}$. Then $c_i \neq c_j$, so $\neg (\sigma \text{ sub } c)$, and thus $c \not\in \pi_\sigma[D^n]$, by Lemma 5.9.5.

On the other hand, consider $\tau \in \Sigma_{n2}$ such that $\tau(i) = 2$, $\tau(j) = 1$ for $j \neq i$. Then $a = \pi_\tau(aa)$, $c = \pi_\tau(b_1b_2)$, and thus $aR^m c$ by $s$-functoriality. Therefore $c \in (R^m(\pi_\tau[D^n])) - \pi_\tau[D^n]$, which contradicts our assumption.

(If.) If $\mathcal{F}$ is $s'$-functorial, it satisfies $(mm_2)$ (Lemma 5.11.2). Then by Lemma 5.11.3, $\pi_\sigma[D^n]$ is $R^m$-stable.

Let us now reformulate the condition $(1\sigma)$. Note that for an $s$-functorial metaframe $\mathcal{F}$, $(1\sigma)$ means that $\pi_\sigma$ is a quasi-p-morphism from $(D^n, R^n)$ onto $(D^m, R^m)$.

**Lemma 5.16.3** Every $s$-functorial metaframe satisfies $(1\sigma)$ for any permutation $\sigma$.

**Proof** $s$-functoriality implies permutability (cf. Definition 5.10.1), so by 5.10.2, for any $\sigma \in \Phi_n$, $\pi_\sigma$ is an automorphism of $F_n$, and thus a (p-)morphism.

**Lemma 5.16.4** Let $\mathcal{F}$ be an $s$-functorial metaframe

1. If $\mathcal{F}$ satisfies $(1\sigma^n_+)$ for any simple embedding $\sigma^n_+$, then it satisfies $(1\sigma)$ for any injection $\sigma$.

2. If $\mathcal{F}$ satisfies $(1\sigma^n_+)$ and $(1\sigma^n_-)$ for every $n$, then it satisfies $(1\sigma)$ for any $\sigma$.

**Proof** Follows from 5.16.3 and the observation that the composition of quasi-morphisms is a quasi-morphism.

**Lemma 5.16.5** Let $\mathcal{F}$ be an $s$-functorial metaframe. Then

1. $\mathcal{F}$ satisfies $(1\sigma)$ for any injection $\sigma$ iff it satisfies $(12.3)$ and $(12.4)$;

2. $\mathcal{F}$ satisfies $(1\sigma)$ for any $\sigma$ iff it satisfies $(12.3)$, $(12.4)$, and $(15.1)$;

3. If $\mathcal{F}$ is $s'$-functorial, then

   a. $\mathcal{F}$ satisfies $(15.1)$;

   b. $(1\sigma)$ holds for any $\sigma$ iff it holds for any injection $\sigma$ (iff $\mathcal{F}$ satisfies $(12.3)$ and $(12.4)$, by (1)).

---

\(^{29}\text{For the notation } a^m, \text{ see the Introduction.}\)
(I2.3) can also be replaced with its particular case corresponding to $n = 1$, cf. Lemma 5.14.18.

Note that in (I2.3) the worlds of $a$ and $b$ are $\simeq^0$-equivalent, but not necessarily equal.

Proof

(1) (I2.3) is the quasi-lift property (1$\varnothing_n$) and (I2.4) is the quasi-lift property (1$\sigma^n_+$) for $n > 0$. Since $\varnothing_1$ is $\sigma^n_+$, we can apply Lemma 5.16.4(1).

(2) Similarly, we can apply 5.16.4(2). Now (I5.1) is $(2\sigma^n_+)$, up to the conjunct $b_n \neq b_{n+1}$ in the premise. But $b_n = b_{n+1}$ obviously implies the conclusion of (I5.1) — just put $c = b$; thus (I5.1) $\Leftrightarrow (1\sigma^n_+)$.

(3) (a) By $(mm_{n+1})$, $(aa_{n+1})R^{n+1}b$ implies $b_n = b_{n+1}$, so the premise of (I5.1) is false.
   (b) Obviously follows from (2) and (3)(a).

Now let us turn to properties (2$\sigma$), (3$\sigma$). It is clear that (3$\sigma$) is dual to (2$\sigma$), so it can be analysed in a similar way.

Lemma 5.16.6 Let $F$ be an s-functorial S4-metaframe. Then

$$ (2\sigma) \& (2\tau) \Rightarrow (2(\sigma \cdot \tau)) $$

and

$$ (3\sigma) \& (3\tau) \Rightarrow (3(\sigma \cdot \tau)). $$

Proof We prove only the first implication; the second one is an easy exercise for the reader.

So assume $(2\sigma) \& (2\tau)$. Let $\sigma \in \Sigma_{mn}$, $\tau \in \Sigma_{km}$; then $\sigma \cdot \tau \in \Sigma_{kn}$. Let $a \in D^n$, $c = c'c_{k+1} \in D^{k+1}$, and assume that

$$(\pi_{\sigma,\tau} a) = ((\pi_{\sigma} a) \cdot \tau)R^k(\pi^k_+ c) = c'.$$

Then by $(2\tau)$, there exists $d = d'd_{m+1} \in D^{m+1}$ such that

$$(\pi_{\sigma} a)R^m(\pi^m_+ d) = d', \ (d \cdot \tau^+)R^{k+1}c.$$ 

Now by $(2\sigma)$, there exists $b = b'b_{n+1} \in D^{n+1}$ such that $aR^n(\pi^n_+ b) = b'$ and $\quad (\pi_{\sigma} b) = (b' \cdot \sigma)b_{n+1}R^{m+1}d$. Thus

$$\pi_{(\sigma,\tau)+} b = b \cdot (\sigma \cdot \tau)^+ = ((b' \cdot \sigma) \cdot \tau)b_{n+1} = ((\pi_{\sigma} b) \cdot \tau^+)R^{k+1}(d \cdot \tau^+)$$

by s-functoriality. So by the transitivity of $R^{k+1}$, it follows that $(\pi_{(\sigma,\tau)+} b)R^{k+1}c$. 

\]
Lemma 5.16.7 Let $F$ be an $s$-functorial $S_4$-metaframe. Then (2σ), (3σ) hold for any permutation $σ ∈ Υ_n$.

Proof Let us check (3σ). Let

$$a ∈ D^n, c = c′c_{n+1} ∈ D^{n+1}, \pi^+_n c = c′R^n(a \cdot σ).$$

Put $b = (c′ \cdot σ^{-1})c_{n+1}$. Then by s-functoriality,

$$\pi^+_n b = (c′ \cdot σ^{-1})R^n((a \cdot σ) \cdot σ^{-1}) = a.$$

We also have:

$$\pi_{σ+b} = ((c′ \cdot σ^{-1}) \cdot σ)c_{n+1} = c′c_{n+1} = c,$$

and thus $cR^{n+1}(\pi_{σ+b})$ by the reflexivity of $R^{n+1}$. □

So to prove (2σ), (3σ) for all injective $σ$ (respectively, for all $σ$), it suffices to check them only for simple embeddings $σ^n_+$ (respectively for all $σ^n_+, σ^n_-$). This is the same as with (1σ) in Lemma 5.16.7.

Lemma 5.16.8 Let $F$ be an $s$-functorial $S_4$-metaframe. Then

(1) (2σ) & (3σ) holds for any $σ$ iff (I4.1) & (I4.2).

(2) (2σ) & (3σ) holds for any $σ$ iff (I4.1) & (I4.2) & (I5.2) & (I5.3).

(3) If $F$ is $s^-$-functorial, then

(a) $F$ satisfies (I5.2) & (I5.3).

(b) (2σ) & (3σ) holds for any $σ$ iff it holds for any injection $σ$ (i.e. iff $F$ satisfies (I4.2) & (I4.2), by (2)).

Note that $aR^{n+1}b$ iff $(aσ_n)R^{n+1}(bb_σ)$ by s-functoriality; so we can respectively change (I5.2); similarly with (I5.3).

Proof (1) As we noticed in the proof of 5.16.1, (2σ1) is equivalent to (I4.1), and (2σ^+_n) is equivalent to (I4.2).

(3σ) is obtained from (2σ) by replacing every $R^k$ with its converse and permuting $a$ with $c$. But this replacement does not change (I4.1), (I4.2); so (3σ^+_n) is equivalent to (2σ^+_n).

(2) (2σ^n) is

$$∀a ∈ D^n ∀b ∈ D^{n+2} ((aσ_n)R^{n+1}(π^{n+1}_n b) ⇒ ∃g ∈ D^{n+1} (aR^{n}(π^n_+ g) & (π^{σ_n}_n + g)R^{n+2}b)).$$

If $g = cd, c ∈ D^n, d ∈ D(c)$, then

$$c = π^n_+ g, π^{σ_n}_n + g = (c \cdot σ^n_n)d = cc_n d,$$
so \((2\sigma^n)\) is equivalent to

\[
\forall a \in D^n \ \forall b \in D^{n+2} \ ((a_{\sigma n}) R^{n+1} (\pi^{n+1}_+ b) \Rightarrow
\exists c \in D^n \ \exists d \in D(c) (a R^n c \ & (c c_n d) R^{n+2} b)).
\]

Note that this implication holds if \(b_n = b_{n+1}\). In fact, put

\[
c := b_1 \ldots b_n = \pi^n_+ \pi^{n+1}_+ b, \ d := b_{n+2};
\]

then \((a a_n) R^{n+1} (\pi^{n+1}_+ b)\) implies \(a R^n c\) by \(s\)-functoriality, and \(c c_n d = c b_{n+1} d = b R^{n+2} b\) by reflexivity. Therefore \((2\sigma^n)\) is equivalent to \((I5.2)\).

Thus an equivalent of \((3\sigma^n)\) is obtained from \((I5.2)\) by replacing \(R^{n+1}, R^{n+2}\) with their converses, which is an equivalent of \((I5.3)\).

\[\text{(3)(b) By 5.14.20 \(s\)-functorial metaframes satisfy} \ (mm_{n+1}). \text{ Thus} \ (aa_n) R^{n+1} (\pi^{n+1}_+ b)\), \(\text{implies} \ (aa_n) \ sub (\pi^{n+1}_+ b), \text{and hence} \ b_n = b_{n+1}. \text{ So} \ (I5.2) \text{ holds trivially. Similarly we obtain} \ (I5.3). \]

**Lemma 5.16.9** Let \(F\) be an \(s\)-functorial metaframe satisfying. Then

1. \((I5.1) \ & (I5.2)\) is equivalent to each of \((I5), \ (I5').\)

2. If \(F\) also satisfies \((I2.4)\), then each of these conditions implies \((I5.3)\) and thus

\[ (I5) \Leftrightarrow (I5') \Leftrightarrow (I5.1) \ & (I5.2) \ & (I5.3). \]

**Proof** Let us first show that \((I5)\) is equivalent to \((I5')\).

- \((I5)\) for \(n = 1 \Rightarrow (I5')\) for \(n = 1.\)

Assume \((I5)\) for \(n = 1:\)

\[
\forall a \in D^1 \ \forall d \in D^3 \ ((a, a) R^2 (d_2, d_3) \ & d_2 \neq d_3 \Rightarrow \exists e \in D^2 \ d \approx^3 e e_2). \quad (*)
\]

To check \((I5')\), suppose \((a, a) R^2 (b_1, b_2), b_1 \neq b_2.\) Let us find \(c \in D^1\) such that \((b_1, b_2) \approx^2 (c, c).\) Put \(d := b_1 b_2 b_1.\) By \((*)\) there exists \(e \in D^2\) such that \(d \approx^3 e e_2 = e_1 e_2 e_2.\) If \(\sigma \in \Upsilon_{23}\) is such that \(\sigma(1) = 2, \ \sigma(2) = 3,\) then \(b_1 b_2 b_2 = b_1 b_2, e_2 \cdot e_2 = e_2,\) so by \(s\)-functoriality it follows that \(b_1 b_2 \approx^2 e_2 e_2,\) and we can put \(c = e_2.\)

- Suppose \((I5), \ (I5')\) hold for \(n,\) and let us prove \((I5')\) for \(n + 1.\)

So we assume \(b \in D^{n+2}, \ a \in D^1, \ (a, a) R^2 (b_{n+1}, b_{n+2}), \ b_{n+1} \neq b_{n+2},\) and find \(c \in D^{n+1}\) such that \(b \approx^{n+2} c c_{n+1}.\) By \((I5')\) for \(n,\) for \(b_1 = b_2 \ldots b_{n+2}\) there exists \(c' \in D^{n+1}\) such that \(b_1 \approx^{n+1} c' c_n.\) Then we apply \((I5)\) to \(a := c', \ b\) and obtain \(c \in D^{n+1}\) such that \(b \approx^{n+2} c c_{n+1}.\)

- \((I5')\) for \(n + 1 \Rightarrow (I5)\) for \(n.\)

In fact, assume \((I5')\) for \(n + 1.\) Suppose

\[
a \in D^n, \ b \in D^{n+2}, \ (aa_n) R^{n+1} \ b_1, \ b_{n+1} \neq b_{n+2}.
\]

Then \((a_n, a_n) R^{n+1} (b_{n+1}, b_{n+2})\) by \(s\)-functoriality, hence by \((I5'),\) \(\exists c \in D^{n+1} \ b \approx^{n+2} c c_{n+1}.\)
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- (I5.1) & (I5.2) ⇒ (I5).
  
  In fact, assume (I5.1), (I5.2). Suppose
  \[ a \in D^n, \ b \in D^{n+2}, \ b_{n+1} \neq b_{n+2}, \]
  \[(aa_n)R^{n+1}b_1. \]
  Now we can apply (I5.2) to a and \( b_1 = b_2 \ldots b_{n+1}b_{n+2}b_1 \) (this is possible, since \( b_{n+1} \neq b_{n+2} \)). So there exist \( e \in D^n, \ d \in D(e) \) such that \((ee_n)dR^{n+2}(b_1b_1)\). Hence by s-functoriality \((de_n)bR^{n+2}b\), and thus by (I5.1) there exists \( c \in D^{n+1} \) such that \( b \approx^{n+2} cc_{n+1} \).

- (I5') ⇒ (I5.1).
  
  In fact, assume (I5'). Suppose \( a \in D^n, (aa_n)R^{n+1}b, \ b_n \neq b_{n+1} \). Then by s-functoriality, \((a_n,a_n)R^2(b_n,b_{n+1})\). Hence by (I5'), there exists \( c \in D^n \) such that \( b \approx^{n+1} cc_n \). Hence \((aa_n)R^{n+1}(cc_n)\) by transitivity, and eventually \( aR^n c \), by s-functoriality. Therefore (I5.1) holds.

- (I5') ⇒ (I5.2).
  
  In fact, assume (I5'). Suppose \( a \in D^n, (aa_n)R^{n+1}b, \ b_n \neq b_{n+1} \). Then by s-functoriality, \((a_n,a_n)R^2(b_n,b_{n+1})\). Hence by (I5'), there exists \( c \in D^n \) such that \( b \approx^{n+1} cc_n \). Hence \((aa_n)R^{n+1}(cc_n)\) by transitivity, and eventually \( aR^n c \), by s-functoriality. Therefore (I5.1) holds.

Finally let us show that together with (12.4), (I5') implies (I5.3).

Assume (I5'). Let \( a \in D^n, \ b \in D^{n+2} \), and \((\pi^+_n b)R^{n+1}(aa_n)\). Then by (12.4) there exists \( c \in D^{n+2} \) such that

\[ bR^{n+2}c \text{ and } aa_n \approx^{n+1} c_1 \ldots c_{n+1}. \]

Now \((a_n,a_n)R^2(c_n,c_{n+1})\) by s-functoriality, so by (I5') (applied to \( a_n, c_1 \ldots c_{n+1} \)) we obtain \( d \in D^{n+1} \), such that \( c_1 \ldots c_{n+1} \approx^{n+2} dd_{n+1} \). Thus

\[ bR^{n+2}c \approx^{n+2} d_2 \ldots d_{n+1}d_1. \]

Then by s-functoriality, \( c_1 \ldots c_{n+1} \approx^{n+2} d_2 \ldots d_{n+1}d_1 \) again by s-functoriality. By s-functoriality we also have

\[ d_2 \ldots d_{n+1} \approx^n c_1 \ldots c_n \approx^n a, \]

so \((d_2 \ldots d_{n+1})R^n a\), which proves the conclusion of (I5.3) (where \( d_2 \ldots d_{n+1} \) stands for \( c, d_1 \) stands for \( d \)).
Note that again (I5), (I5') become obvious if \( b_n \neq b_{n+1} \) is replaced with 
\( b_n = b_{n+1} \) — take \( c = b \).
Therefore we obtain the following equivalent form of Definitions 5.15.7, 
5.15.8, 5.15.9.

**Proposition 5.16.10** Let \( \mathcal{F} \) be an S4-metaframe. Then

(1) \( \mathcal{F} \) is a wi-metaframe iff \( \mathcal{F} \) is an s-functorial metaframe satisfying (I2.3) \& (I2.4) \& (I4.1) \& (I4.2);

(2) \( \mathcal{F} \) is an i-metaframe iff \( \mathcal{F} \) is a wi-metaframe satisfying (I5) (or equivalently, 
\( (I5') \));

(3) \( \mathcal{F} \) is a wi\(-\)-metaframe iff \( \mathcal{F} \) is an i\(-\)-metaframe iff \( \mathcal{F} \) is a wi-metaframe 
satisfying (mm2).

**Proof** (1) If a metaframe is s-functorial, then by 5.16.5, the quasi-lift property 
for injections is equivalent to (I2.3) \& (I2.4), and by 5.16.8, the 2-lift properties 
for injections are (together) equivalent to (I4.1) \& (I4.2).

(2) By 5.16.5, in s-functorial metaframes the quasi-lift property for all maps 
is equivalent to (I2.3) \& (I2.4) \& (I5.1). By 5.16.8, the 2-lift properties for all 
maps are equivalent to (I4.1) \& (I4.2) \& (I5.2) \& (I5.3).

So if \( \mathcal{F} \) is quasi-functorial, it is weakly functorial and satisfies (I5.1) \& (I5.2).
By 5.16.9, the latter implies (I5).

The other way round, if \( \mathcal{F} \) is weakly functorial and satisfies (I5), then by 
5.16.9, it satisfies (I5.1), (I5.2), (I5.3). Also by (1), it satisfies (I2.3) \& (I2.4) \& 
(I4.1) \& (I4.2), which yields the quasi-lift and the 2-lift properties.

(3) By 5.16.5, for monotonic\(-\) metaframes the quasi-lift property for 
all injections implies it for all maps. By 5.16.8, the same happens to the 2-lift 
properties.

**Proposition 5.16.11** Let \( \mathcal{F} \) be an S4-metaframe such that \( \mathcal{QH} \subseteq \mathcal{IL}^{(=)}(\mathcal{F}) \).
Then \( \mathcal{F} \) is an i\((-\)=)-metaframe.

**Proof** By Proposition 5.14.16, \( \mathcal{F} \) is monotonic\((-\)=). So by Proposition 5.16.10 
it is sufficient to check the properties (I2.3), (I2.4), (I4.1), (I4.2), (I5):

(I2.4) \( aR^n b \Rightarrow \forall d \in D(a) \exists c \in D^{n+1} \ ((ad)R^{n+1} c \& (\pi^c_+ c) \approx^n b) \);

(I2.3) \( uRv \Rightarrow \forall d \in D_u \exists t \approx^0 v \exists c \in D_t \ dR^1 c \).

The proofs in both cases are quite similar. Suppose \( aR^n b \), \( n > 0 \ d \in D(a) \) 
(or \( uRv \), \( d \in D_u \)). Consider the \( \mathcal{QH} \)-theorem 
\[ A_1 := \exists y p \supset p \]
for \( p \in PL^0 \) and its substitution instance.

\[ A := \exists y (P(x) \supset Q(x)) \supset \bullet P(x) \supset Q(x), \]
where \( |x| = n, \ y \notin x \). Let \( M := (\mathbb{F}, \xi) \) be an intuitionistic model such that
\[
\xi^+(P) = R^n(b), \ \xi^+(Q) = D^n - (R^n)^{-1}(b).
\]
Thus
\[
\xi^+(P) - \xi^+(Q) = \approx^n(b),
\]
in particular \( b \in \xi^+(P) - \xi^+(Q) \), hence \( M, a \not\models P(x) \supset Q(x)[x] \), so from the assumption \( M \models A \) we obtain
\[
M, a \not\models \exists y(P(x) \supset Q(x))[x],
\]
and thus
\[
M, ad \not\models (P(x) \supset Q(x))[xy].
\]
So there exists \( c \) such that
\[
(ad)R^{n+1}c \& c \models P(x)[xy] \& c \not\models Q(x)[xy].
\]
Since \( x = (xy) \cdot \sigma^n_+ \), by definition of forcing it follows that
\[
\pi^n_+c \in (\xi^+(P) - \xi^+(Q)) = \approx^n(b),
\]
which proves (I2.4). The changes for (I2.3) are now obvious: \( n = 0 \) and
\[
\xi^+(P) = R(v), \ \xi^+(Q) = W - R^{-1}(v).
\]
(I4.1) \( \pi_\varnothing(a)R^0\pi_\varnothing(c) \Rightarrow \exists b_1, b_2 \ (\pi_\varnothing(b_1) = \pi_\varnothing(b_2) \& aR^1b_1 \& b_2R^1c)\),
(I4.2) \( (\pi^n_+a)R^n(\pi^n_+c) \Rightarrow \exists b \in D^n \exists d, e \in D(b) (aR^{n+1}(bd) \& (be)R^{n+1}c)\).
For these properties the proofs are also similar, so we check only (I4.2). Suppose \( (\pi^n_+a)R^n(\pi^n_+c) \) and consider the intuitionistic model \( M := (\mathbb{F}, \xi) \) such that \( \xi^+(P) = D^{n+1} - (R^{n+1})^{-1}(c) \). By the assumption, the \( \text{QH} \)-theorem (the substitution instance of the same \( A_1 \))
\[
B := \exists y \forall z P(x, z) \supset \forall z P(x, z),
\]
where \( |x| = n, \ y, z \notin x, \ y \neq z \), is true in \( M \).
But \( M, c \not\models P(x, z)[xz] \), and so since \( (\pi^n_+a)R^n(\pi^n_+c) \), we have \( M, \pi^n_+a \not\models \forall z P(xz)[x] \). Now \( M \models B \) implies \( M, \pi^n_+a \not\models \exists y \forall z P(x, z)[x] \); hence \( M, a \not\models \forall z P(x, z)[xy] \). So there exists \( bde \in D^{n+2} \) such that \( aR^{n+1}(bd) \) and \( M, bde \not\models P(x, z)[xyz] \). The latter is equivalent to \( be \notin \xi^+(P) \), i.e. to \( (be)R^{n+1}c \), by the definition of \( \xi \). Thus (I4.2) holds.

(I5) \( (aa_n)R^{n+1}c_1 \& c_n \neq c_{n+1} \Rightarrow \exists b \in D^{n+1} c \approx^{n+2} bb_{n+1} \).
Recall that by Proposition 5.16.10, this property is essential only for the case without equality.
Suppose \((aa_n) R^{n+1}_1\). Take a distinct list \(x\) of length \(n\), \(y, z \not\in x\), \(y \neq z\) and consider the following formulas:

\[
B_1 := \forall y (P_1(y, x, z) \supset P_2(y, x, z)), \\
B_2 := Q_1(x, z) \supset Q_2(x), \\
B := B_1 \land B_2, \; A := [x_n/z]B \supset \exists zB.
\]

Obviously, \(A\) is a \(QH\)-theorem (a substitution instance of the axiom \(A_2 := P(x_n) \supset \exists z P(z)\)). Consider the intuitionistic model \(M := (F, \xi)\) such that

\[
\xi^+(P_1) = R_{n+2}(c), \\
\xi^+(P_2) = D_{n+2} - (R_{n+2})^{-1}(c), \\
\xi^+(Q_1) = D_{n+1} - (R_{n+1})^{-1}(aa_n), \\
\xi^+(Q_2) = D^n - (R^n)^{-1}(a).
\]

Then obviously

\[
(0) \; \xi^+(P_1) - \xi^+(P_2) = \approx^{n+2}(c).
\]

Now we claim that

\[(0.1) \; M, a \not \models \exists zB[x].\]

In fact, suppose the contrary. Then there exists \(d \in D^{n+1}\) such that

\[(1) \; d \supset R\langle a \rangle \text{ and } d \models (Q_1(x, z) \supset Q_2(x))[xz].\]

Hence

\[(2) \; d \models (Q_1(x, z) \supset Q_2(x))[xz].\]

By the choice of \(M\), \(d \supset R\langle a \rangle\) implies \(d \not \models \xi^+(Q_2)\) and thus

\[(3) \; d \not \models Q_2(x)[xz].\]

Now from (2), (3) we obtain

\[(4) \; d \not \models Q_1(x, z)[xz],\]

which according to the choice of \(M\), is equivalent to

\[(5) \; (de) R^{n+1}(aa_n).\]

By (1) we also have

\[(6) \; (de) \models \forall y (P_1(y, x, z) \supset P_2(y, x, z))[xz].\]

By our initial assumption, \((aa_n) R^{n+1}_1\), hence \((de) R^{n+1}_1\), by (5). Now by (6) we obtain

\[(7) \; \bar{c}_1 c_1 \not \models (P_1(y, x, z) \supset P_2(y, x, z))[xy],\]
On the other hand, by (0)
\[ c = c_1 \tilde{c}_1 \in \xi^+(P_1) - \xi^+(P_2), \]
so
\[ \tilde{c}_1 c_1 \models P_1(y, x, z) \ [xzy] \]
and
\[ \tilde{c}_1 c_1 \not\models P_2(y, x, z) \ [xzy]. \]
This contradicts (7) and the reflexivity of \( R^{n+2} \).
So we have proved that \( a \not\models \exists z B \ [x] \). But \( M \models A \), therefore
\[ (8) \ a \not\models [x_n/z]B \ [x]. \]
However note that
\[ (9) \ M \models [x_n/z]B_2 \ [x]. \]
In fact, if (in \( M \)) \( d \not\models Q_2(x) \ [x] \), then by definition, \( dR^n a \), which implies \( (dd_n)R^n(aa_n) \) by \( s \)-functoriality. But the latter means \( (dd_n) \not\in \xi^+(Q_1) \), i.e. \( d \not\models Q_1(x, x_n) \ [x] \). Eventually from (8) and (9) it follows that
\[ a \not\models [x_n/z]B_1 \ [x]. \]
So there exists \( e \in D^{n+1} \) such that
\[ e \models P_1(y, x, x_n) \ [xy] \] and \( e \not\models P_2(y, x, x_n) \ [xy], \]
which is equivalent to
\[ (e_{n+1} ee_n) \in (\xi^+(P_1) - \xi^+(P_2)) = \approx^{n+2}(c) \]
by (0). Then \( b := e_{n+1}e \) fits for the conclusion of (15).

**Remark 5.16.12** So we see that \( F \) is an \( i(=) \)-metaframe if \( IL^{(=)}(F) \) contains \( A_0 := (p \supset \top \supset p) \) (cf. the proof of 5.14.12) and \( A_1, A_2 \) from the proof above.

**Theorem 5.16.13 (Maximality theorem)** Let \( F \) be an \( S4 \)-metaframe. Then the following conditions are equivalent:

1. \( IL^{(=)}(F) \) is an s.p.l (=);
2. \( QH \subseteq IL^{(=)}(F); \)
3. \( F \) is an \( i(=) \)-metaframe.

**Proof** In fact, (2) \( \Rightarrow \) (3) by Proposition 5.16.11; (3) \( \Rightarrow \) (1) by Theorem 5.15.26. (1) \( \Rightarrow \) (2) is trivial. \( \blacksquare \)
Therefore we can introduce the intuitionistic semantics of metaframes \(M\mathcal{F}^{(=)}_{\text{int}}\) generated by \(i^{(=)}\)-metaframes.

Theorem 5.16.13 shows that (using the terminology from Section 2.12) this is the greatest sound semantics of \(S4\)-metaframes for superintuitionistic logics, because it is generated by all \(S4\)-metaframes strongly validating \(QH^{(=)}\)-theorems. Therefore the list of properties of \(i^{(=)}\)-metaframes, yields a precise criterion of intuitionistic soundness for metaframes. In the next section we will apply this criterion to describe intuitionistic soundness in Kripke-quasi-bundles.

Note that the classes of weak \(i\)-metaframes, \(i\)-metaframes and \(i^{-}\)-metaframes are actually different. Let us consider two simple examples.

**Example 5.16.14** Let \(\mathbb{F}_1 = ((F_n)_{n \in \omega}, D)\), be an \(S4\)-metaframe such that

- \(D^0 = \{u_0, u_1\}, (D^0, R^0)\) is a two-element chain;
- \(D_{u_0} = \{d_0\}, D_{u_1} = \{d_1, d_2\};\)
- \((D^n, R^n)\) is a tree of height 2 with the root \(d^n_0\), i.e. the tuple \(d^n_0\) sees all \(n\)-tuples from \(D_{u_1}\) (which are incomparable).

On one hand, by Proposition 5.16.10, \(\mathbb{F}_1\) is a \(wi\)-metaframe: the \(s\)-functoriality, \((I2.3), (I2.4), (I6.1), (I6.2)\) hold obviously. For example, for \((I6.2)\): if \(n > 0, (a_1, \ldots, a_n, a_{n+1}) \in (D_u)^{n+1}, (c_1, \ldots, c_n, c_{n+1}) \in (D_v)^{n+1}, (a_1, \ldots, a_n) R_n(c_1, \ldots, c_n)\), then we can put \((b_1, \ldots, b_n, d, e) := (d_0, \ldots, d_0, d_0, c_{n+1})\) if \(u = u_0\) and \((b_1, \ldots, b_n, d, e) := (a_1, \ldots, a_n, a_{n+1}, c_{n+1})\) if \((a_1, \ldots, a_n) = (c_1, \ldots, c_n)\).

On the other hand, \(\mathbb{F}_1\) is not an \(i\)-metaframe, since \((I5)\) fails for \(n = 1\). In fact, \((d_0, d_0) R^2(d_1, d_2)\), but there does not exist \(b' = (b_0, b_1, b_1) \approx^3 (d_1, d_1, d_2)\).

**Example 5.16.15** Let \(\mathbb{F}_2\) be the metaframe with the same \(D\) as \(\mathbb{F}_1\) and with the universal \(R^0, R^n = R^n \cup (R^n)^{-1}\), see the following figure:

Again, \(\mathbb{F}_2\) is a \(wi\)-metaframe, and even an \(i\)-metaframe: \((I5)\) holds since \((d_0, d_0, d_0) \approx^3 (d_1, d_2, d_2)\), etc. On the other hand, \(\mathbb{F}_2\) is not an \(i^{-}\)-metaframe — \(mm_2)\) fails, since \((d_0, d_0) R^2(d_1, d_2)\), but \(d_1 \neq d_2\).
Therefore, the class of $i$-metaframes (which are sound for the intuitionistic logic without equality) is larger than the class of $i^=\text{-}metaframes$. Recall that in the modal case there is no difference between logics with or without equality in this respect; sound metaframes are same (see Theorem 5.12.13). Recall also that $\text{IL}(\mathcal{F}) = \text{IL}^=\text{(F)} \cap IF$ is the fragment without equality for any $i^=\text{-}metaframe$ $\mathcal{F}$ (by Proposition 5.15.22).

Let us also show that the class of $i^=\text{-}metaframes$ is larger than the class of $\text{QS4}\text{-}metaframes$ (which are sound for the modal case).

**Example 5.16.16** Consider an $\text{S4}\text{-}metaframe} \mathcal{F}$ such that

$$W = \{u_0, u_1\}, \; u_0 \approx u_1, \; D_{u_0} = \{d_0\}, \; D_{u_1} = \{d_1, d_2\},$$

$$aR^n b \text{ if and only if } a = (b_1, \ldots, b_n) \land \exists i \leq 1(a = (d_i, \ldots, d_i) \land b = (d_{1-i}, \ldots, d_{1-i})).$$

Obviously, $R^n = \approx^n$. On the one hand, $\mathcal{F}$ is not a modal metaframe, because (I2.3) fails; in fact, $(a_1, a_2) \in D^n_{u_1}, a_1 R^1 a_0$, and $\exists b \in D_{u_0} (a_1 a_2) R^2 (a_0, b)$.

On the other hand, $\mathcal{F}$ is an $i^=\text{-}metaframe$. E.g. the condition (I2.3) holds: one can take $(b_1', \ldots, b_n', b_0') = (a_1, \ldots, a_n)$ since $R^n = \approx^n$. And (I6) also holds, because the $(n + 1)$-tuple $(a_0, \ldots, a_0, a_0) \in D^n_{u_0}$ has an $\approx^{n+1}$-copy $(a_1, \ldots, a_1, a_1) \in D^{n+1}_{u_1}$, and we can stick together $(n + 1)$-tuples in $D_{u_1}$.

Note that we can consider $\mathcal{M}_m$ as a semantics for superintuitionistic logics (with or without equality) and $\mathcal{M}_m^=\text{int}$ also as a semantics for superintuitionistic logics without equality. We still do not know if all these semantics are equal to $\mathcal{M}_m^=\text{int}$.

### 5.17 Kripke quasi-bundles

Recall that according to Definition 5.5.19, a Kripke quasi-bundle is a quasi-p-morphism between $\text{S4}\text{-}frames$, i.e. a monotonic surjective map with the quasi-lift property.

Every Kripke quasi-bundle $\pi : F_1 \longrightarrow F_0$, where $F_i = (W_i, R_i)$, is associated with a system of domains $D_u := \{\pi^{-1}(u) \mid u \in F_0\}$ and a family of inheritance relations $\rho_{uv} := R_1 \cap (D_u \times D_v)$. It also corresponds to a metaframe constructed
as in the case of Kripke bundles, cf. Definition 5.3.2; the n-levels are \( F_n = (D^n, R^n) \), with

\[
a R^n b \text{ iff } \forall j \ a_j R_1 b_j \text{ & } a \text{ sub } b.
\]

So \( R_i = R^i \) for \( i = 0, 1 \).

**Proposition 5.17.1** Let \( F = (F, (D_n)) \) be a metaframe corresponding to an intuitionistic Kripke quasi-bundle. Then \( F \) is an \( i \)-metaframe iff \( F \) is an \( i^\sim \)-metaframe iff the following conditions hold:

1. \( a R^1 b \Rightarrow \exists v \exists c, d \in D_n \ (a R^1 c \& d R^1 b) \);
2. if \( n > 0 \), and \( (ad) \in D^{n+1} \), all \( a_i \) and \( d \) are distinct; \( b \in D^n \ (b_i \text{ are not necessarily distinct}), \) and \( \forall s a_s R^1 b_s \), then there exists \( c \in D^{n+1} \) such that \( d R^1 c_{n+1} \& \forall s \leq n \ (b_s \approx c_s) \& \forall s, t \leq n \ (b_s = b_t \Leftrightarrow c_s = c_t) \);
3. if \( n > 0 \), \( a, c \in D^{n+1} \), all \( a_i \) are distinct, all \( c_i \) are distinct and \( \forall s a_s R^1 c_s \), then there exists \( (bde) \in D^{n+2} \) such that \( d, b_1, \ldots, b_n \) are distinct (but perhaps \( e = d \) or \( e = b_i \)) such that

\[
\forall s (a_s R^1 b_s \& b_s R^1 c_s \& a_{n+1} R^1 d \& e R^1 c_{n+1}).
\]

These conditions allow us to 'move' individuals from one world to another.

**Proof** First, \( F \) is an S4-metaframe: all \( R^n \) are reflexive and transitive, since the relations \( R^1, R^0, \text{sub} \) are reflexive and transitive.

Now let us check the properties of \( i^\sim \)-metaframes stated in Proposition 5.16.10.

The monotonicity is almost obvious. In fact, \( \forall i a_i R^1 b_i \) implies \( \forall i a_{\sigma(i)} R^1 b_{\sigma(i)} \); and \( a \text{ sub } b \) implies \((\sigma a) \text{ sub } \sigma b\).

(12.4) is the quasi-lift property, which follows from the definition of a Kripke quasi-bundle.

(16.1) is the same as (1).

(12.3) follows from (2). In fact, \( b = (b_1, \ldots, b_n) \approx^n c = (c_1, \ldots, c_n) \) iff \( \forall s \leq n \ (b_s \approx c_s) \& b \text{ sub } c \& c \text{ sub } b \); we also have \((ad)R^{n+1}c\), since \((a_1, \ldots, a_n)R^n(c_1, \ldots, c_n) \) and \( d R^1 c_{n+1} \). The requirement in (2) that \( d, a_1, \ldots, a_n \) are distinct, is overcome due to the 'local functionality'.

It remains to check (16.2) or its equivalent version 5.16.8 (16.2~). So let \( a, c \in D^{n+1}, \pi^+_a \approx^n \pi^+_c \). Then \( a \text{ sub } c \) and \( c \text{ sub } a \). The cases when \( a_{n+1} \in \{a_1, \ldots, a_n\} \), \( c_{n+1} \in \{c_1, \ldots, c_n\} \) are obvious. Thus we may assume that all \( a_j \) and all \( b_j \) are distinct. Now (3) yields us a tuple \((bde) \in D^{n+2} \) such that \( a R^{n+1} (bd), \ (be) R^{n+1} c \).

Also note that the conditions (1), (2), (3) are necessary: they definitely hold if \( F \) is an i-metaframe. In fact, they respectively follow from (16.1), (12.3), (16.2).

Therefore, the logic \( \mathbf{II}^{(=)}(F) \) of a Kripke quasi-bundle \( F \) is superintuitionistic iff the conditions (1), (2), (3) hold. We call such quasi-bundles intuitionistic (or,
briefly, i-quasi-bundles); they generate a sound semantics for superintuitionistic logics. Every intuitionistic Kripke bundle is clearly an i-quasi-bundle.

Note that a metaframe associated with a Kripke quasi-bundle is not necessarily modal. For instance, the i'-metaframe \( F \) from Example 5.16.16, which is not modal, corresponds to the Kripke quasi-bundle \( (W, D, \bar{D}) \), \( W = \{ u_0, u_1 \}, u_0 \approx u_1, D_{u_0} = \{ d_0 \}, D_{u_1} = \{ d_1, d_2 \} \rho_{u_1, u_1} = \{ id_{d_0}, (i = 0, 1), \rho_{d_1, u_1} = \{ (d_1, d_1) \} \). This explains why the construction of the \( C \)-set corresponding to a Kripke bundle described in the proof of Proposition 7.8.11, fails for Kripke quasi-bundles.

### 5.18 Some constructions on metaframes

**Definition 5.18.1** Let \( \mathbb{F} = (F, (D_n)) \) be an \( N \)-metaframe based on \( F = (W, R_1, \ldots, R_N) \) and let \( V \subseteq W \). Then the submetaframe \( \mathbb{F}|V \) is the restriction of \( \mathbb{F} \) to \( V \), i.e. it has a system of domains \( D|V := (D_u \mid u \in V) \) with relations \( R^n_i|V := R^n_i \cap (D|V)^n \); in particular, \( R^0|V = R^0 \mid V \).

A submetaframe \( \mathbb{F}|V \) is called generated if \( V \subseteq W \) is \( R_i \)-stable for every \( i = 1, \ldots, N \).

**Lemma 5.18.2**

1. If \( \mathbb{F}|V \) is a generated submetaframe of \( \mathbb{F} \) and all the maps \( \pi_{\sigma_n} \) in \( \mathbb{F} \) are monotonic (in particular, if \( \mathbb{F} \) is a modal or a \( \omega \)-metaframe), then each of its \( n \)-level \( (\mathbb{F}|V)_n \) is a generated subframe of \( F_n \).

2. Every submetaframe of an \( N \)-modal metaframe is an \( N \)-modal metaframe.

3. Every generated submetaframe of an \( i \)- (resp., \( i^- \), \( \omega \)-) metaframe is also an \( i \)- (resp., \( i^- \), \( \omega \)-) metaframe.

**Definition 5.18.3** A cone \( \mathbb{F}^u \) of a metaframe \( \mathbb{F} \) (for \( u \in F^0 \)) is its restriction to the subset (cone) \( R^*(u) \) in the base \( F^0 \).

Recall that \( R^* \) is the reflexive transitive closure of \( \bigcup_{i=1}^{N} R_i \).

**Definition 5.18.4** A metaframe model \( M = (\mathbb{F}, \xi) \) over \( \mathbb{F} \) (\( N \)-modal or intuitionistic) gives rise to the model \( M|V := (\mathbb{F}|V, \xi|V) \) (the restriction of \( M \) to \( V \)) such that \( (\xi|V)^+(P^n_j) = \xi^+(P^n_j) \cap (D|V)^n \).

**Lemma 5.18.5** Let \( \mathbb{F} \) be an \( N \)-modal metaframe satisfying (12.1), or a \( \omega \)-metaframe, let \( M = (\mathbb{F}, \xi) \) be a model (\( N \)-modal or intuitionistic, respectively), \( \mathbb{F}|V \) a generated submetaframe, and let \( M|V \) be the corresponding submodel of \( M \). Then

1. \( M, a \models (\exists x)A[x] \) iff \( M|V, a \models (\exists x)A[x] \)

   for any appropriate assignment \( (a, x) \) (modal or intuitionistic, respectively).
(2) \( M \models (\forall)A \Rightarrow M[V] \models (\forall)A. \)

**Proof** By induction on the length of \( A \). Note that in the intuitionistic case it is essential that in a wi-metaframe an inductive clause for \( \exists \) can be rewritten with \( \approx \) instead of \( R \), see Section 5.11. Thus we cannot state (1) for an arbitrary metaframe.

Note that every valuation (N-modal or intuitionistic) in \( \mathbb{F}[V] \) is also a valuation in \( \mathbb{F} \) and it coincides with its restriction to \( \mathbb{F}[V] \). Hence we obtain

**Proposition 5.18.6**

1. \( \mathbf{ML}^{(=)}(\mathbb{F}) \subseteq \mathbf{ML}^{(=)}(\mathbb{F}[V]) \) and \( \mathbf{ML}^{(=)}(\mathbb{F}) \subseteq \mathbf{ML}^{(=)}(\mathbb{F}[V]) \) for a generated submetaframe of an N-modal metaframe \( \mathbb{F} \).
2. \( \mathbf{IL}^{(=)}(\mathbb{F}) \subseteq \mathbf{IL}^{(=)}(\mathbb{F}[V]) \) and \( \mathbf{IL}^{(=)}(\mathbb{F}) \subseteq \mathbf{IL}^{(=)}(\mathbb{F}[V]) \) for a generated sub-metaframe of an i- (wi-, i\textsuperscript{-}) metaframe \( \mathbb{F} \).

**Proposition 5.18.7**

1. \( \mathbf{ML}^{(=)}(\mathbb{F}) = \bigcap_{u \in \mathbb{W}} \mathbf{ML}^{(=)}(\mathbb{F}[u]) \) and \( \mathbf{ML}^{(=)}(\mathbb{F}) = \bigcap_{u \in \mathbb{W}} \mathbf{ML}^{(=)}(\mathbb{F}[u]) \) for an N-modal metaframe \( \mathbb{F} \).
2. \( \mathbf{IL}^{(=)}(\mathbb{F}) = \bigcap_{u \in \mathbb{W}} \mathbf{IL}^{(=)}(\mathbb{F}[u]) \) and \( \mathbf{IL}^{(=)}(\mathbb{F}) = \bigcap_{u \in \mathbb{W}} \mathbf{IL}^{(=)}(\mathbb{F}[u]) \) for an i- (wi-, i\textsuperscript{-}) metaframe \( \mathbb{F} \).

**Proof** If \( M = (\mathbb{F}, \xi) \not\models A \) then \( M, a \not\models A[x] \) for some \( (a, x) \), and thus \( M[u, a \not\models A[x], \text{for } u = \pi_\emptyset(a) \).

**Definition 5.18.8** Let \((\mathbb{F}_j)_{j \in J}\) be a family of metaframes of the same type (N-modal or intuitionistic), \( \mathbb{F}_j = ((\mathbb{F}_j^n), D_j) \), \( \mathbb{F}_n = (D^n, R^n_1, \ldots, R^n_N) \). Their disjoint sum (union) \( \bigsqcup_{j \in J} \mathbb{F}_j \) is defined as the metaframe \( \mathbb{F} = ((\mathbb{F}_n), D) \) such that

- \( F_0 = \bigsqcup_{j \in J} F_{j0} \);
- \( D_{(u, j)} = (D_j)_u \times \{j\} \);
- \( F_n = (D^n, R^n_1, \ldots, R^n_N) \), where \( D^n = \bigcup_{w \in F_0} D^n_w \), and for \( a, b \in D^n_j \)

\[(a_1, j), \ldots, (a_n, j), (b_1, j), \ldots, (b_n, j) \text{ iff } a R^n_{jk} b.\]

**Lemma 5.18.9** Let \( \mathbb{F} = \bigsqcup_{j \in J} \mathbb{F}_j \). Then

1. \( F_n \cong \bigsqcup_{j \in J} F_{jn} \);
2. \( \mathbb{F} \) is an N-modal metaframe, or respectively an i- (wi-, i\textsuperscript{-}) metaframe iff all \( \mathbb{F}_j \) are.
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Proof

(1) The map \( \bigcup_{j \in J} F_j \longrightarrow F_n \) sending \((a, j)\) (for \(a \in D_j^n\)) to \(((a_1, j), \ldots, (a_n, j))\), is an isomorphism.

(2) An exercise.

Lemma 5.18.10

(1) \( \text{ML}_-(\bigcup_j F_j) = \bigcap_j \text{ML}_-(F_j) \) for \(N\)-modal metaframes.

(2) \( \text{IL}_-(\bigcup_j F_j) = \bigcap_j \text{IL}_-(F_j) \) for \(i\)-\((wi, i^w)\)-metaframes.

(3) Similarly for \(\text{IL}, \text{ML}\).

Proof On one hand, each \(F_j\) is isomorphic to a generated submetaframe of \(\bigcup_j F_j\). On the other hand, every cone in \(\bigcup_j F_j\) is isomorphic to a cone in the corresponding \(F_j\).

Therefore the semantics \(\mathcal{M}F_m, \mathcal{M}F_{int}\) satisfy the collection property \((\text{CP})\) (see Section 2.16).

5.19 On semantics of intuitionistic sound metaframes

Note that all definitions related to \(i^=\)-soundness can be readily extended to arbitrary (not necessarily \(S4\)) \(1\)-metaframes. Namely, a valuation \(\xi\) in a \(1\)-metaframe is called intuitionistic if \(R^n(\xi^+(P^n_j)) \subseteq \xi^+(P^n)\) for any predicate letter \(P^n, n \geq 0\), cf. 5.14.1. Now the definitions of forcing, validity, and strong validity are rewritten in a straightforward way (cf. Definitions 5.14.2, 5.14.3).

The notations \(\text{IL}_-, \text{IL}^=\) are also used in this case. Finally a metaframe \(F\) is called \(i^=\)-sound if \(\text{IL}^=(F)\) is an s.p.l., cf. Definition 5.14.4.

To conclude our description of intuitionistic sound metaframes, we now prove the following statement.

Proposition 5.19.1 For every \(i^=\)-sound \(1\)-metaframe \(F\) there exists an \(S4\)-metaframe \(F'\) such that

\[ \text{IL}^=(F') = \text{IL}^=(F). \]

Hence \(F'\) is also \(i^=\)-sound, so it is an \(i^=\)-metaframe, by 5.16.13.

Therefore we can conclude that the semantics \(\mathcal{M}F_{int}^=\) of arbitrary \(i^=\)-metaframes actually equals the semantics of all \(i^=\)-sound metaframes.

To prove Proposition 5.19.1, we need two auxiliary notions.
Definition 5.19.2 A propositional frame \( F = (W, R) \) is called weakly reflexive (or \( w \)-reflexive, for short) if the transitive closure of \( R \) is reflexive (cf. [Došen, 1993]). A 1-metaframe \( \mathcal{F} \) is called \( w \)-reflexive if all its levels \( F_n \) are \( w \)-reflexive.

Definition 5.19.3 A propositional frame \( F = (W, R) \) is called co-serial if \( \forall u \exists v \, v R u \). A 1-metaframe \( \mathcal{F} \) is co-serial if all \( F_n \) are co-serial.

Obviously, every reflexive propositional frame (and 1-metaframe) is \( w \)-reflexive and co-serial.

Lemma 5.19.4 Let \( \mathcal{F} \) be a 1-metaframe. Consider the formulas \( B_1 := \exists z \, T \) and \( B_2 := (\top \supset p) \supset p \).

(1) If \( B_1 \in \text{IL}(\mathcal{F}) \), then \( \mathcal{F} \) is co-serial.

(2) If \( \mathcal{F} \) is co-serial and \( B_2 \in \text{IL}(\mathcal{F}) \), then \( \mathcal{F} \) is \( w \)-reflexive.

Proof

(1) Suppose \( B_1 \in \text{IL}(\mathcal{F}) \), \( a \in D^n \). Let \( x = (x_1, \ldots, x_n) \), \( z \notin x \). Consider an arbitrary intuitionistic model \( M = (\mathcal{F}, \xi) \). Then \( M, a \vdash B_1[x] \), so there exists \( b \in (R^n)^{-1}(a) \) satisfying the condition: \( \exists c \in D(b) \, M, bc \vdash \top \supset [xz] \). This implies co-seriality.

(2) Suppose \( \mathcal{F} \) is co-serial, \( B_2 \in \text{IL}(\mathcal{F}) \). Given \( a \in D^n \), choose \( b \in (R^n)^{-1}(a) \), by co-seriality. Consider the substitution instance \( B'_2 := (\top \supset P^n(x)) \supset P^n(x) \)

of \( B_2 \), and an intuitionistic model \( M = (\mathcal{F}, \xi) \) such that \( \xi^+(P^n) = ((R^n)^* \circ R^n)(a) \). Then

\[
M, a \vdash (\top \supset P^n(x))[x].
\]

In fact, if \( c \in R^n(a) \) and \( M, c \vdash \top \), then \( c \in \xi^+(P^n) \), and so \( M, c \vdash P^n(x) \) \([x]\). Thus \( M, a \vdash P^n(x) \), since \( M, b \vdash B'_2[x] \) and \( b R^n a \). Thus \( a \in \xi(P^n) \) \([x]\), so \( a((R^n)^* \circ R^n)a \). Hence the \( w \)-reflexivity of \( \mathcal{F} \) follows.

\[\blacksquare\]

Definition 5.19.5 A 1-metaframe \( \mathcal{F} \) is monotonic(\( = \)) if it satisfies

\[(im) \; M, a \vdash A[x] \land a R^n b \Rightarrow M, b \vdash A[x]\]

for all for \( A \in \text{IF}(=) \), \( x \supseteq FV(A) \) and \( a, b \in D^n \), cf. 5.14.14.

Proposition 5.19.6 Let \( \mathcal{F} \) be a co-serial-metaframe, \( B := p \supset (\top \supset p) \). If \( B \in \text{IL}(=)(\mathcal{F}) \), then \( \mathcal{F} \) is monotonic(\( = \)).
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Proof 30 Let us check (im) for a formula \( A \in IF(=) \). Consider a substitution instance \( B' := A \supset (\top \supset A) \) of \( B \).

Assume \( aR^n b \) and \( M, a \vDash A [x] \). Due to co-seriality, there exists \( cR^n a \). Since \( M, c \vDash B' [x] \), it follows that \( M, a \vDash (\top \supset A) [x] \). Now \( M, b \vDash T [x] \) implies \( M, b \vDash A [x] \).

Corollary 5.19.7 If for a 1-metaframe \( \mathbb{F} \), \( QH \subseteq IL^{(=)}(\mathbb{F}) \) and even, if \( \{ p \equiv (\top \supset p), \exists z T \} \subseteq IL^{(=)}(\mathbb{F}) \), then \( \mathbb{F} \) is w-reflexive, co-serial, and monotonic(=).

Proof Note that \( (p \equiv (\top \supset p)) \in IL^{(=)}(\mathbb{F}) \) iff \( p \supset (\top \supset p), (\top \supset p) \supset p \in IL^{(=)}(\mathbb{F}) \). Then apply 5.19.4, 5.19.6.

Definition 5.19.8 For a w-reflexive metaframe \( \mathbb{F} = ((D^n, R^n)_{n \in \omega}, D) \) we define its transitive closure, the S4-metaframe \( \mathbb{F}^* = ((D^n, R^n^*)_{n \in \omega}, D) \), where \( R^n^* = R^n^+ \) is the transitive closure of \( R^n \).

Lemma 5.19.9 Let \( \mathbb{F} \) be a w-reflexive monotonic(=)-metaframe. Then

\[ IL^{(=)}(\mathbb{F}) = IL^{(=)}(\mathbb{F}^*). \]

Proof It is clear that \( \xi \) is an intuitionistic valuation in \( \mathbb{F} \) iff \( \xi \) is an intuitionistic valuation in \( \mathbb{F}^* \), since \( \xi^+(P^n) \) is \( R^n \)-stable iff it is \( R^n^* \)-stable.

So it is sufficient to show that for every model \( M = (\mathbb{F}, \xi) \) and the corresponding model \( M^* = (\mathbb{F}^*, \xi) \), for every ordered assignment \((x, a)\) such that \( x \supset FV(A) \), the following holds:

\[ M, a \vDash A [x] \Leftrightarrow M^*, a \vDash A [x]. \]

We proceed by induction and consider only three non-trivial cases.

1. \( A = B \supset C \). If \( M, a \not\vDash A [x] \), then there exists \( b \in R^n(a) \) such that \( M, b \vDash B [x] \) and \( M, b \not\vDash C [x] \). Then \( M^*, b \vDash B [x] \) and \( M^*, b \not\vDash C [x] \) by the induction hypothesis, and obviously \( b \in R^n^*(a) \). So \( M^*, a \not\vDash A [x] \).

The other way round, let \( M^*, a \not\vDash A [x] \). Then there exists \( b \in R^n^*(a) \) such that \( M^*, b \vDash B [x] \) and \( M^*, b \not\vDash C [x] \). So \( M, b \vDash B [x] \) and \( M, b \not\vDash C [x] \). Take \( d \in D^n \) such that \( aR^*dR^n b \). Then \( M, d \not\vDash A [x] \) by monotonicity \( M, a \not\vDash A [x] \) as well.

2. \( A = \exists y B [x] \). If \( M, a \vDash A [x] \), then there exists \( b \in (R^n)^{-1}(a) \) and \( c \in D(b) \) such that \( M, \pi_{x\|y}(bc) \vDash B [x\|y] \). So \( M^*, \pi_{x\|y}(bc) \vDash B [x\|y] \) and \( M^*, a \vDash A [x] \), since \( aR^*b \).

The other way round, let \( M^*, a \vDash A [x] \). Then there exist \( b, d \in D^n \) and \( c \in D(b) \) such that \( bR^*dR^n a \) and \( M^*, \pi_{x\|y}(bc) \vDash B [x\|y] \). Hence \( M, \pi_{x\|y}(bc) \vDash B [x\|y] \), so \( M, d \vDash A [x] \), and thus \( M, a \vDash A [x] \) by monotonicity.

31Note that \( R^* \circ R = R \circ R^* \) is the transitive closure of \( R \).
(3) The case \( A = \forall y B [x] \) is similar.

Lemma 5.19.9 and Corollary 5.19.7 obviously imply Proposition 5.19.1; namely we put \( F' = F^* \).

The previous consideration in this section allows us to extend Lemma 5.14.11(3) to arbitrary 1-metaframes:

**Proposition 5.19.10** Let \( F \) be a 1-metaframe. Then the following conditions are equivalent:

(1) \( F \) is \( i^{(=)} \)-sound;

(2) \( \text{QH} \subseteq \text{IL}^{(=)}(F) \);

(3) \( F \) is \( w \)-reflexive and monotonic\(^{(=)} \), and \( F^* \) is an \( i^{(=)} \)-metaframe.

Moreover, these conditions are equivalent to

(4) the following formulas are in \( \text{IL}^{(=)}(F) \):

\[
p \equiv (\top \supset p), \exists z \top, A_1 := \exists y p \supset p, A_2 := P(x) \supset \exists z P(z).
\]

Here \( A_1 \) and \( A_2 \) are the formulas used in the proof of Proposition 5.16.11. Note that all formulas in (4) are \( \text{QH} \)-theorems, so (2) readily implies (4).

**Proof** (3)\( \Rightarrow \)(1). By Lemma 5.19.9 and Theorem 5.15.26 (soundness).

(2)\( \Rightarrow \)(3). By Corollary 5.19.7, Lemma 5.19.9, and 5.16.11.

(4)\( \Rightarrow \)(3). Assume (4). Then \( F \) is \( w \)-reflexive and monotonic\(^{(=)} \) by 5.19.7, so by 5.19.9 \( \text{IL}^{(=)}(F) = \text{IL}^{(=)}(F^*) \). So formulas from (4) are in \( \text{IL}^{(=)}(F^*) \), and therefore \( F^* \) is an \( i^{(=)} \)-metaframe, cf. the proof of Proposition 5.16.11 and Lemma 5.14.15, or cf. Remark 5.16.12.

By the way, basing on Proposition 5.19.10 (1) \( \Leftrightarrow \) (3), we can give an explicit description of the class of all \( i^{(=)} \)-sound metaframes. Viz., to check that \( F^* \) is an \( i^{(=)} \)-metaframe, we can rewrite all conditions from the definition of an \( i^{(=)} \)-metaframe (cf. Section 5.15 and 5.16) with \( R^{n*} \) replacing \( R^n \). In particular, the condition \((0\sigma)\) becomes

\[
aR^{n*}b \Rightarrow (\pi_\sigma a)R^{m*}(\pi_\sigma b),
\]

or equivalently

\[
aR^n b \Rightarrow (\pi_\sigma a)R^{m*}(\pi_\sigma b).
\]

Here is a description of monotonicity\(^{(=)} \) for arbitrary metaframes, which we state without a proof.

**Lemma 5.19.11**
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(1) A 1-metaframe \( \mathbb{F} \) is monotonic iff it satisfies

(i) \( aR^m n b \Rightarrow (\pi_\sigma a)R^{m*}(\pi_\sigma b) \) for any \( \sigma \in \Sigma_{mn} \);
(ii) \( aR^m n bR^m c \Rightarrow \exists c'(aR^m c' & cR^{m*}c & c'R^{m*}c) \);
(iii) \( aR^m n bR^m c \Rightarrow \forall e \in D(c) \exists d \exists g \in D(c) (aR^m d & (dg)R^{(n+1)}(ce)) \);
(iv) \( cR^m n bR^m a \Rightarrow \forall e \in D(c) \exists d \exists g \in D(d) (dR^m a & (ce)R^{(n+1)}(dg)) \).

(2) \( \mathbb{F} \) is monotonic \( = \) iff it is monotonic and satisfies \( (mm_2) \).

Here the condition (i) corresponds to \( (0\sigma) \) in \( \mathbb{F}^* \) and expresses the monotonicity for atomic formulas \( P(\pi_\sigma x) \), cf. the proof of 5.14.11. The condition (ii) means that all levels \( F_n \) are weakly transitive.\(^{32}\)

Actually the conditions (b), (c), (d) express the monotonicity for the implication and the quantifiers \( \forall, \exists \) respectively, i.e. for formulas \( P(x) \supset Q(x), \forall yP(x, y), \exists yP(x, y) \).

The interested reader can try to restore the missing details. We point out that the description of \( i(=)\)-soundness in 5.19.11 is more complicated than the notion of an \( i(=)\)-metaframe, but the semantics is the same.

5.20 Simplicial frames

Introduction

In the last section of this chapter we briefly describe the next step in generalising Kripke-type semantics.

Recall that in the usual Kripke semantics for predicate logics we have a system of nested domains \( D = (D_u \mid u \in F) \), and every individual from \( D_u \) is considered as its own inheritor in the domains \( D_v \) of all worlds \( v \) accessible from \( u \). In Kripke sheaves, Kripke bundles and functor semantics accessibility relations (or functions) between individuals are introduced; they describe inheritors of individuals in accessible worlds. Accessibility relations between \( n \)-tuples of individuals (for \( n > 1 \)) can be derived from these accessibility relations on individuals.

At the next step, in metaframes, accessibility relations \( R^n \) (or \( R^n_t \), in the polymodal case) between \( n \)-tuples of individuals for different \( n \geq 1 \) may be arbitrary, and only the requirement of soundness puts some constraints on these relations. But \( n \)-tuples \( a = (a_1, \ldots, a_n) \) are obtained from individuals existing in the same world. So all \( n \)-tuples are taken from the set \( D^n = \bigcup \{ D^n_u \mid u \in F \} \).

Now let us consider more general kind of frames, in which \( n \)-tuples are 'abstract'; so the sets \( D^n \) for \( n \geq 1 \) are a priori independent, and unlike the case of metaframes, \( D^n \) is not constructed from the set of 'actual' individuals \( D^1 = \)

\[^{32}\text{A propositional frame } F = (W, R) \text{ is called weakly transitive if } \forall u, v, w \in W (uRvRw \Rightarrow \exists t (uRt & wR^* t & tR^* w)).\]
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∪{D_u \mid u \in F}. But then for any \( \sigma \in \Sigma_{mn} = (I_n)^I_m \), we should introduce a map \( \pi_\sigma : D^n \rightarrow D^m \) transforming ‘abstract’ tuples. Their metaframe analogues are ‘jections’ \( \pi_\sigma \) transforming ‘actual’ tuples: \( \pi_\sigma(a_1, \ldots, a_n) = (a_{\sigma(1)}, \ldots, a_{\sigma(m)}) \). But in the ‘abstract’ case \( \pi_\sigma \) are chosen arbitrarily. The definitions of valuations, forcing, validity, and strong validity can be given quite similarly to metaframe semantics.\(^{33}\) And again the predicate logic \( \text{ML}(=)(F) \) or \( \text{IL}(=)(F) \) for such a frame \( F \) is the set of formulas strongly valid in \( F \). Next, we can find constraints corresponding to logical soundness or to other natural properties of forcing, as we did in Chapter 5 for metaframes.

An ‘abstract’ n-tuple \( a \in D^n \) corresponds to the ‘real’ n-tuple \( (\pi_{\lambda^F_1}(a), \ldots, \pi_{\lambda^F_n}(a)) \), where \( \lambda^F_i \in \Sigma_{1n} \), \( \lambda^F_i(1) = i \), see the Introduction. \( \pi_{\lambda^F_i}(a) \) can be regarded as the ith ‘component’ of \( a \). But this correspondence in general is not bijective — different tuples may have the same ‘components’; moreover, for a permutation \( \sigma \in \Sigma_n \), ‘n-tuples’ \( a \) and \( \pi_\sigma(a) \) may have different (and even disjoint!) sets of ‘components’. Nevertheless, this correspondence is useful; as we shall see in Volume 2, this helps associate logically sound simplicial frames with metaframes.

As in metaframes, we can identify \( D^0 \) with the ‘underlying propositional frame’ \( F = (W, R_1, \ldots, R_N) \). And we can introduce the individual domains of a world \( u \in W \) as \( D_u := \{ a \in D^1 \mid \pi_{\sigma_1}(a) = u \} \). More generally, put \( D^n_u := \pi_{\sigma_n}^{-1}(u) \) for \( n \geq 1 \), where \( \sigma_n \in \Sigma_{0n} \) is the empty function. Then \( D^n = \bigcup\{D^n_u \mid u \in F\} \) is a partition. But in general, an ‘abstract’ n-tuple \( a \) and its ‘components’ \( \pi_{\sigma_n}(a) \) for \( 1 \leq i \leq n \) may be in domains of different worlds from \( F \).

Forcing in simplicial frames

Now let us turn to precise definitions. We begin with the case without equality.

**Definition 5.20.1** A (formal) simplicial N-frame based on a propositional Kripke frame \( F = (W, R_1, \ldots, R_N) \) is a tuple \( F = (F, D, R, T, \pi) \), where \( D = (D^n \mid n \in \omega) \) is a sequence of (non-empty) sets, \( R = (R_i^n \mid n \in \omega, 1 \leq i \leq N) \) is a family of relations \( R_i^n \subseteq D^n \times D^n \); so we have propositional N-modal frames: \( F_0 := (D^1, R_1^n, \ldots, R_N^n) \). As usual, we assume that \( F_0 := (W, R_1, \ldots, R_N) \) is the original frame \( F \). And finally, \( \pi = (\pi_\sigma \mid \sigma \in \bigcup \Sigma_{mn}) \) is a family of mappings (‘abstract jections’) \( \pi_\sigma : D^n \rightarrow D^m \) for \( \sigma \in \Sigma_{mn} \).

So \( D^n \) (for \( n \geq 1 \)) are sets of ‘abstract’ n-tuples (in particular, \( D^1 \) is the set of ‘individuals’), \( R_i^n \) are accessibility relations between n-tuples, and \( \pi_\sigma \) are mappings (‘jections’) transforming ‘abstract’ tuples.

Put \( D^n_u := \pi_{\sigma_n}^{-1}(u) \) for \( n > 0, u \in W \), where \( \sigma_n \) is the empty function from \( \Sigma_{0n} \); it may be called the n-tuple domain of the world \( u \). In particular, for \( n = 1 \) we sometimes write \( D_u \) rather than \( D^1_u \) and call this set the individual domain of the world \( u \). Obviously \( \{D^n_u \mid u \in W\} \) is a partition of \( D^n \).

\(^{33}\) Up to some details about the equality, which are briefly discussed later on.
The \(i\)th component of an abstract \(n\)-tuple \(a \in D^n\), \(n > 0\) is \([a]_i := \pi_{\sigma_i}(a) \in D^1\), where \(\sigma_i \in \Sigma_1\), \(\sigma_i(1) = i\) (i.e. \((a_1, \ldots, a_n) \cdot \sigma_i = a_i\)). In general, we do not assume that \(a\) and its components 'live' in the same world, i.e. that \(\pi_{\sigma_i}(a) = \pi_{\sigma_i}([a]_i)\). We do not even assume that \([a]_1 = a\) for \(a \in D^1\) — an 'individual' may be non-equal to its own 'component'; they can even 'live' in different worlds. But as we shall see later on, in natural semantics (for modal or superintuitionistic logics) we may consider only frames, in which \(\pi_{\sigma_i}(a) = a\) for any \(n\); such frames are called \(\pi\)-identical. In these frames we also have \([a]_1 = a\) for \(a \in D^1\), and \(\pi_{\sigma}(u) = u\) for \(u \in W\), i.e. \(D^0_u = \pi_{\sigma}^{-1}(u) = \{u\}\).

As usual, for the monomodal (and the intuitionistic) case we denote accessibility relations on \(n\)-tuples by \(R^n\), without the subscript \(i = 1\).

The notations \(R^n+\) (respectively, \(R^n*\)) for the transitive (respectively, reflexive transitive) closure of \(\bigcup_i R^n_i\) are usual, cf. Chapter 1. We also consider the corresponding equivalence relations \(\approx_n := \approx_{R^n}\) on \(D^n\).

Obviously, every \(N\)-metaframe gives rise to a simplicial \(N\)-frame, with \(D^n, R^n_i, \pi\sigma\) described in Section 5.9.

**Definition 5.20.2** A valuation in a simplicial \(N\)-frame \(\mathcal{F}\) is a function \(\xi\) sending every \(n\)-ary predicate letter \(P^n_j\) to a subset \(\xi^+(P^n_j) \subseteq D^n\); in particular, \(\xi^+(P^n_0) \subseteq W\) (cf. Definition 5.9.2).

A valuation \(\xi\) in a simplicial 1-frame (or more briefly, a simplicial frame) is intuitionistic if every \(\xi^+(P^n_j)\) is an \(R^n\)-stable subset of \(D^n\):

\[
aR^n b \land a \in \xi^+(P^n_j) \Rightarrow b \in \xi^+(P^n_j),
\]

cf. Definition 5.14.1. A simplicial model is a pair \(M = (\mathcal{F}, \xi)\).

**Definition 5.20.3** An assignment of length \(n\) in a simplicial frame \(\mathcal{F}\) is a pair \((x, a)\), where \(a \in D^n\), and \(x\) is a distinct list of variables of length \(n\).

Now we can define forcing for modal formulas without equality, cf. Definition 5.9.4.

**Definition 5.20.4** A simplicial \(N\)-model \(M = (\mathcal{F}, \xi)\) gives rise to the forcing relation \(M, a \models A [x]\), where \(A \in MF_N\), \((x, a)\) is an assignment in \(\mathcal{F}\), \(FV(A) \subseteq r(x)\). The definition is by induction, modifying 5.9.4 in a natural way. We assume that \((x, a)\) is of length \(n\).

\[
(At1) \quad M, a \nvdash \bot [x];
\]
\[
(At2) \quad M, a \vdash P^n_j (x \cdot \sigma) [x] \text{ iff } \pi_\sigma a \in \xi^+(P^n_j) \text{ (for } \sigma \in \Sigma_{mn})
\]
\[
(\lor) \quad M, a \vdash (B \lor C) [x] \text{ iff } M, a \nvdash B [x] \text{ or } M, a \nvdash C [x];
\]

and similarly for the other Boolean connectives;

\[
(\Box) \quad M, a \vdash \Box_i B [x] \text{ iff } \forall b \in R^n_i(a) \ M, b \vdash B [x];
\]
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\[(Q1)\] \(M, a \models \exists y B [x] \text{ iff } \exists c \in D^{n+1}(\pi_{\sigma}^+ c = a & M, c \models B [xy]),\)

\(M, a \models \forall y B [x] \text{ iff } \forall c \in D^{n+1}(\pi_{\sigma}^+ c = a \Rightarrow M, c \models B [xy])\)

(for \(y \not\in x\));

\[(Q2)\] \(M, a \models \exists x_i B [x] \text{ iff } M, \pi_{\delta}^+[a] \models \exists x_i B [x - x_i],\)

\(M, a \models \forall x_i B [x] \text{ iff } M, \pi_{\delta}^+[a] \models \exists x_i B [x - x_i],\)

where \(x - x_i = x \cdot \delta_i^\uparrow\).

Hence we obtain the truth conditions for \(\neg\) and \(\Box_i\) similar to those in Section 5.9.

One can see that in a simplicial frame corresponding to a metaframe the clauses \((\Box 2), (\Box), (Q1), (Q2)\) become equivalent to \((2), (8), (9), (10)\) from Definition 5.9.4. In fact, for \((Q1)\) note that in a metaframe \(\pi_{\sigma}^+ c = a \iff c = ad\) for some \(d \in D(a)\).

Let us also note that in \('\pi\)-identical' simplicial frames (where \(\pi_{id_a} a = a\) for any \(a \in D^n\)) both cases of the inductive clause for the quantifier can be presented in the following uniform way (cf. \((9+10)\) in Section 5.9):

\(M, a \models \exists y B [x] \text{ iff } \exists c \in D^{m+1}(\pi_{\sigma}^+ c = \pi_{\varepsilon_x-y} a & M, c \models B [x\|y])\)

(where \(m = |x - y|\)). In fact, if \(y = x_i\), this clause is equivalent to \((Q2)\), since \(\varepsilon_{x-y} = \delta_i^\uparrow\) and \(x\|y = (x - x_i)x_i\); and if \(y \not\in x\), then it is equivalent to \((Q1)\), since \(\pi_{\varepsilon_x-y} a = \pi_{id_a} a = a\) and \(x\|y = xy\).

Remark 5.20.5 Again we can propose a reasonable alternative definition of forcing \(\models^*\) differing in the clause \((Q2)\):

\(M, a \models^* \exists x_i B [x] \text{ iff } \exists c \in D^n(\pi_{\delta}^+ c = \pi_{\delta}^+ a & M, c \models B [x]),\)

and similarly for \(\forall x_i B\).

But this definition actually leads to the same natural semantics of modal simplicial frames as Definition 5.20.4; thus both versions are equivalent. However we do not know if these definitions give equal (or equivalent) maximal logically sound semantics.

Similarly we can define forcing for the intuitionistic case.

Definition 5.20.6 An (intuitionistic) simplicial model \(M = (\mathbb{F}, \xi)\) gives rise to forcing for intuitionistic formulas \(A\) and assignments \((x, a)\) with \(r(x) \supseteq \text{FV}(A)\) defined by the following inductive clauses (cf. Definition 5.14.2):

- \(M, a \not\models \perp [x]\);  
- \(M, a \models P^m_j(\pi_{\sigma} x) [x] \text{ iff } \pi_{\sigma} a \in \xi^+(P^m_j);\)
- \(M, a \models (B \land C) [x] \text{ iff } M, a \models B [x] \text{ and } M, a \models C [x];\)
- \(M, a \models (B \lor C) [x] \text{ iff } M, a \models B \text{ or } M, a \models C [x];\)
5.20. SIMPLICIAL FRAMES

- $M, a \vDash (B \supset C) [x]$ iff $\forall b \in D^n(a R^n b \& M, b \vDash B [x]) \Rightarrow M, b \vDash C [x]$;
- $M, a \vDash \forall y B [x]$ iff $\forall c \in D^{n+1}(a R^n \pi_{\sigma 1}(c) \Rightarrow M, c \vDash B [xy])$ (for $y \not= x$);
- $M, a \vDash \exists y B [x]$ iff $\exists c \in D^{n+1}(\pi_{\sigma 2}(c) R^n a \& M, c \vDash B [xy])$;
- $M, a \vDash \forall x_i B [x]$ iff $M, \pi_{\sigma x} a \vDash \forall x_i B [x - x_i]$ for a quantifier $\forall$.

Again we have uniform presentations of quantifier clauses in $\pi$-identical simplicial frames.

**Sound and natural simplicial frames**

**Definition 5.20.7** An $N$-modal formula $A$ is true in a simplicial model (notation: $M \vDash A$) if $M, a \vDash A [x]$ for any assignment $(x, a)$ such that $FV(A) \subseteq \tau(x)$. The definition for the intuitionistic case is similar.

**Definition 5.20.8** An $N$-modal formula $A$ is valid in a simplicial frame $F$ (notation $F \vDash A$) if it is true in all models over $F$. The definition for the intuitionistic case is similar.

$\text{ML}_- (F)$ denotes the set of all $N$-modal formulas (without equality) valid in $F$; the notation $\text{IL}_- (F)$ is similar.

**Definition 5.20.9** A formula is strongly valid in $F$ if all its substitution instances (without equality) are valid.

The set of all formulas strongly valid in $F$ is denoted by $\text{ML}(F)$ in the modal case and by $\text{IL}(F)$ in the intuitionistic case.

A simplicial frame $F$ is logically sound if $\text{ML}(F)$ or $\text{IL}(F)$ is a modal or a superintuitionistic logic respectively; we call such a frame $m$-sound or $i$-sound respectively. Logically sound simplicial frames generate 'maximal semantics'. But unlike the case of metaframes, we do not know an explicit description of these semantics, and we may conjecture that they are rather complicated (cf. Section 5.16, 5.19 for intuitionistic sound metaframes). Therefore in Volume 2 we shall describe more convenient classes of simplicial frames, where forcing satisfies natural properties of 'logical invariance' similar to those we had for metaframes in Sections 5.10, 5.11, 5.15. These classes generate 'almost maximal' semantics (and maybe even maximal, but this is yet unknown).

**Definition 5.20.10** A simplicial frame $F$ is called modally transformable (or $m$-transformable) if the following condition$^{34}$

$$M, a \vDash A [x] \iff M, \pi_{\sigma x} a \vDash A [\pi_{\sigma x}]$$

holds for any injection $\sigma \in Y_{mn} (n \geq m)$, for any model $M = (F, \xi)$, and for any assignment $(x, a)$ such that $FV(A) \subseteq \tau(\pi_{\sigma x})$.

$^{34}$Cf. 5.10.6(*).
The definition of \( i(=) \)-transformable frames for the intuitionistic case is similar — they satisfy the condition\(^{35}\)

\[
M, a \vdash A [x] \iff M, \pi \sigma a \vdash A [\pi \sigma x]. \tag{trfi}
\]

for intuitionistic models \( M \) and formulas \( A \).

Recall that the case \( n = 0 \) (when \( A \) is a sentence and \( \sigma = id_0 = \Lambda_0 \)) is trivial.

Loosely speaking, in transformable frames forcing \( M, a \vdash (i-)A[x] \) does not depend on the choice of a list of variables \( x \) containing \( FV(A) \). More precisely, \( M, a \vdash A [x] \) with \( r(x) \supset FV(A) \) can be reduced to \( M, a \vdash A[x] \) with \( r(x') = FV(A) \), where \( x' = \pi \sigma x \), \( \sigma \) is an injection. Moreover, the choice of \( x' \) is inessential, since for any enumerations \( x, x' \) of \( FV(A) \), there exists a permutation \( \sigma \) such that \( x' = \pi \sigma x \). This observation allows us to use only the first case in the inductive clause for quantifiers, cf. Section 5.6.

**Lemma 5.20.11** Let \( F \) be an \( m \)-transformable simplicial frame, \( M = (\mathbb{F}, \xi) \) a simplicial model. Then for any congruent formulas \( A, A' \) and for any \( x, a \) such that \( FV(A) \subseteq r(x) \):

\[
M, a \vdash A [x] \iff M, a \vdash A [x], \tag{*}
\]

and similarly for the intuitionistic case.

**Definition 5.20.12** A simplicial frame \( F \) is called modally \( s \)-transformable (or \( ms \)-transformable) if for any \( \sigma \in \Sigma_{mn} \), for any distinct lists \( x \) of length \( n, m \) respectively, for any modal formula \( A \) such that \( FV(A) \subseteq r(y) \), for any model \( M \) over \( F \), and for any \( a \in D^n \):

\[
M, a \vdash ([\pi \sigma x/y] A) [x] \iff M, \pi \sigma a \vdash A [y]. \tag{trfms}
\]

Similarly for the intuitionistic case, we define \( is \)-transformable simplicial frames with the following condition:

\[
M, a \vdash ([\pi \sigma x/y] A) [x] \iff M, \pi \sigma a \vdash A [y]. \tag{trfis}
\]

This condition expresses the invariance of forcing under variable substitutions (cf. Lemma 5.11.7 for metaframes).

Recall that variable substitutions are defined up to congruence, and the congruent versions of \( A \) are all the results of applying the identity substitution \([/]\) to \( A \) (cf. Section 2.3).

So in \( ms \)-transformable simplicial frames forcing is congruence invariant:

\[
M, a \vdash A [x] \iff M, a \vdash A^1 [x] \tag{*}
\]

for any congruent formulas \( A, A^1 \) (apply \( (trfsm) \) with \( y = x \), \( \sigma = id_n \)), and similarly in the intuitionistic case. On the other hand, we can show that forcing is congruence invariant in every transformable simplicial frame as well (cf. 35Cf. 5.15.14(*i).
Lemmas 5.10.11 and 5.15.16 for metaframes in the modal and the intuitionistic cases respectively.

Anyway, if the condition (*) holds, then the choice of any congruent version of \([\pi_\sigma x/y]A\) in (trf sm) does not matter.

**Definition 5.20.13** A simplicial frame \(F\) is called \(m\)-natural (respectively, \(i\)-natural) if it is logically sound and transformable (respectively, \(s\)-transformable).

The classes of all \(m\)-natural and \(i\)-natural simplicial frames generate the 'natural' semantics of simplicial frames \(SF_m\) and \(SF_{int}\).

In Volume 2 we will also describe other classes of simplicial frames (called 'modal' and 'intuitionistic') generating the same semantics \(SF_m\) and \(SF_{int}\); this description generalises the work done in Sections 5.11 and 5.15 above.

Theorems 5.12.13, 5.16.13, along with 5.10.6, 5.11.7, 5.15.14, 5.15.17, show that for every logically sound metaface the corresponding simplicial frame is natural; thus \(MF_m \leq SF_m\) and \(MF_{int} \leq SF_{int}\).

On the other hand, we do not know if every logically sound simplicial frame is natural and even if the semantics of natural simplicial frames equals the 'maximal' semantics of all logically sound simplicial frames. It is also unknown if the semantics of simplicial frames (natural or 'maximal') are stronger than the semantics of metaframes.

**Equality in simplicial frames**

To conclude this preliminary exposition, we briefly discuss problems with interpretation of equality in simplicial frames.

Recall that in metaframes the clause for equality has the following form:

\[ a \vdash x_j = x_k \ [x] \iff a_j = a_k, \]

where \(a \in D^n, |x| = n\). This definition admits several reasonable generalisations for simplicial frames; in metaframes all these versions are equivalent.

First, for the formula \(x_1 = x_2, x = (x_1, x_2)\), and \(a \in D^2\), we can put

\[ a \vdash x_1 = x_2 \ [x] \iff \pi_{\sigma_{12}}(a) = \pi_{\sigma_{22}}(a), \]

where \(\sigma_{j2}(1) = j\) (so \((a_1, a_2) \cdot \sigma_{j2} = a_j\) in metaframes).

Second, we can define

\[ a \vdash x_1 = x_2 \ [x] \iff a = \pi_\sigma(b) \text{ for some } b \in D^1, \]

where \(\sigma^1 \in \Sigma_{21}\) is the 'diagonal' map; \(\sigma^1(j) = 1\) for \(j \in I_2\) (so \(\pi_{\sigma^1}(b) = (b, b)\) in metaframes). These two definitions are clearly equivalent for 'real' tuples, but they differ for 'abstract' tuples. For example, in general we cannot assert that \(\pi_{\sigma_{j2}}(\pi_{\sigma^1}(b)) = b\) for \(b \in D^1\) or that \(\pi_{\sigma^1}(\pi_{\sigma_{12}}(a)) = a\) if \(\pi_{\sigma_{12}}(a) = \pi_{\sigma_{22}}(a)\).

---

36We do not mention a model \(M\) as equality does not depend on the valuation.
There exist other more 'exotic' versions of forcing, e.g.

\[ a \models x_1 = x_2 [x] \text{ iff } \pi_{\sigma_{12}}(a) = \pi_{\sigma_{22}}(a) \& a = \pi_{\sigma_1}(\pi_{\sigma_{12}}(a)). \]

All these interpretations are equivalent to \((=)\) (with \(n = 2, \ i = 1, \ j = 2\)) in metaframes, although they are non-equivalent in simplicial frames.

To express these definitions (i.e. generalisations of \((=)\) to simplicial frames) for arbitrary \(n\) and \(i, j \in I_n\) we use projections \(\pi_{\sigma_{1i}}\) ‘extracting’ components of an \(n\)-tuple \((\pi_{\sigma_{1n}}(a_1, \ldots, a_n) = a_i)\) and ‘pairingjections’ \(\pi_{\lambda_{jk}}\), where \(\lambda_{jk}^n \in \Sigma_{2n}\), \(\lambda_{jk}^n(1) = j, \ \lambda_{jk}^n(2) = k\), i.e. \(\pi_{\lambda_{jk}^n}(a_1, \ldots, a_n) = a_ja_k\).

Thus we obtain the following interpretations of equality in simplicial frames

\[ a \models ^+ x_j = x_k [x] \text{ iff } \pi_{\sigma_{jn}}(a) = \pi_{\sigma_{kn}}(a), \quad (=^+) \]

the ‘componentwise’ or the upper interpretation,

\[ a \models ^- x_j = x_k [x] \text{ iff } \exists b \in D^1 \pi_{\lambda_{jk}^n}(a) = \pi_{\sigma_1}(b) \quad (=-) \]

the ‘diagonal’ or the lower interpretation, and also the ‘combined’ interpretation

\[ a \models ^\pm x_i = x_j [x] \text{ iff } \pi_{\sigma_{jn}}(a) = \pi_{\sigma_{kn}}(a) \& \pi_{\lambda_{jk}^n}(a) = \pi_{\sigma_1}(\pi_{\sigma_{1n}}(a)). \quad (=^\pm) \]

We suppose that the upper interpretation is the most straightforward generalisation of the condition \((=)\) in metaframes. However, some logical properties of the lower interpretation \(\models^-\) seem better.

In Volume 2 we will consider a general approach to interpretation of equality in simplicial frames covering all these versions — various interpretations of equality \(a \models x_i = x_j [x]\) depend on triples of the form \((\pi_{\sigma_{1n}}(a), \pi_{\sigma_{jn}}(a), \pi_{\lambda_{jk}^n}(a))\).

But such a general approach seems too complicated, so we simplify it by using ‘pairs’ \(\pi_{\lambda_{jk}^n}(a) \in D^2\). Thus, for example, the upper interpretation has a simplified version:

\[ a \models ^+ x_j = x_k [x] \text{ iff } \pi_{\lambda_{jk}^n} \in I^+, \]

where \(I^+ = \{b \in D^2 | \pi_{\sigma_{12}}(b) = \pi_{\sigma_{22}}(b)\}\).

Although this condition is not equivalent to \((=^+)\) in arbitrary simplicial frames, for ‘natural’ simplicial frames this reduction works quite well.
Part III

Completeness
Chapter 6

Kripke completeness for varying domains

Completeness proofs in this chapter are based on various canonical model constructions. They originate from canonical models in modal propositional logic and Henkin's completeness proof in classical predicate logic. The main idea is that worlds in canonical models are consistent (or even syntactically complete) theories, and individuals are identified with individual constants of these theories.

6.1 Canonical models for modal logics

Recall (Definition 2.7.7) that for a first-order modal theory $\Gamma$, $D_\Gamma$ denotes the set of individual constants occurring in $\Gamma$, $L(=)(\Gamma)$ is the set of all $D_\Gamma$-sentences in the language of $\Gamma$.

**Definition 6.1.1** A (first-order) modal theory $\Gamma$ is called $L$-consistent if

\[ \not\vdash_L \bigwedge_{i=1}^k A_i \text{ for any } A_1, \ldots, A_k \in \Gamma, \text{ or equivalently, if } \Gamma \not\vdash_L \bot. \]

A modal theory is called $L$-complete if it is maximal (by inclusion) among $L$-consistent theories in the same language.

**Lemma 6.1.2** Let $\Gamma$ be a modal theory.

1. If $\Gamma$ is $L$-consistent and $\Gamma \vdash_L \neg A$, then $\Gamma \not\vdash_L A$.

2. If $\Gamma$ is $L$-complete, $A \in L(=)(\Gamma)$, then

\[ \Gamma \vdash_L A \text{ iff } A \in \Gamma; \]

in particular, $L \subseteq \Gamma$ (where $L$ denotes the set of all sentences in $L$).
(3) If $\Gamma$ is $L$-complete, then for any $A \in \mathcal{L}(=)(\Gamma)$
\[-A \in \Gamma \text{ iff } A \notin \Gamma.\]

(4) Every $L$-consistent $\Gamma$ satisfying the equivalence (3), is $L$-complete.

**Proof**

(1) If $\Gamma \vdash_L \neg A$ and $\Gamma \vdash_L A$, then $\Gamma \vdash_L \bot$, by MP.

(2) 'If' is obvious. To show 'only if', suppose $A \notin \Gamma$. Then $\Gamma \subseteq \Gamma \cup \{A\}$, hence $\Gamma \cup \{A\} \vdash_L \bot$, and thus $\Gamma \vdash_L A \supset \bot (= \neg A)$ by 2.8.1. Hence $\Gamma \not\vdash_L A$ by (1).

(3) 'Only if' readily follows from (1) and (2). To show 'if', suppose the contrary — that both $A$, $\neg A$ are not in $\Gamma$. Then as in the proof of (2), we obtain $\Gamma \vdash_L \neg A$ and similarly, $\Gamma \vdash_L \neg \neg A$, which implies the $L$-inconsistency of $\Gamma$.

(4) Suppose the equivalence in (3) holds for any $A \in \mathcal{L}(=)(\Gamma)$, but $\Gamma$ is $L$-incomplete. Then $\Gamma \subseteq \Gamma'$ for some $\Gamma'$ in the same language, so there exists $A \in (\Gamma' - \Gamma)$. By (3), we have $\neg A \in \Gamma' \subseteq \Gamma'$, thus $\Gamma'$ is $L$-inconsistent. 

Similarly to the propositional case, we have the following

**Lemma 6.1.3** Let $\Gamma$ be an $L$-complete theory. Then for any $A, B \in \mathcal{L}(=)(\Gamma)$:

(i) $(A \land B) \in \Gamma \text{ iff } (A \in \Gamma \text{ and } B \in \Gamma)$;

(ii) $(A \lor B) \in \Gamma \text{ iff } (A \in \Gamma \text{ or } B \in \Gamma)$;

(iii) $(A \supset B) \in \Gamma \text{ iff } (A \notin \Gamma \text{ or } B \in \Gamma)$.

**Proof**

(i) (If.) $A \land B \supset A$ is an instance of a tautology, so if $(A \land B) \in \Gamma$, we have $\Gamma \vdash_L A$ by MP. Hence $A \in \Gamma$ by 6.1.2 (2). Similarly $B \in \Gamma$.

(Only if.) If $A, B \in \Gamma$, we apply the instance of a tautology $A \supset (B \supset A \land B)$, MP and 6.1.2 (2).

(ii) (If.) Similarly to (i), use $A \supset A \lor B$ or $B \supset A \lor B$.

(Only if.) Suppose $A, B \notin \Gamma$. Then $\neg A, \neg B \in \Gamma$ by 6.1.2 (3). Since $\neg A \supset (\neg B \supset \neg (A \lor B))$ is an instance of a tautology, we obtain $\neg A, \neg B \vdash_L \neg (A \lor B)$, hence $\neg (A \lor B) \in \Gamma$, and thus $(A \lor B) \not\in \Gamma$ by 6.1.2 (3).

(iii) (Only if.) Supposing $(A \supset B), A \in \Gamma$, we obtain $B \in \Gamma$ by MP and 6.1.2 (2).

(If.) If $A \notin \Gamma$, then $\neg A \in \Gamma$ by 6.1.2 (3). Since $\neg A \supset (A \supset B)$ is an instance of a tautology, by MP and 6.1.2 (2), it follows that $(A \supset B) \in \Gamma$.

If $B \in \Gamma$, we use $B \supset (A \supset B)$ in the same way.

Recall that $\mathcal{L}$ denotes the set of all sentences in a logic $L$,

$\Box^* \Gamma := \{ \Box^* A \mid A \in \Gamma, \alpha \in I_N^\infty \}$ (if $N$ is fixed).

**Lemma 6.1.4** Let $L$, $L_1$ be $N$-m.p.l.(=), $\Gamma$ an $L$-complete theory. Then
6.1. CANONICAL MODELS FOR MODAL LOGICS

(1) \( \Gamma \) is \( L_1 \)-complete iff \( \overline{L_1} \subseteq \Gamma \).

(2) Suppose \( L_1 = L + \Theta \) for a set of sentences \( \Theta \). Then \( \Gamma \) an \( L_1 \)-complete iff
\[ \Box^* \text{Sub}(\Theta) \subseteq \Gamma. \]

**Proof**

(1) 'Only if' follows from Lemma 6.1.2 (2).

To show 'if', assume \( \overline{L_1} \subseteq \Gamma \). By Lemma 6.1.2 (3), (4), it suffices to check that \( \Gamma \) is \( L_1 \)-consistent. Suppose the contrary; then for a finite \( X \subseteq \Gamma, \vDash L_1 \neg \bigwedge X \). Put \( A := \neg \bigwedge X \). Then by Definition 2.7.1, \( A \) has a generator \( A_1 \in L_1 \). Hence \( \forall A_1 \in \overline{L_1} \subseteq \Gamma \). But then \( \vDash L_\neg A_1 \), whence \( \vDash L A_1 \), and therefore \( \vDash L A \) implying the \( L \)-inconsistency of \( \Gamma \).

(2) By (1), \( \Gamma \) is \( L_1 \)-complete iff \( \overline{L_1} \subseteq \Gamma \). Since \( \Box^* \text{Sub}(\Theta) \subseteq \overline{L_1} \), the 'only if' part of (2) follows readily.

For the converse, we assume \( \Box^* \text{Sub}(\Theta) \subseteq \Gamma \) and show that \( \overline{L_1} \subseteq \Gamma \). In fact, by the deduction theorem 2.8.4, every \( A \in \overline{L_1} \subseteq L + \Theta \) is \( L \)-provable in \( \Box^* \text{Sub}(\Theta) \), hence \( \vDash L A \), therefore \( A \in \Gamma \), by 6.1.2. 

**Lemma 6.1.5 (Lindenbaum lemma)** Every \( L \)-consistent theory can be extended to an \( L \)-complete theory in the same language.

**Proof** Similar to the propositional case. If \( \mathcal{L}(=)(\Gamma) \) is countable, we consider its enumeration \( A_0, A_1, \ldots \) and construct a sequence of \( L \)-consistent theories \( \Gamma = \Gamma_0 \subseteq \Gamma_1, \ldots \) such that \( \Gamma_{n+1} = \Gamma_n \cup \{ A_n \} \) or \( \Gamma_{n+1} = \Gamma_n \cup \{ \neg A_n \} \). This is possible, since one of \( \Gamma_n \cup \{ A_n \}, \Gamma_n \cup \{ \neg A_n \} \) is \( L \)-consistent (otherwise \( \vDash L \neg A_n, \neg \neg A_n \) by Lemma 2.8.1, and thus \( \Gamma_n \) is \( L \)-inconsistent). Eventually the union \( \Gamma_\omega := \bigcup_{n} \Gamma_n \) is \( L \)-consistent and \( L \)-complete by 6.1.2(4).

If the language is uncountable, we apply transfinite induction, but actually we do not need this case in the sequel.

**Definition 6.1.6** An \( L \)-complete theory \( \Gamma \) is called \( L \)-Henkin if for any \( D_\Gamma \)-sentence \( \exists x A(x) \) there exists a constant \( c \in D_\Gamma \) such that
\[ (\exists x A(x) \supset A(c)) \in \Gamma. \]

We say that \( \Gamma \) is \((L,S)\)-Henkin if \( \Gamma \) is \( L \)-Henkin with \( D_\Gamma = S \).

Due to Lemma 6.1.3 (iii), every \( L \)-Henkin theory \( \Gamma \) has the following existence property:

(EP) if \( \exists x A(x) \in \Gamma \), then for some \( c \in D_\Gamma \), \( A(c) \in \Gamma \).

Moreover, we have

**Lemma 6.1.7** Every \( L \)-Henkin theory \( \Gamma \) satisfies the condition
\[ (EP') \exists x A(x) \in \Gamma \text{ iff } \text{for some } c \in D_\Gamma, A(c) \in \Gamma. \]
Proof Since $L \vdash A(y) \supset \exists x A(x)$ for a new variable $y$ (Lemma 2.6.15(ii)), we have $\Gamma \vdash A(c) \supset \exists x A(x)$, and so $(A(c) \supset \exists x A(x)) \in \Gamma$. Thus Lemma 6.1.3 implies the converse of (EP), therefore (EP') holds.

Exercise 6.1.8 Show that every $L$-complete theory with the existence property is $L$-Henkin.

Lemma 6.1.9 Let $\Gamma$ be an $L$-consistent theory, $D_\Gamma \subset S$, $|S| = |S - D_\Gamma| = \aleph_0$. Then there exists an $(L, S)$-Henkin theory $\Gamma' \supset \Gamma$.

Proof The set of all $S$-sentences is countable, so let us enumerate all $S$-sentences of the form $\exists x A(x) : \exists x_1 A_1(x_1), \exists x_2 A_2(x_2), \ldots$ Choose distinct constants $c_k$ from $(S - D_\Gamma)$ such that $c_k$ does not occur in $\exists x_i A_i(x_i)$ for $i < k$, and put

$$\Gamma_\omega := \Gamma \cup \{\exists x_k A_k(x_k) \supset A_k(c_k) \mid k \in \omega\}.$$ 

Let us show that $\Gamma_\omega$ is $L$-consistent. Suppose the contrary. Then for some $k$

$$\Gamma \vdash_L \neg \bigwedge_{i \leq k} (\exists x_k A_k(x_k) \supset A_k(c_k)).$$

Take the minimal $k$ with this property. Let

$$B := \bigwedge_{i < k} (\exists x_i A_i(x_i) \supset A_i(c_i)).$$

Then

$$\Gamma, B, \exists y A_k(y) \supset A_k(c) \vdash_L \bot$$

(where $y = x_k$, $c = c_k$). Hence by Lemma 2.8.1,

$$\Gamma, B \vdash_L \neg (\exists y A_k(y) \supset A_k(c)).$$

Then from the propositional tautology $\neg(p \supset q) \equiv p \land \neg q$ we obtain

1. $\Gamma, B \vdash_L \exists y A_k(y)$;
2. $\Gamma, B \vdash_L \neg A_k(c)$.

Since $c$ does not occur in $\Gamma \cup \{B\}$, by Lemma 2.7.11, it follows that

3. $\Gamma, B \vdash_L \forall y \neg A_k(y)$,

and thus by 2.6.15(xiii),

4. $\Gamma, B \vdash_L \neg \exists y A_k(y)$.

From (1) and (4) it follows that $\Gamma \vdash_L \neg B$, contrary to the choice of $k$.

Therefore $\Gamma_\omega$ is $L$-consistent, and by Lemma 6.1.5, we can extend it to $\Gamma'$. ■
Lemma 6.1.10 Let $A$ be a modal sentence, $|S| = \aleph_0$.
Then $L \vdash A$ iff $(A \in \Gamma$ for any $(L, S)$-Henkin theory $\Gamma$).

Proof ‘Only if’ follows from Lemma 6.1.3. To prove ‘if’, suppose $L \not\vdash A$. Then $\{\neg A\}$ is an $L$-consistent subset of $MF_N^{(=)}(\emptyset)$, and so by Lemma 6.1.9, there exists an $(L, S)$-Henkin theory $\Gamma \supseteq \{\neg A\}$.

For $S \subseteq S'$ and $\Gamma' \subseteq MF_N^{(=)}(S')$, let $\Gamma'|S := \Gamma \cap MF_N^{(=)}(S)$ (the restriction of $\Gamma'$ to the domain $S$). Obviously, $\Gamma'|S$ is consistent (complete) if $\Gamma'$ is consistent (respectively, complete), but for a Henkin $(L, S')$-theory $\Gamma'$, $\Gamma'|S$ may be not a Henkin $(L, S)$-theory.

Henceforth we fix a denumerable set $S^*$, the ‘universal set of constants’. We call a set $S \subset S^*$ small if $(S^* - S)$ is infinite.

Definition 6.1.11 An $L$-place is a Henkin $L$-theory with a small set of constants.
The set of all $L$-places is denoted by $VP_L$.

Lemma 6.1.12 Let $u$ be a world in a Kripke model $M$ over an $N$-modal predicate Kripke frame $F = (F, D)$ and assume that $M \models L$. Then the set of $D_u$-sentences that are true at $u$

$$
\Gamma_u := \{A \in MF_N^{(=)}(D_u) \mid M, u \models A\}
$$

is a Henkin $L$-theory.

Proof This is an easy consequence from the definitions. Let us only give an argument for $L$-consistency. If $\vdash L \neg \bigwedge_{i=1}^k A_i$, then $B := \bigwedge_{i=1}^k A_i$ can be presented as $[c/x] B_0$ for $B_0 \in L$, $c \in D_u^\infty$, and then $\forall x B_0 \in L$. Hence by assumption $M, u \models \forall x B_0$, which implies $M, u \models [c/x] B_0 = \neg \bigwedge_{i=1}^k A_i$ by Lemma 3.2.18. Thus one of the $A_i$ must be false at $M, u$.

Definition 6.1.13 Let $M$ be a Kripke model for a modal logic $L$, in which every individual domain is a small subset of $S^*$. The map from (the worlds of) $M$ to $VP_L$, sending $u$ to $\Gamma_u$ is called canonical and denoted by $\nu_{M, L}$ (or by $\nu_M$, or by $\nu$ if there is no confusion).

Lemma 6.1.12 extends to Kripke sheaves, Kripke bundles, or other kinds of models for $L$. Moreover, $VP_L$ is the set of all ‘places’ $\Gamma_u$ of all possible Kripke models (and of metaframe models) for $L$ with small domains. To show this, we shall now construct a certain predicate Kripke frame (the canonical frame), whose worlds are $L$-places and in which every ‘world’ $\Gamma$ has the individual domain $D_\Gamma$, while $\Gamma$ is the set of all $D_\Gamma$-sentences true at this world. Since all the domains are small, there are infinitely many spare constants at every world.
For an $N$-modal theory $\Gamma$ and $i = 1, \ldots, N$, put

\[
\Diamond_i \Gamma := \{ \Diamond_i A \mid A \in \Gamma \},
\]

\[
\Box_i \Gamma := \{ B \mid \Box_i B \in \Gamma \}.
\]

**Definition 6.1.14** For $L$-places $\Gamma, \Gamma'$ we define canonical accessibility relations:

\[
\Gamma R_{Li} \Gamma' := \Box_i \Gamma \subseteq \Gamma'.
\]

**Lemma 6.1.15** $\Gamma R_{Li} \Gamma'$ iff $D_{\Gamma} \subseteq D_{\Gamma'}$ & $\Diamond_i \Gamma' \cap \mathcal{L}(\Gamma) \subseteq \Gamma$.

**Proof** (Only if.) Assume $\Gamma R_{Li} \Gamma'$. For any $c \in D_{\Gamma}$ we have

\[
\vdash_{\Gamma} \Box_i (P(c) \supset P(c)),
\]

so $\Box_i (P(c) \supset P(c)) \in \Gamma$ by 6.1.2(2), and thus

\[
(P(c) \supset P(c)) \in \Box_i \Gamma \subseteq \Gamma'.
\]

Therefore $c \in D_{\Gamma'}$, which shows $D_{\Gamma} \subseteq D_{\Gamma'}$.

(If.) Assume $D_{\Gamma} \subseteq D_{\Gamma'}$, $\Diamond_i \Gamma' \cap \mathcal{L}(\Gamma) \subseteq \Gamma$. Next, assume $\Box_i B \in \Gamma$. Then $B \in \Gamma'$, since otherwise $\neg B \in \Gamma'$, $\Diamond_i \neg B \in \Diamond_i \Gamma' \cap \mathcal{L}(\Gamma) \subseteq \Gamma$, and thus $\Gamma$ is inconsistent. Therefore $\Box_i \Gamma \subseteq \Gamma'$.

The crucial property of canonical relations is the following

**Lemma 6.1.16** For any $L$-place $\Gamma$ and for any $D_{\Gamma}$-sentence $A$ such that $\Box_i A \in \Gamma$, there exists an $L$-place $\Gamma'$ such that $\Gamma R_{Li} \Gamma'$ and $A \in \Gamma'$.

**Proof** Let us show that the theory $\Gamma_0 := \Box_i \Gamma \cup \{ A \}$ is $L$-consistent if $\Box_i A \in \Gamma$.

Suppose $\vdash_{\Gamma_0} \bigwedge_j B_j \supset \neg A$ for some formulas $B_j \in \Box_i \Gamma$. Then $\vdash_{\Gamma} \Box_i (\bigwedge_j B_j) \supset \Box_i \neg A$, hence $\vdash_{\Gamma} \bigwedge_j \Box_i B_j \supset \neg \Box_i A$, $\vdash_{\Gamma} \neg \Box_i A$, and thus by Lemma 6.1.2 (2), $\neg \Box_i A \in \Gamma$. This is a contradiction.

Now by Lemma 6.1.9, we can construct an $L$-place $\Gamma'$ such that $\Gamma_0 \subseteq \Gamma'$ (remember that $D_{\Gamma}$ is small, so there exists a small $S$ such that $|S - D_{\Gamma}| = n_0$). By Definition 6.1.14, $\Gamma R_{Li} \Gamma'$.

**Lemma 6.1.17** For any $\Gamma \in V P_L$ and $A \in \mathcal{L}(\Gamma)$ we have

1. $\Diamond_i A \in \Gamma$ iff ($A \in \Gamma'$ for some $\Gamma' \in R_{Li}(\Gamma)$);
2. $\Box_i A \in \Gamma$ iff ($A \in \Gamma'$ for each $\Gamma' \in R_{Li}(\Gamma)$).

**Proof**

1. 'Only if' follows from 6.1.16. The other way round, if $\Diamond_i A = \neg \Box_i \neg A \notin \Gamma$, then $\Box_i \neg A \in \Gamma$, hence $\neg A \in \Gamma'$ (i.e. $A \notin \Gamma'$) for any $\Gamma' \in R_{Li}(\Gamma)$.

2. 'Only if' follows from Definition 6.1.14. To show 'if', assume $\Box_i A \notin \Gamma$, then $\neg \Box_i A \in \Gamma$, and since $\vdash_{\Gamma} \neg \Box_i A \supset \Diamond_i \neg A$, this implies $\Diamond_i \neg A \in \Gamma$. Hence by (1), $\neg A \in \Gamma'$ for some $\Gamma' \in R_{Li}(\Gamma)$.

**Definition 6.1.18** The canonical Kripke frame with varying domains of an $N$-modal logic $L$ without equality is $VF_L := (V P_L, R_{L1}, \ldots, R_{LN}, D_L)$, where $(D_L)_{\Gamma} := D_{\Gamma}$.
Definition 6.1.19 For an N-modal logic $L$ with equality we define the canonical Kripke frame with equality as

$$V F^e_L := (V P_L, R_{L1}, \ldots, R_{LN}, D_L, \sim_L),$$

where

$$c (\sim_L)_\Gamma d := (c = d) \in \Gamma$$

for $c, d \in D_\Gamma$.

Due to the standard properties of equality 2.6.16(i), (ii), $(\sim_L)_\Gamma$ is an equivalence relation on $D_\Gamma$ (sometimes we denote it just by $\sim_\Gamma$). We also have

$$\Gamma R_{L_i}\Gamma' \land (c = d) \in \Gamma \Rightarrow (c = d) \in \Gamma',$$

since $L \vdash (x = y) \supset \Box_i(x = y)$; thus $\sim_L$ satisfies the $R_{L_i}$-stability condition from Definition 3.5.1.

Definition 6.1.20 The canonical model of an m.p.l. ($=) L$ is

$$V M^e_L := (VF^e_L, \xi_L),$$

where

$$(\xi_L)_\Gamma(P^m_k) := \{c \in (D_\Gamma)^m \mid P^m_k(c) \in \Gamma\}.$$}

Note that $V M^e_L$ is well-defined, since $P^m_k(c_1, \ldots, c_m), (c_1 = d_1), \ldots, (c_m = d_m) \in \Gamma$ implies $P^m_k(d_1, \ldots, d_m) \in \Gamma$, thanks to 2.6.16(v).

The main property of canonical models is the following

Theorem 6.1.21 (Canonical model theorem)

$$V M^e_L, \Gamma \models A \text{ iff } A \in \Gamma$$

for any $L$-place $\Gamma \in V P_L$ and $A \in \mathcal{L}^{(=)}(\Gamma)$.

Proof By induction on the length of $A$.

The base readily follows from the definitions.

The inductive step for classical connectives and quantifiers follows from 6.1.3 (i)–(iii) and 6.1.7 (EP'), properties (i)–(iv), (vi) of Henkin theories. The case $A = \Box_i B$ follows from Lemma 6.1.17 (1).

Corollary 6.1.22 For any modal formula $A$, $V M^e_L \models A$ iff $L \models A$.

Proof If $A$ is a modal sentence, then by Corollary 6.1.10, $A \in L$ implies $A \in \Gamma$ for any $L$-place $\Gamma$; hence $V M^e_L, \Gamma \models A$ by Theorem 6.1.21.

The other way round, if $A \not\models L$, then by 6.1.10, $A \not\models \Gamma$ for some Henkin $(L, S)$-theory $\Gamma$, where $S$ is infinite and small. So $\Gamma$ is an $L$-place, and thus $V M^e_L, \Gamma \not\models A$ by Theorem 6.1.21.

The claim for an arbitrary formula $A$ is reduced to the claim for $\Box A$, since $V M^e_L \models A$ iff $V M^e_L \models \Box A$, and $A \in L$ iff $\Box A \in L$. 


Definition 6.1.23  An m.p.l.\((=)\) \(L\) is called V-canonical if \(VF_L^{(=)} \models L\).

This property is sufficient for completeness:

Corollary 6.1.24  Every V-canonical m.p.l. is strongly Kripke complete. Every V-canonical m.p.l.\(=\) is strongly KFE (or Kripke sheaf) complete.

Proof  In fact, let \(L\) be a V-canonical m.p.l. \((=)\), \(\Gamma_0\) an \(L\)-consistent theory; by Lemma 6.1.9, there exists an \(L\)-place \(\Gamma \supseteq \Gamma_0\). By 6.1.21, \(VM_L^{(=)}\), \(\Gamma \models \Gamma\), so \(\Gamma_0\) is satisfied the \(L\)-frame \(VF_L^{(=)}\).

Now let us consider canonical models for two \(N\)-modal logics \(L_1 \subseteq L_2\). Every \(L_2\)-consistent theory is clearly \(L_1\)-consistent, thus \(VP_{L_1} \subseteq VP_{L_2}\). Moreover, we obtain

Lemma 6.1.25  Let \(L_1 \subseteq L_2\) be \(N\)-m.p.l.\((=)\). Then

1. \(VP_{L_2} = \{\Gamma \in VP_{L_1} | VM_{L_1}, \Gamma \models \overline{L}_2\}\).
2. \(VM_{L_2}\) is a generated submodel of \(VM_{L_1}\).
3. If \(L_1\) is canonical, then \(VF_{L_2} \models L_1\).

Proof  (1) Let \(\Gamma\) be an \(L_1\)-place. Then \(VM_{L_1}, \Gamma \models \overline{L}_2\) iff \(\overline{\overline{\Gamma}} \subseteq \Gamma\) (by the canonical model theorem) iff \(\Gamma\) is \(L_2\)-complete (by 6.1.4(1)). This is equivalent to \(\Gamma \in VP_{L_2}\), since \(\Gamma \in VP_{L_1}\) already enjoys the Henkin property and has a small set of constants.

(2) In fact, \(\Gamma \models \overline{\overline{\Gamma}}\) implies \(\Gamma \models \square_i A\) for any \(A \in \overline{\overline{\Gamma}}\); thus \(\Delta \models \overline{\overline{\Gamma}}\) for any \(\Delta \in R_{L_1}(\Gamma)\). So due to (1), \(VP_{L_2}\) is stable in \(VF_{L_1}\). \(VM_{L_2}\) is a submodel of \(VM_{L_1}\), since \(VM_{L_1}, \Gamma \models P(a)\) iff \(P(a) \in \Gamma\) iff \(VM_{L_2}, \Gamma \models P(a)\).

(3) Follows from (2) and the generation lemma 3.3.18.

Proposition 6.1.26  If \(L\) is a V-canonical \(N\)-m.p.l.\((=)\), \(\Gamma\) is a set of \(N\)-modal pure equality formulas, then \(L_1 = L + \Gamma\) is also V-canonical.

Proof  By Lemma 6.1.25, \(VF_{L_1}\) is a generated subframe of \(VF_L\), so \(L \subseteq ML^{(=)}(VF_{L_1}) \subseteq ML^{(=)}(VF_{L_1})\). On the other hand, \(VM_{L_1} \models \Gamma\) by the canonical model theorem, thus since all formulas in \(\Gamma\) are constant, \(VF_{L_1} \models \Gamma\). Therefore \(VF_{L_1} \models L_1\).

Corollary 6.1.22 shows that \(VM_{L}^{(=)}\) is an exact model for \(L\). The next proposition is a further refinement of this fact.

Proposition 6.1.27

1. Every m.p.l. \((=)\) \(L\) has a countable exact Kripke model (respectively, KFE-model).

2. Moreover, if \(L \supseteq L_0\) for a \(\Delta\)-elementary canonical m.p.l.\((=)\) \(L_0\), such a model exists over an \(L_0\)-frame.
Moreover, every \( L \)-consistent theory is satisfiable in some model described in (2).

**Proof** We can apply Corollary 3.12.11 to \( L = L_0 \), \( M = VM_L^{(=)} \), \( F = VF_L^{(=)} \) and \( u_0 \) such that \( M, u_0 \models \Gamma \) (for a given \( L \)-consistent \( \Gamma \)).

Note that \( F \models L_0 \) by 6.1.33. So by 3.12.11 there exists a countable reliable \( M_0 \subseteq M \) over an \( L_0 \)-frame such that \( MT(M_0) = MT(M) = L \) and \( u_0 \in M_0 \). Then \( M_0, u_0 \models \Gamma \) by reliability.

Obviously, the logics \( \mathbf{QK}_N \) and \( \mathbf{QK}^\sim_N \) are \( \mathcal{V} \)-canonical. Let us also show \( \mathcal{V} \)-canonicity for some other simple modal predicate logics.

For an \( N \)-modal theory \( \Gamma, \alpha \in I_N^{\infty} \) let

\[
\Box^\alpha \Gamma := \{ B \mid \Box^\alpha B \in \Gamma \},
\]

\[
\Diamond^\alpha \Gamma := \{ \Diamond^\alpha A \mid A \in \Gamma \}.
\]

Also let

\[
R_{L \alpha} := R_{L_{i_1} \cdots i_k} := R_{L_{i_1}} \circ \cdots \circ R_{L_{i_k}},
\]

and let \( R_{L,\alpha} \) be the equality relation \( Id_{VP_L} \), cf. Proposition 1.11.5.

**Lemma 6.1.28** Let \( \alpha \in I_N^{\infty}, \Gamma R_{L,\alpha} \Delta \). Then

(1) \( \Box^\alpha \Gamma \subseteq \Delta \),

(2) \( \Diamond^\alpha \Delta \cap L(\Gamma) \subseteq \Gamma \).

**Proof** (1) By induction on the length of \( \alpha \). If \( \alpha = \lambda \), everything is obvious. If (i) holds for \( \beta, \alpha = k\beta \) and \( \Gamma R_{L,\alpha} \Delta \), then there exists \( \Gamma' \) such that \( \Gamma R_{L,\beta} \Gamma', \Gamma' R_{L,\beta} \Delta \). By Definition 6.1.14 and the induction hypothesis we obtain

\[
\Box^\alpha \Gamma = \{ B \mid \Box^\beta B \in \Gamma \} = \{ B \mid \Box^\beta B \in \Box^\alpha \Gamma \} \subseteq \Box^\beta \Gamma' \subseteq \Delta.
\]

(2) Ad absurdum. Suppose \( A \in \Delta \), \( \Diamond^\alpha A \in (\neg \Gamma) \). Then \( \neg \Diamond^\alpha A \in \Gamma \), and thus \( \Box^\alpha \neg A \in \Gamma \). By (1), this implies \( \neg A \in \Delta \), which makes \( \Delta \) inconsistent.

**Theorem 6.1.29** Let \( \Lambda \) be a propositional one-way PTC-logic. Then the logics \( \mathbf{Q}\Lambda, \mathbf{Q}\Lambda^{=} \) are \( \mathcal{V} \)-canonical.

**Proof** Every constant axiom is valid in the canonical frame, since it is true in the canonical model.

If \( A = \Box_k p \supset \Box_\beta p \in L = \mathbf{Q}\Lambda^{(=)} \), then \( R_{L,\beta} \subseteq R_{L,k} \). In fact, suppose \( \Gamma R_{L,\beta} \Delta, \Box_k B \in \Gamma \). Since \( \models L \Box_k B \supset \Box_\beta B \), we have \( \Box_\beta B \in \Gamma \). By Lemma 6.1.28, it follows that \( B \in \Delta \).

By Proposition 1.11.5 and Lemma 3.2.34, we obtain \( VF_L \models A \).
Example 6.1.30 The following counterexample shows that Theorem 6.1.29 does not extend to arbitrary PTC-logics, because they may be $K\Sigma$-incomplete. Consider the logic $K5 := K + \Diamond \Box p \supset \Box p$; its Kripke frames are characterized by the 'Euclidian' condition

\[(\varepsilon) \quad \forall u \forall v \forall w (uRv \land uRw \supset vRw).\]

Then $QK5^=\varepsilon$ is $K\Sigma$-incomplete. To see this, first note that

$$QK5^= \vdash K\Sigma \Diamond AU_1 \supset \Box AU_1.$$ 

In fact, consider a Kripke sheaf model $M$ over a Euclidian frame, and suppose $M, u \vDash \Diamond AU_1$, i.e. for some $v \in R(u) M, v \vDash AU_1$, which means that $|D_v| = 1$. Then $|D_w| = 1$ for any $w \in R(u)$. In fact, by ($\varepsilon$) we have $vRw \land wRv$, so there are transition functions $\rho_{uv}, \rho_{vw}$; and by the properties of Kripke sheaves it follows that they are bijections. So $|D_w| = 1$.

However $QK5^= \nvdash \Diamond AU_1 \supset \Box AU_1$. To see this, consider a Kripke bundle $F$ over the base $F = (W, R)$, where

$$W = \{u, v_1, v_2\}, \quad R = W \times (W - \{u\}),$$

$$D_u = \{a\}, \quad D_{v_1} = \{b\}, \quad D_{v_2} = \{c_1, c_2\}, \quad \rho = D^+ \times (D^+ - \{a\}).$$

It is clear that $F$ is Euclidian. Every level $F_n = (D^n, R^n)$ is also Euclidian, because on $n$-tuples from $D_{v_1}$ or $D_{v_2}$ the relation $R^n$ is equivalent to $\underline{c} \leq \underline{d} \land \underline{c} \leq \underline{d}$ (which is an equivalence relation) and $\underline{a}R^n\underline{d}$ for any $\underline{d} \neq \underline{a}^n$. Therefore $F \vDash QK5^=\varepsilon$ by 5.3.7 and soundness of Kripke bundle semantics. On the other hand, $F \nvdash \Diamond AU_1 \supset \Box AU_1$, since $v_1 \vDash AU_1$ and $v_2 \nvdash AU_1$. Therefore $QK5^= \nvdash \Diamond AU_1 \supset \Box AU_1$ by soundness.

Actually many simple and natural predicate logics are not $V$-canonical. So they are either Kripke-incomplete, or the completeness proof requires more work. One of the options may be to take an appropriate submodel of $VM_L$, for which an analogue of Lemma 6.1.21 still holds.

This leads us to the following definition.

Definition 6.1.31 A relation $R_i \subseteq R_{Li}$ on $VP_L$ is called (i-)selective if for any $L$-place $\Gamma$ and for any $A \in L(\Gamma)$ such that $\Diamond_i A \in \Gamma$, there exists an $L$-place $\Gamma'$ such that $\Gamma R_i \Gamma'$ and $A \in \Gamma'$.

So we obtain an analogue of Lemma 6.1.17:

Lemma 6.1.32 For any selective $R_i$, for any $\Gamma \in VP_L$ and $A \in L(\Gamma)$ we have

1. $\Diamond_i A \in \Gamma$ iff ($A \in \Gamma'$ for some $\Gamma' \in R_i(\Gamma)$);
2. $\Box_i A \in \Gamma$ iff ($A \in \Gamma'$ for each $\Gamma' \in R_i(\Gamma)$).

Proof The same as in 6.1.17, but ‘only if’ in (1) now follows from 6.1.31. ■
Lemma 6.1.16 means that the relation \( R_{Li} \) is \( i \)-selective, and thus it is the greatest \( i \)-selective relation on \( VP_L \).

Now similarly to the canonical frame, we can introduce quasi-canonical Kripke frames \((VP_L, R_1, \ldots, R_N, D_L [], \preceq_L)\) with arbitrary \( i \)-selective relations \( R_i \) and the corresponding quasi-canonical models (with the valuation \( \xi_L \)). Then we obtain the following analogues of Lemma 6.1.21 and Corollary 6.1.22:

**Lemma 6.1.33** Let \( M \) be a quasi-canonical Kripke model for an m.p.l.(=) \( L \). Then

1. for any \( L \)-place \( \Gamma \) and \( A \in \mathcal{L}(\Gamma) \)
   
   \[ M, \Gamma \models A \iff A \in \Gamma; \]

2. for any formula \( A \)
   
   \[ M \models A \iff L \vdash A. \]

Another option is to use subsets of \( VP_L \) still satisfying Corollary 6.1.10 (and thus, Corollary 6.1.22). Various modifications will be considered in this chapter later on.

### 6.2 Canonical models for superintuitionistic logics

Now let us turn to the intuitionistic case. We would again like to represent possible worlds in Kripke (or other) models by sets of formulas. But now we should take false formulas into account as well, because in intuitionistic logic the falsity of \( A \) is not equivalent to the truth of \( \neg A \). For this purpose we need double theories introduced in 2.7.12.

**Definition 6.2.1** If \( L \) is a superintuitionistic logic (with or without equality), a theory \((I', A)\) is called \( L \)-consistent if \((r, A) \models_l \), i.e.

\[ \forall L \bigwedge \Gamma_1 \supset \bigvee \Delta_1 \]

for any finite \( \Gamma_1 \subseteq \Gamma \), \( \Delta_1 \subseteq \Delta \).

It is obvious that \( \Gamma \cap \Delta = \emptyset \) whenever \((\Gamma, \Delta)\) is \( L \)-consistent.

**Definition 6.2.2** An \( L \)-consistent theory \((\Gamma, \Delta)\) is called \( L \)-complete if \( \Gamma \cup \Delta = \mathcal{L}(\Gamma, \Delta) \)

So in this case \( \Delta = \mathcal{L}(\Gamma, \Delta) - \Gamma \), which is abbreviated as \((-\Gamma)\). Also note that for any \( c \in \mathcal{L}(\Gamma, -\Gamma) \) we have \( (P(c) \supset P(c)) \in \Gamma \), since otherwise \( (P(c) \supset P(c)) \in (-\Gamma) \), while the theory \((\emptyset, \{P(c) \supset P(c)\})\) is inconsistent. So it follows that \( D_{-\Gamma} \subseteq D_{\Gamma} \). A similar argument using \( P(c) \wedge \neg P(c) \) shows that \( D_{\Gamma} \subseteq D_{-\Gamma} \).

So we can write \( D_{\Gamma} \) rather than \( D_{(\Gamma, -\Gamma)} \).

Moreover, we shall often say that \( \Gamma \) (rather than \((\Gamma, -\Gamma)\)) is an \( L \)-complete intuitionistic theory.
Definition 6.2.3 An L-complete intuitionistic theory $\Gamma$ is called \(L3\)-complete if it satisfies the existence property:

\[(EP) \text{ if } \exists x A(x) \in \Gamma, \text{ then } A(c) \in \Gamma \text{ for some } c \in D_{\Gamma}.
\]

More specifically, an \(L3\)-complete $\Gamma$ is called \((L3, S)\)-complete if $D_{\Gamma} = S$.

Definition 6.2.4 An \(L3\)-complete (respectively, an \((L3, S)\)-complete) theory $\Gamma$ is called \(L3\V\)-complete (respectively, an \(L3\V, S\)-complete) if it satisfies the following coexistence property:

\[(Av) \text{ if } \forall x A(x) \in (-\Gamma) \text{ and } A(d) \in \Gamma \text{ for all } d \in D_{\Gamma}, \text{ then } \forall x (A(x) \lor C) \in \Gamma \text{ for some formula } C \in (-\Gamma).
\]

The intended meaning of this property (after we prove the canonical model theorem) is the following. If at a world $F$ of the canonical model $VxA(x)$ is false, but $A(d)$ is true for every individual $d$, then obviously $\forall x A(x)$ must be false at some strictly accessible world. $\,(Av)$ allows us to find a formula $C$ such that $\Gamma \Vdash \forall x (A(x) \lor C)$; so $C$ should be true at all strongly accessible worlds, where $A(d)$ is refuted for some individual $d$.

For example, this property holds if $C$ exactly defines the set of all strictly accessible worlds (and the canonical model theorem holds).

For a logic $L$ containing the formula $CD$ the condition $(Av)$ may be simplified:

\[(Ac) \text{ if } \forall x A(x) \in (-\Gamma), \text{ then } A(d) \in (-\Gamma) \text{ for some } d \in D_{\Gamma}.
\]

In fact, let $CD \in L$. If $(Av)$ holds and $\forall x A(x) \in (-\Gamma)$, but $A(d) \in \Gamma$ for all $d \in D_{\Gamma}$, then $\forall x (A(x) \lor C) \in \Gamma$ for some $C \in (-\Gamma)$. Since $CD \in \Gamma$, we obtain $(\forall x A(x) \lor C) \in \Gamma$, which leads to a contradiction. Thus $(Ac)$ holds.

Conversely, if $(Ac)$ holds then the premise of $(Av)$ is false, so $(Av)$ holds.

Note that $(Ac)$ is an analogue to the property $(\forall)$ of forcing in Kripke models with constant domains.

Since the Gödel-Tarski translation for $\exists$ is just $\exists$, the definition of intuitionistic forcing for the $\exists$-case is the same as classical. This explains, why the existence property is defined in the same way. On the other hand, the intuitionistic $\forall$ translates as modal $\Box \forall$, and thus the intuitionistic definition for the $\forall$-case consists of the 'classical' part dealing with individuals from a certain world and of the 'modal' part dealing with accessibility relations between worlds. So property $(Av)$ is responsible for the classical part of intuitionistic universal quantification. In the case of constant domains $(\forall x B)^T$ is equivalent to $\forall x B_T$, and thus $(Ac)$ is just the dual of $(E)$.

$\exists\forall$-complete theories seem to be a better intuitionistic analogue of Henkin theories than $\exists$-complete theories. But the latter can be used for completeness proofs in the simplest cases (for example, for $QH^{(m)}$).

Lemma 6.2.5 Every $L$-complete theory $\Gamma$ has properties 6.1.3(i), (iii), 6.1.2(2):

- $(A \land B) \in \Gamma$ iff $(A \in \Gamma$ and $B \in \Gamma)$;
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- \((A \lor B) \in \Gamma \iff (A \in \Gamma \text{ or } B \in \Gamma)\);
- \(\Gamma \vdash_{L} A \iff A \in \Gamma\) (for \(A \in \mathcal{L}^{(=)}(\Gamma)\)).

Every \(L_{3}\)-complete \(\Gamma\) also satisfies 6.1.7(EP'):

- \(\exists x A(x) \in \Gamma \iff (A(c) \in \Gamma \text{ for some } c \in D_{\Gamma})\).

**Proof**  An exercise.  

Now let us prove the crucial property of \(L_{3}\)-complete theories analogous to Lemma 6.1.9 for Henkin theories.

For intuitionistic theories \((\Gamma, \Delta), (\Gamma', \Delta')\) put

\[(\Gamma, \Delta) \preceq (\Gamma', \Delta') := \Gamma \subseteq \Gamma' \& \Delta \subseteq \Delta'.\]

For a complete theory \(\Theta\) also let

\[(\Gamma, \Delta) \preceq \Theta := \Gamma \subseteq \Theta \& \Delta \cap \Theta = \emptyset,\]

which means \((\Gamma, \Delta) \preceq (\Theta, -\Theta)\).

**Lemma 6.2.6** Let \((\Gamma, \Delta)\) be an \(L\)-consistent theory, \(D_{(\Gamma, \Delta)} = S \subseteq S', \ |S'| = |S' - S| = \aleph_{0}\). Then there exists an \((L_{3}, S')\)-complete (and even an \((L_{3}V, S')\)-complete) theory \(\Theta\) such that \((\Gamma, \Delta) \preceq \Theta\).

**Proof** The idea of the proof is quite standard. Namely, we can successively extend the pair \((\Gamma, \Delta)\) by adding \(S'\)-sentences either to \(\Gamma\) or to \(\Delta\). Moreover, together with adding \(\exists x A(x)\) to \(\Gamma\), we always add \(A(c)\) for some new constant \(c\) from \((S' - S)\). And together with adding \(\forall x A(x)\) to \(\Delta\), we add \(A(c)\) for a new \(c\) — but only if this is consistent. This construction provides the existence and the coexistence properties. If we need only an \(L_{3}\)-complete theory \(\Theta \succeq (\Gamma, \Delta)\), we can use the same procedure, but without mentioning \(\forall\)-formulas.

Let \(IF^{(=)}(S') = \{B_{k} \mid k \in \omega\}\). Choose distinct constants \(c_{k}\) from \((S' - S)\) such that \(c_{k}\) does not occur in \(B_{0}, \ldots, B_{k}\). Next, define a sequence of \(L\)-consistent theories

\[(\Gamma, \Delta) = (\Gamma_{0}, \Delta_{0}) \preceq \ldots \preceq (\Gamma_{k}, \Delta_{k}) \preceq \ldots\]

as follows.

- If \(B_{k} = \exists x A(x)\), then

\[(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} \{ (\Gamma_{k} \cup \{B_{k}, A(c_{k})\}, \Delta_{k}) \mid \text{ if } (\Gamma_{k} \cup \{B_{k}\}, \Delta_{k}) \text{ is } L\text{-consistent,} \} \\ \{ (\Gamma_{k}, \Delta_{k} \cup \{B_{k}\}) \} \text{ otherwise} \end{cases}\]

- If \(B_{k}\) is not of the form \(\exists x A(x)\), then

\[(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} \{ (\Gamma_{k} \cup \{B_{k}\}, \Delta_{k}) \mid \text{ if } (\Gamma_{k} \cup \{B_{k}\}, \Delta_{k}) \text{ is } L\text{-consistent,} \} \\ \{ (\Gamma_{k}, \Delta_{k} \cup \{B_{k}\}) \} \text{ otherwise.} \end{cases}\]
Let us show that \((\Gamma_{k+1}, \Delta_{k+1})\) is \(L\)-consistent if \((\Gamma_k, \Delta_k)\) is.

First, if \((\Gamma_k \cup \{B_k\}, \Delta_k)\) and \((\Gamma_k, \Delta_k \cup \{B_k\})\) are both inconsistent, then \((\Gamma_k, \Delta_k)\) is also inconsistent. In fact, in this case

\[
\Gamma_k \vdash_L B_k \supset \bigvee \Delta', B_k \lor \bigvee \Delta''
\]

for some finite \(\Delta', \Delta'' \subseteq \Delta_k\), and thus

\[
\Gamma_k \vdash_L B_k \supset \bigvee \Delta, B_k \lor \bigvee \Delta,
\]

where \(\Delta = \Delta' \cup \Delta''\). Now since \(p \supset q\), \(p \lor q \vdash \Delta q\), we obtain \(\Gamma_k \vdash_L \bigvee \Delta\), i.e. \((\Gamma_k, \Delta_k)\) is inconsistent.

Second, if \(B_k = \exists x A(x)\) and \((\Gamma_k \cup \{\exists x A(x), A(c_k)\}, \Delta_k)\) is inconsistent, then

\[
\Gamma_k \vdash_L A(c_k) \supset \bigvee \Delta
\]

for some finite \(\Delta \subseteq \Delta_k\); therefore by 2.7.11

\[
\Gamma_k \vdash_L \forall x(A(x) \supset \bigvee \Delta),
\]

since \(c_k\) does not occur in \(\Gamma_k \cup \Delta_k\), which implies

\[
\Gamma_k \vdash_L \exists x A(x) \supset \bigvee \Delta
\]

by 2.6.10 and MP, and thus \((\Gamma_k \cup \{B_k\}, \Delta_k)\) is inconsistent.

Next, put \(\Theta := \bigcup_{k \in \omega} \Gamma_k\). By construction, the theory \((\Theta, -\Theta)\) is \(L\)-consistent and has the existence property.

To obtain an \(L\exists\forall\)-complete theory, we should slightly refine the above construction. Viz., if \(B_k = \forall x A(x)\), we put

\[
(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} 
(\Gamma_k \cup \{B_k\}, \Delta_k) & \text{if this theory is } L\text{-consistent}, \\
(\Gamma_k, \Delta_k \cup \{B_k, A(c_k)\}) & \text{if this theory is } L\text{-consistent and} \\
(\Gamma_k \cup \{B_k\}, \Delta_k) & \text{is } L\text{-inconsistent,} \\
(\Gamma_k, \Delta_k \cup \{B_k\}) & \text{otherwise.}
\end{cases}
\]

In other words, we add \(B_k\) either to \(\Gamma_k\) or to \(\Delta_k\) keeping consistency, and also add \(A(c_k)\) to \(\Delta_k\) whenever this is possible.

The consistency of \((\Gamma_k, \Delta_k)\) again implies the consistency of \((\Gamma_{k+1}, \Delta_{k+1})\), since either \((\Gamma_k \cup \{B_k\}, \Delta_k)\) or \((\Gamma_k, \Delta_k \cup \{B_k\})\) is consistent, as we have seen above.

Finally, let us check \((\forall \forall)\) for \(\Theta\). Suppose \(B_k = \forall x A(x) \notin \Theta\) and \(A(c) \in \Theta\) for all \(c \in D_\Theta\). Then \(B_k \in \Delta_{k+1}\), and \(A(c_k) \notin \Delta_{k+1}\), so by construction, \((\Gamma_k, \Delta_k \cup \{B_k, A(c_k)\})\) is \(L\)-inconsistent, and thus

\[
\Gamma_k \vdash_L \bigvee \Delta \lor A(c_k)
\]
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for some finite $\Delta \subseteq \Delta_k$. Hence by 2.7.11

$$\Gamma_k \vdash_L \forall x(\bigvee \Delta \vee A(x)).$$

Since $\Theta$ is $L^3$-complete, it satisfies 6.1.3(v), so it follows that $\forall x(C \vee A(x)) \in \Theta$, where $C = \bigvee \Delta \in (-\Theta)$. ■

**Lemma 6.2.7** Let $A$ be an intuitionistic sentence, $|S| = \aleph_0$. Then:

$$L \vdash A \iff (A \in \Gamma \text{ for any } (L, S)\text{-place } \Gamma)$$

$$\iff (A \in \Gamma \text{ for any } (L\forall, S)\text{-place } \Gamma).$$

**Proof** Cf. Lemma 6.1.10 for the modal case. ■

Let $\Gamma|S := \Gamma \cap IF_S^{(=)}$ for $S \subseteq S'$ and $\Gamma \subseteq IF_S^{(=)}$. We also fix a denumerable universal set of constants $S^*$ and consider small subsets of $S^*$ as in Section 7.1.

**Definition 6.2.8** An $L$-place (respectively, an $L\forall$-place) is an $L^3$-complete (respectively, an $L^3\forall$-complete) theory with a small set of constants.

$VPL$ and $UPL$ denote the sets of all $L$-places and $L\forall$-places respectively.

Obviously, $UPL \subseteq VPL$. We also have an analogue of Lemma 6.1.12:

**Lemma 6.2.9** Let $u$ be a world in an intuitionistic Kripke model $M$ over a frame $F = (F, D)$ and assume that $M \models L$. Then

$$\Gamma_u := \{A \in IF^{(=)}(D_u) \mid M, u \models A\}$$

is an $L^3$-complete theory.

**Proof** Almost obvious. As for $L$-consistency, note that if $A_i \in \Gamma_u$ for $i = 1, \ldots, k$ and $L \vdash \bigwedge_{i=1}^k A_i \supset \bigvee_{j=1}^m B_j$, then the latter implication is true in $M$, and so $M, u \models B_j$ for some $j$. Thus $(\Gamma_u, -\Gamma_u)$ is $L$-consistent. ■

Note that $\Gamma_u$ is not necessarily $L\forall$-complete.

Similarly to the modal case we define canonical maps:

**Definition 6.2.10** Let $M$ be a Kripke model for a superintuitionistic logic $L$, in which every individual domain is a small subset of $S^*$. The map from (the worlds of) $M$ to $VPL$ sending $u$ to $\Gamma_u$ is called canonical and denoted by $\nu_{M,L}$ (or by $\nu_{M}$, or by $\nu$).

Lemma 6.2.9 also extends to Kripke sheaves, bundles and to metaframe models in modal metaframes. The intuitionistic canonical model theorem (see below) shows that $VPL$ is the set of all $L$-places $\Gamma_u$ in Kripke models for $L$ with small domains.

Now let us define some accessibility relations on $VPL$ and $UPL$.

\[\text{For intuitionistic metaframes the situation may be different.}\]
Definition 6.2.11 For $\Gamma, \Gamma' \in V P_L$, we say that

- $\Gamma'$ is conservative over $\Gamma$ (notation: $\Gamma \preceq \Gamma'$) if $\Gamma = \Gamma' \cup D_{\Gamma}$;
- $\Gamma'$ is proper over $\Gamma$ (notation: $\Gamma < \Gamma'$) if $\Gamma \subset \Gamma' \cup D_{\Gamma}$.

Let $\leq$ be the reflexive closure of $<$, i.e.

$$\Gamma \leq \Gamma' \iff \Gamma < \Gamma' \text{ or } \Gamma = \Gamma'.$$

The inclusion relation $\subseteq$ on $V P_L$ (i.e. the set $\{(\Gamma, \Gamma') \in V P^2_L \mid \Gamma \subseteq \Gamma'\}$) is also denoted by $R_L$.

It is clear that $\Gamma \subseteq \Gamma'$ iff $\Gamma < \Gamma' \lor \Gamma \subseteq \Gamma'$.

Definition 6.2.12 Let $W \subseteq V P_L$. A partial ordering $R \subseteq R_L$ is called selective on $W$ if for any $\Gamma \in V P_L$ the following two conditions hold:

(>) if $(A_1 \supseteq A_2) \in (-\Gamma)$, then $A_1 \in \Gamma'$ and $A_2 \notin \Gamma'$ for some $\Gamma' \in R(\Gamma) \cap W$;

(∀) if $\forall x A(x) \in (-\Gamma)$, then $A(c') \notin \Gamma'$ for some $\Gamma' \in R(\Gamma) \cap W$ and some $c' \in D_{\Gamma'}$.

These conditions can be rewritten in an equivalent form:

(>) $(A_1 \supseteq A_2) \in \Gamma$ iff for any $\Gamma' \in R(\Gamma) \cap W$ ($A_1 \in \Gamma'$ and $A_2 \notin \Gamma'$);

(∀) $\forall x A(x) \in \Gamma$ iff $A(c') \in \Gamma'$ for any $\Gamma' \in R(\Gamma) \cap W$, $c' \in D_{\Gamma'}$.

Lemma 6.2.13

(1) The relation $R_L$ is selective on $V P_L$ and $U P_L$.

(2) The relation $\leq$ is selective on $U P_L$.

Proof (1) (>) If $(A_1 \supseteq A_2) \notin \Gamma$, then $\Gamma \nvdash_L (A_1 \supseteq A_2)$ (Lemma 6.2.5), i.e. the theory $\langle \Gamma \cup \{A_1\}, \{A_2\} \rangle$ is $L$-consistent. So by Lemma 6.2.6, there exists an $L\forall$-place $\Gamma' \succeq (\Gamma \cup \{A_1\}, \{A_2\})$. Thus $\Gamma \subseteq \Gamma'$, $A_1 \in \Gamma$, $A_2 \notin \Gamma'$.

(∀) Suppose $\forall x A(x) \notin \Gamma$, then $\Gamma \nvdash_L \forall x A(x)$. Take $c \in (S^* - D_\Gamma)$; then the theory $\langle \Gamma, \{A(c)\} \rangle$ is $L$-consistent (otherwise $\Gamma \vdash_L A(c)$, which implies $\Gamma \vdash_L \forall x A(x)$). Hence by Lemma 6.2.6, there exists $\Gamma' \in U P_L$ such that $\Gamma' \succeq (\Gamma, \{A(c)\})$, i.e. $\Gamma' \supseteq \Gamma$ and $A(c) \notin \Gamma'$.

(2) (>) Suppose $(A_1 \supseteq A_2) \in (-\Gamma)$ and consider two cases.

(a) If $A_1 \notin \Gamma$, then by (1) we obtain $\Gamma' \supseteq \Gamma$ such that $A_1 \in \Gamma'$, $A_2 \notin \Gamma'$. So $A_1 \in ((\Gamma') \cup D_{\Gamma'} - \Gamma)$, and thus $\Gamma < \Gamma'$.

(b) If $A_1 \in \Gamma$, then to show (IV), we can take $\Gamma' = \Gamma$, since $A_2 \vdash_L (A_1 \supseteq A_2)$, and thus $(A_1 \supseteq A_2) \notin \Gamma$ implies $A_2 \notin \Gamma$.

(∀) Suppose $\forall x A(x) \in (-\Gamma)$ and consider two cases.

(a) If $A(c) \notin \Gamma$ some $c \in D_\Gamma$, there is nothing to prove: one can take $\Gamma' = \Gamma$.

(b) If $A(c) \in \Gamma$ for all $c \in D_\Gamma$, we can apply 6.2.4 (Av). So there exists a formula $C \in (-\Gamma)$ such that $\forall x (A(x) \lor C) \in \Gamma$. Now as in the proof of (1), we
obtain $\Gamma' \supseteq \Gamma$ and $e \in (D_{\Gamma'} - D_{\Gamma})$ such that $A(e) \notin \Gamma'$. Since $\forall x(A(x) \lor C) \in \Gamma \subseteq \Gamma'$ and $\forall x(A(x) \lor C) \vdash_L A(e) \lor C$, we also have $(A(e) \lor C) \in \Gamma'$, and thus $C \in \Gamma'$ by Lemma 6.2.5 (iii). Hence $C \in (\Gamma'|D_{\Gamma'}) - \Gamma$, and eventually $\Gamma < \Gamma'$.

Note that the previous argument fails for $VPL$.

**Definition 6.2.14** The canonical frame of a superintuitionistic predicate logic $L$ is defined as $VF_L := (VPL, RL, DL)$, where

$$\Gamma RL \Gamma' := \Gamma \subseteq \Gamma', \ (DL)_{\Gamma} := D_{\Gamma}$$

similarly to the modal case (cf. Definition 6.1.18).

The canonical frame with equality of an s.p.l.(=) $L$ is

$$VF_L^{=} := (VPL, RL, DL, \approx_L),$$

where, as in Definition 7.1.10,

$$c(\approx_L)_\Gamma d := (c = d) \in \Gamma.$$ 

The stability condition for $\approx_L$ holds trivially, due to the definition of $RL$.

**Definition 6.2.15** The canonical model of an s.p.l.(=) $L$ is $VM^{=} := (VF^{=}^{}, \xi_L)$, where

$$(\xi_L)_{\Gamma}(P^m_k) := \{c \in (D_{\Gamma})^m \mid P^m_k(c) \in \Gamma\}.$$ 

This valuation is obviously intuitionistic.

**Definition 6.2.16** We also introduce two kinds of quasi-canonical frames and models:

$$UF^{=} := VF^{=}^{, }UPL, \ U^{=} := VM^{=}^{, }UPL,$$

$U \leq F^{=}$ (respectively, $U \leq M^{=}$) is the same as $UF^{=}$ (respectively, $UM^{=}$), but with the relation $\leq$ instead of $RL$.

**Theorem 6.2.17** (Canonical model theorem) Let $L$ be an s.p.l.(=). Then for any $L$-place $\Gamma$ and $A \in L(\Gamma)$,

$$VM^{=}^{, }_{\Gamma} \models A \text{ iff } A \in \Gamma;$$

similarly for both quasi-canonical models $UM_L, U \leq M_L$ and for all their selective submodels.

**Proof** By induction on the length of $A$ (cf. Lemma 6.1.21). The inductive step for $\wedge, \lor, \exists$ uses Lemma 6.2.5. In the cases of $\supset$ and $\forall$ use 6.2.12 ($\supset$), ($\forall$).
Corollary 6.2.18 If $M = VM_L^{(=)}$, $UM_L^{(=)}$ or $U \subseteq M_L^{(=)}$, then

\[ M \models A \iff L \models A \]

for any $A \in IF^{(=)}$.

Now we obtain an analogue of 6.1.25:

Lemma 6.2.19 Let $L_1 \subseteq L_2$ be s.p.l.($\cong$). Then

(1) $VP_{L_2} = \{ \Gamma \in VP_{L_1} \mid VM_{L_1}(, \Gamma) \models L_2 \}$,

(2) $VM_{L_2}$ is a generated submodel of $VM_{L_1}$.

Proof Almost the same as in the modal case. The proof of (1) uses a slightly different argument about consistency. Viz., let us show the $L_2$-consistency for an $L_1$-place $\Theta \supseteq L_2$. Suppose the contrary: $L_2 \models \bigwedge \Delta \supset \bigvee \nabla$ for finite $\Delta \subseteq \Theta$, $\nabla \subseteq -\Theta$; then $\Xi \models RH \bigwedge \Delta \supset \bigvee \nabla$ for a finite $\Xi \subseteq L_2 \subseteq \Theta$, and hence $QH \models \bigwedge (\Delta \cup \Xi) \supset \bigvee \nabla$, thus $(\Theta, -\Theta)$ is $L_1$-inconsistent, which contradicts the assumption that $\Theta$ is an $L_1$-place.

For the proof of (2) note that $\Gamma \models L_2$ implies $\Delta \models L_2$ for any $\Delta \supseteq \Gamma$ since the model is intuitionistic.

Definition 6.2.20 An s.p.l.$(\cong)$ $L$ is called $V$-canonical if $L \subseteq IL^{(=)}(VF_L^{(=)})$. $U$-canonicity and $U \leq$-canonicity are defined analogously.

Similarly to 6.1.24, we obtain

Corollary 6.2.21 Every (quasi-) canonical superintuitionistic logic is strongly Kripke complete (Kripke sheaf complete in the case of a logic with equality).

By definition we readily have

Proposition 6.2.22 The logics $QH$, $QH^\cong$ are $V$-canonical.

We also have an analogue of Proposition 6.1.26:

Proposition 6.2.23 If $L$ is a $V$-canonical s.p.l.$\cong$, $\Gamma$ is a set of pure equality intuitionistic sentences, then $L + \Gamma$ is also $V$-canonical.

Proof The same as in the modal case, but using Lemma 6.2.19.

Corollary 6.2.24 The logics with decidable and with stable equality $QH^d = QH^\cong + DE$ and $QH^s = QH^\cong + SE$ are $V$-canonical.

Therefore, we obtain the following completeness result.

Theorem 6.2.25

(1) Intuitionistic predicate logic $QH$ is strongly Kripke frame complete.
(2) The logics with equality $\mathsf{QH}^=, \mathsf{QH}^{-d}, \mathsf{QH}^{-s}$ are strongly Kripke sheaf complete.

Note that (ii) can also be deduced from (i) by applying Theorems 3.8.8, 3.8.4.

Remark 6.2.26 In Section 6.4 we shall show that $\mathsf{QH}^{-d}$ is actually Kripke-complete.

However $V$-canonicity is rare. In general one may expect that superintuitionistic predicate logics with simple axioms are not $V$-canonical. For example, classical logic $\mathsf{QCL}$ is obviously not $V$-canonical; in fact, every $\mathsf{QCL}$-place (as well as every $L$-place for any $L$) is not maximal in the corresponding canonical frame, since it may be extended after adding new constants to the language. Another trivial observation is that $\mathsf{QCL} \vdash CD$, but $CD$ is not valid in $VF_{\mathsf{QCL}}$, since as we have just noted, every $\mathsf{QCL}$-place sees another place with a larger domain.

Moreover, the same argument shows that $VF_L$ is of infinite depth (for any logic $L$). Thus every superintuitionistic logic containing a propositional finite depth formula $P_k$, $k \geq 1$ (see Section 1.1.2) is not $V$-canonical.

Nevertheless for logics with additional axioms listed in Chapter 2 we can sometimes prove $U^\leq$-canonicity. Let us now consider such examples.

6.3 Intermediate logics of finite depth

Let us begin with some simple, but useful remarks.

Lemma 6.3.1 Let $L$, $L'$ be superintuitionistic logics, $(\Gamma, \Delta)$ an intuitionistic theory.

(1) If $(\Gamma, \Delta)$ is $\mathsf{QH}$-consistent, $\Gamma \subseteq \Gamma$, then $(\Gamma, \Delta)$ is $L$-consistent.

(2) If $(\Gamma, \Delta)$ is $L\exists$-complete, then it is $L'\exists$-complete iff $\Gamma \subseteq \Gamma$.

(3) If $(\Gamma, \Delta)$ is $L\forall$-complete, then it is $L'\forall$-complete iff $L' \subseteq \Gamma$.

Proof (i) Suppose $(\Gamma, \Delta)$ is $L$-inconsistent, $\Gamma \subseteq \Gamma$, and let us show that $(\Gamma, \Delta)$ is $\mathsf{QH}$-inconsistent. We have $\Gamma \models \Gamma \cup \bigvee \Delta_1$ for some finite $\Gamma_1 \subseteq \Gamma_1, \Delta_1 \subseteq \Delta_1$. So a generator $A$ of this formula is in $L$; hence $\forall A \in \Gamma \subseteq \Gamma$. Thus $\Gamma \models \mathsf{QH} A$, and so $\Gamma \models \mathsf{QH} \Gamma \models \bigvee \Delta_1$, which implies $\Gamma \models \mathsf{QH} \bigvee \Delta_1$, i.e. the $\mathsf{QH}$-inconsistency of $(\Gamma, \Delta)$.

(ii) (If.) Assume that $\Gamma \subseteq \Gamma$ and $(\Gamma, \Delta)$ is $L\exists$-complete. Then it is $\mathsf{QH}$-consistent, and thus $L'$-consistent, by (i). Now $L'$-completeness and the existence property readily follow from $L\exists$-completeness.

(Only if.) Suppose $\Gamma \not\subseteq \Gamma$, and take $A \in \Gamma - \Gamma$. Since $\mathsf{QH} \models \top \supset A$, it follows that $\Gamma \models \top \supset A$, and thus $(\Gamma, \Delta)$ is $L'$-inconsistent.

(iii) The proof is similar to (ii).
Remark 6.3.2 Every QCL-place Γ has all the properties from Lemma 6.1.3 (with \( L = \text{QCL} \)), so it obviously satisfies the conditions 6.2.12 (\( \supset \)), (\( \forall \)). In both cases we can take \( \Gamma' = \Gamma \). In fact, \( (A_1 \supset A_2) \not\in \Gamma \) implies \( \neg(A_1 \supset A_2) \in \Gamma \), and thus \( A_1 \in \Gamma, A_2 \not\in \Gamma \). Next, if \( \forall x A(x) \not\in \Gamma \), then \( \neg \forall x A(x) \in \Gamma \), and thus \( \exists x \neg A(x) \in \Gamma \), by classical logic (cf. 2.6.17(ii)). Hence by the existence property, for some \( c \in S \) we have \( \neg A(c) \in \Gamma \), i.e. \( A(c) \not\in \Gamma \).

Now let us consider Kuroda's axiom. Recall that \( KF \) is valid in Kripke frames \( F = (W, R, D) \) with the McKinsey property:

\[
\forall u \in W \exists v \in W (uRv \& v \text{ is maximal}),
\]

and similarly for Kripke sheaves. As noted at the end of the previous section, there are no maximal worlds in the canonical frame \( VF_L \), so \( VF_L \) does not have the McKinsey property (for any logic \( L \)). However let us show that the McKinsey property holds in \( \mathcal{U} < F \mathcal{L} \) whenever \( L \) contains \( KF \).

Lemma 6.3.3 Let \( L \) be an s.p.1. (=) containing \( KF \), \( (\Gamma, \emptyset) \) an \( L \)-consistent theory. Then \( (\Gamma, \emptyset) \) is QCL-consistent.

Proof In fact, suppose \( \vdash_{\text{QCL}} \neg(\bigwedge \Gamma_1) \). Then \( (\bigwedge \Gamma_1) \) has a generator \( A \) such that \( \neg A \in \text{QCL} \). By the Glivenko Theorem 2.12.1, then \( \neg A \in \text{QH} + KF \subseteq L \), and thus \( \vdash_{L} \neg(\bigwedge \Gamma_1) \). So the QCL-inconsistency of \( (\Gamma, \emptyset) \) implies its \( L \)-inconsistency.

Lemma 6.3.4 Let \( L \) be an s.p.1. (=).

1. \( \Gamma \) is a QCL-place iff \( \Gamma \) is maximal in \( U\leq F_L \).

2. If \( KF \in L \), then for any \( L \)-place \( \Gamma \) there exists an QCL-place \( \Gamma' \geq \Gamma \).

Proof

1. Every QCL-complete theory is maximal among consistent theories in the same language, so every extension of \( \Gamma \) is conservative. By 6.3.2, \( \Gamma \) is a QCL\( ^{\forall} \)-place, i.e. \( \Gamma \in UP_L \).

2. If already \( \text{QCL} \subseteq \Gamma \), then according to 6.3.2, \( \Gamma \) is a QCL\( ^{\forall} \)-place, so we can take \( \Gamma' = \Gamma \).

The other way round, if an \( L^{\forall} \)-place \( \Gamma \) is not maximal in \( U\leq F_L \), then \( \Gamma \prec \Delta \) for some \( \Delta \in UP_L \), so there exists \( A \in ((\Delta|D_{\Gamma}) - \Gamma) \). Then \( A = B(a) \) for some \( B(x) \in IF(=) \) and a tuple \( a \) from \( D_{\Gamma} \). By 6.2.17 it follows that \( U\leq M_L^{=} (\Gamma), \Gamma \not\models A \lor \neg A \), and thus

\[
U\leq M_L^{=} (\Gamma), \Gamma \not\models \forall x(B(x) \lor \neg B(x)).
\]

Hence \( \forall x (B(x) \lor \neg B(x)) \not\in \Gamma \), and thus \( \text{QCL} \subseteq \Gamma \) (again by 6.2.17).

Otherwise there exists \( B \in (\text{QCL} - \Gamma) \). Obviously, the theory \( (\Gamma, \emptyset) \) is \( L \)-consistent, and thus QCL-consistent by 6.3.3. By Lemma 6.2.6, there exists a QCL\( ^{\forall} \)-place \( \Gamma' \supseteq \Gamma \). We also have \( \Gamma \prec \Gamma' \) since \( B \in (\Gamma' - \Gamma) \cap IF \).
The argument for logics with equality is similar.

So we readily obtain completeness:

**Theorem 6.3.5** The logic $\text{QH} + KF$ (respectively, $\text{QH}^− + KF$) is strongly determined by Kripke frames (respectively, Kripke sheaves) satisfying the McK- insey property.

Now let us prove completeness for intermediate logics $\text{QHP}_k^+$ introduced in Section 2.4. Recall that $\text{QHP}_k^+ = \Delta^{k-1}(\text{QCL})$, where the logic $\Delta L$ is axiomatised by formulas $\delta A = p \lor (p \supset A)$ for $A \in L$, $p$ not occurring in $A$; $\text{QHP}_k^+$-frames are the intuitionistic frames of depth $\leq k$.

**Lemma 6.3.6** If $\Gamma$, $\Gamma'$ are $L$-places, $\Gamma < \Gamma'$ and $\Delta L \subseteq \Gamma$, then $L \subseteq \Gamma'$.

**Proof** Let $B \in L(\Gamma) \cap (\Gamma' - \Gamma)$ and suppose $L \not\subseteq \Gamma'$, $A \in L - \Gamma'$. Then $B \lor (B \supset A)$ is a substitution instance of $\delta A$, so $B \lor (B \supset A) \in \Delta L$.

On the other hand, $(B \supset A) \not\in \Gamma$; in fact, otherwise $(B \supset A) \in \Gamma \subseteq \Gamma'$, which together with $B \in \Gamma'$, implies $A \in \Gamma'$ contradicting the choice of $A$. Now since also $B \not\in \Gamma$, we obtain $B \lor (B \supset A) \not\in \Gamma$, by Lemma 6.2.5.

Hence it follows that $\Delta L \not\subseteq \Gamma$, in a contradiction to the assumption.

**Lemma 6.3.7** If $\text{QHP}_k^+ \subseteq L$, then the frame $U \subseteq F_L$ is of depth $\leq k$.

**Proof** Suppose $\Gamma_1 < \ldots \Gamma_k$ is a chain in $U \subseteq F_L$. We have $L \subseteq \text{QHP}_k^+ = \Delta^{k-1}(\text{QCL}) \subseteq \Gamma_1$, which yields $\text{QCL} = \text{QHP}_1^+ \subseteq \Gamma_k$, by applying Lemma 6.3.6 ($k - 1$) times. Thus $\Gamma_k$ is maximal in $U \subseteq F_L$, by Lemma 6.3.4(i).

**Theorem 6.3.8** The logic $\text{QHP}_k^{+ (=)}$ for $k \geq 1$ is determined by Kripke frames (respectively, Kripke sheaves) of depth $k$.

**Proof** We already know that $L = \text{LP}_k^+$ is sound for frames of depth $k$, so it suffices to show that $U \subseteq F_L$ is of depth $k$. By Lemma 6.3.7, $U \subseteq F_L$ is of depth $\leq k$. On the other hand, $\text{LP}_k^+ \not\models P_{k-1}$, hence $U \subseteq F_L \not\models AP_{k-1}$ by Corollary 6.2.18. Therefore by Proposition 1.4.17 and Lemma 3.2.23 $U \subseteq F_L$ is of depth $\geq k$.

**Remark 6.3.9** For some logics without equality canonical models allow us to prove $\mathcal{K}\mathcal{E}$-completeness rather than $\mathcal{K}$-completeness. Indeed, let $L$ be a non-canonical predicate logic without equality, $L^- = \text{QH}^- + L$ its minimal extension with equality, and suppose the canonical model for $L^-$ validates $L$ (unlike the canonical model for $L$). Then $L^-$ is $\mathcal{K}\mathcal{E}$-complete (recall that axioms of $\text{QH}^-$ are valid in all Kripke sheaves), as well as $L$. In fact, if a formula $A$ without equality is not $L$-provable, then by conservativity (Proposition 2.9.2), $A$ is not $L^-$-provable and thus is refuted in the canonical model of $L^-$. Although we do not know how to apply this idea for a direct construction of a Kripke incomplete logic $L$, later on we will show an indirect construction of this kind.
Definition 6.3.10 Let $L$ be a superintuitionistic logic, $\Gamma$ an $L$-place. We say that $E$ is a characteristic formula for $\Gamma$ if $E \in (\neg)\Gamma$ and $A \vee (A \supset E) \in \Gamma$ for any $A \in \mathcal{L}(\neg)\Gamma$. Or equivalently, this means that $(A \supset E) \in \Gamma$ for any $A \in (\neg)\Gamma$, i.e. $E$ is the weakest $D_{\Gamma}$-sentence outside $\Gamma$.

If a characteristic formula exists, it is clearly unique up to equivalence in $\Gamma$ (the relation $(A \equiv B) \in \Gamma$).

The next lemma uses the operation $\Delta$ introduced in Section 2.13

Lemma 6.3.11 If $\Gamma$ is a $\Delta L$-place, which is not an $L$-place, then there exists a sentence $E \in IF(\equiv)$ (without extra constants), which is characteristic for $\Gamma$.

Proof Let $E \in (L - \Gamma)$. Then $p \vee (p \supset E) \in \Delta L$ for $p$ not occurring in $E$, and thus $A \vee (A \supset E) \in \Gamma$ for any $A \in \mathcal{L}(\Gamma)$.

Corollary 6.3.12 If $QHP_n^+ \subseteq L$ for some $n > 0$, then for any $L$-place there exists a characteristic sentence.

Proof Recall that $QHP_k^+ = \Delta(QHP_{k-1})$ for $k > 0$ and $LP_0^+ = QH + \bot$ is inconsistent. For an $L$-place $\Gamma$, take the least $k$ ($> 0$) such that $\Gamma$ is an $QHP_k^+$-theory. Then apply Lemma 6.3.11.

6.4 Natural models for modal and superintuitionistic logics

As we can see, canonical models work well only for some particular predicate logics. In subtler cases it is convenient to use so-called 'natural models' obtained from canonical models by a sort of 'selective filtration', together with canonical maps (cf. Definition 6.1.13).

Let us begin with some simple remarks on canonical maps.

The following is a trivial reformulation of the canonical model theorem.

Lemma 6.4.1 Let $M$ be a Kripke model for a modal logic $L$, in which every individual domain is a small subset of $S^*$ (cf. Definition 6.1.13). Then for any world $u \in M$, for any $A \in \mathcal{L}(u)$

$$M, u \models A \text{ iff } VM_L, \nu_M(u) \models A,$$

i.e. canonical maps are reliable.

The same holds in the intuitionistic case, with obvious changes.

Lemma 6.4.2 Let $M$ be the same as in the previous lemma, $R_i$ the accessibility relations in $M$. Then for any $u, v \in M$

$$uR_i v \Rightarrow \nu_M(u)R_L \nu_M(v),$$

i.e. canonical maps are monotonic.
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Proof Almost obvious: if $\Box_i A \in \nu_M(u)$, i.e. $M, u \models \Box_i A$, and $u R_i v$, then $M, v \models A$, i.e. $A \in \nu_M(v)$.

In the intuitionistic case, if $u R v$, then $\nu_M(u) \subseteq \nu_M(v)$, by truth-preservation. ■

However note that in general $\nu_M$ is not a morphism of Kripke frames. The lift property does not always hold, because it may happen that $\Gamma_u R_{Li} \Gamma_v$ for incomparable $u, v \in F$, but there does not exist $w \in R_i(u)$ such that $\Gamma_w = \Gamma_v$.

Definition 6.4.3 Let $L$ be a predicate logic ($N$-modal or superintuitionistic, without or with equality), $F = (W, R_1, \ldots, R_N)$ a propositional frame of the corresponding kind. An $L$-map based on $F$ is a monotonic map from $F$ to (the propositional base of) $VF^{(=)}_L$, i.e. a map $h$ such that for any $u, v \in F$

$$u R_i v \Rightarrow h(u) R_{Li} h(v).$$

In the intuitionistic case we say that $h$ is an $L \forall$-map if all $h(u)$ are $L \forall$-places, i.e. $h$ is actually a map to $UF_L$.

Definition 6.4.4 Let $h$ be an $L$-map based on a propositional frame $F$. The predicate Kripke frame associated with $h$ is $F(h) := (F, D)$, where $D_u = (D_L)_{h(u)}$ for $u \in F$.

If $L$ is a logic with equality, the Kripke frame with equality associated with $h$ is $F^=(h) := (F, D, \simeq)$, where

$$c \simeq_u d \iff (c = d) \in h(u)$$

for $u \in F$ and $c, d \in D_u$.

The $L$-model associated with $h$ is $M^{(=)}(h) := (F^{(=)}(h), \xi(h))$, where

$$\xi(h)_u(P^m_k) := \{c \in D^m_u \mid P^m_k(c) \in h(u)\}$$

for $u \in F$.

All these frames and models are well-defined, since $u R_i v$ implies $h(u) R_{Li} h(v)$ (which means $h(u) \subseteq h(v)$ in the intuitionistic case).

Definition 6.4.5 An $L$-map $h : F \rightarrow VF_L$ and the associated $L$-model $M(h)$ are called natural if $h = \nu_M(h), L$, i.e. $(h, id) : M(h) \rightarrow VM_L$ is reliable: for any $u \in F$, $A \in \mathcal{L}(u)$

$$M(h) \models (\vdash) A \iff VM_L(h(u)) \models (\vdash) \iff A \in h(u)).$$

This definition is an analogue to the canonical model theorem for $M(h)$. In this case we readily obtain $M(h) \models L$.

Remark 6.4.6 Lemma 6.4.1 shows that every predicate Kripke model $M$ with small domains can be presented as a natural Kripke model $M(h)$ in a unique way; namely, put

$$h(u) := \Gamma_u = \{A \in MF^{(=)}_N(D_u) \mid M, u \models A\}$$

and similarly in the intuitionistic case.
Definition 6.4.7 A modal L-map $h : F \to VF_L$ based on $F = (W, R_1, \ldots, R_N)$ is called selective if it satisfies the condition

$(\bigcirc)$ if $u \in F$ and $\diamond_i A \in h(u)$, then there exists $v \in R_i(u)$ such that $A \in h(v)$.

Lemma 6.4.8 An L-map $h : F \to VF_L$ is natural iff it is selective.

Proof The 'only if' direction is trivial, since in a natural model $M(h)$ we have $M(h), u \vdash A$ iff $A \in h(u)$ (or $VM_L, h(u) \vdash A$, by the canonical model theorem) for any $A \in \mathcal{L}(u)$.

To prove 'if', we check the above equivalence by induction on the length $A$ (for any $u$). Let us consider the case $A = \square_i B$; other cases easily follow from Lemma 6.1.3.

If $\square_i B \not\in h(u)$, then $h(u) \not\vdash \square_i B$, and thus $h(u) \not\vdash \diamond_i \neg B$. By $(\bigcirc)$, there exists $v \in R_i(u)$ such that $\neg B \in h(v)$, which is equivalent to $B \not\in h(v)$. Hence $M(h), v \not\vdash B$ by the induction hypothesis. Since $v \in R_i(u)$, we obtain $M(h), u \not\vdash \square_i B$.

The other way round, if $h(u) \vdash \square_i B$ and $u \mathcal{R}_i v$, then $h(u) \mathcal{R}_i h(v)$, by monotonicity. So $h(v) \vdash B$, and thus $v \vdash B$, by the induction hypothesis. Since $v$ is arbitrary, this implies $u \vdash \square_i B$.

Note that the condition $(\bigcirc)$ is a generalisation of selectivity from Definition 6.1.31 (where $F$ is a subframe of $VF_L$). Moreover, by $(\bigcirc)$ and monotonicity it follows that the image of $h$ is a selective subframe of $VF_L$.

The intuitionistic analogue of 6.4.7 is the following

Definition 6.4.9 An intuitionistic L-map $h : F \to VF_L$ based on $F = (W, R)$ is called selective if it satisfies the following two conditions (cf. 6.2.12 $(\triangledown)$, $(\forall)$).

$(\bigtriangledown)$ if $(A_1 \supset A_2) \in (-h(u))$, then $A_1 \in h(v)$, $A_2 \not\in h(v)$ for some $v \in R(u)$;

$(\forall)$ if $\forall x A(x) \in (-h(u))$, then $A(c) \not\in h(v)$ for some $v \in R(u)$ and $c \in D_v$.

Lemma 6.4.10 An intuitionistic L-map $h : F \to VF_L$ is natural iff it is selective.

Proof Similar to Lemma 6.4.8. The 'only if' part is trivial. To prove 'if', we show by induction on the length of $A \in \mathcal{L}(u)$ (for any $u$) that

$M(h), u \vdash A$ iff $A \in h(u)$ (or $VM_L, h(u) \vdash A$).

Let us consider only three cases.

Let $A = B \supset C$. If $A \not\in h(u)$, then by $(\bigtriangledown)$, there exists $v \in R(u)$ such that $B \in h(v)$, $C \not\in h(v)$. By the induction hypothesis, we obtain $v \not\vdash B$, $v \not\vdash C$. Hence $u \not\vdash \square_i B \supset C$.

The other way round, if $u \not\vdash B \supset C$, then there exists $v \in R(u)$ such that $v \vdash B$, $v \not\vdash C$. Thus $h(v) \not\vdash B$, $h(v) \not\vdash C$ by the induction hypothesis. Since $h(u) \mathcal{R}_i h(v)$, it follows that $h(u) \not\vdash B \supset C$. 


Let $A = \exists x B$. Then

$$u \Vdash A \text{ iff } \exists c \in D_u \ u \Vdash B(c) \text{ iff } \exists c \in D_u \ h(u) \Vdash B(c)$$

by the induction hypothesis. Since $h(u)R_L h(v)$, the latter is equivalent to $h(u) \Vdash A$.

Let $A = \forall x B$. If $A \not\vDash h(u)$, then by $\forall$, there exist $v \in R(u), c \in D_v$ such that $B(c) \not\vDash h(v)$. By the induction hypothesis, $v \not\Vdash B(c)$. Hence $u \not\Vdash A$.

The other way round, if $u \not\Vdash \forall x B$, then there exist $v \in R(u), c \in D_v$ such that $v \not\vDash B(c)$. Thus $h(v) \not\vDash B(c)$ by the induction hypothesis, and therefore $h(u) \not\vDash A$, since $h(u)R_L h(v)$.

Sometimes we specify the terminology and say, e.g. that $h$ is an $(L, R')$-map if $R'$ is a selective relation on $VP_L$ such that $uRv$ implies $h(u)R'h(v)$. In the intuitionistic case we also use the term $L\forall$-map if $h$ is a map to $UF_L$; $M(h)$ is called a natural $L\forall$-model.

If an $L$-map $h$ is injective and also

$$uR_i v \text{ iff } h(u)R_{L_i} h(v),$$

then $M(h)$ is clearly a selective submodel of the canonical model.

Also note that the naturalness condition $\langle \diamond \rangle$ is a weak analogue of lift property.

In the intuitionistic case an $L$-map $h$ over a p.o. set is called proper if

$$uR^- v \Rightarrow h(u) \neq h(v);$$

in particular, if $h$ is $L, \leq$-natural and proper, then

$$uR^- v \Rightarrow h(u) < h(v).$$

Obviously this condition does not imply the injectivity for $h$. Similarly in the intuitionistic case one can consider proper natural models on a quasi-ordered $F$; in the modal case for extensions of $QT$ proper natural models on reflexive frames can be used. Informally speaking, in these cases in natural models we do not have to 'repeat' the same $L$-place in strictly accessible worlds, thanks to reflexivity.

**Lemma 6.4.11** Let $L$ be a modal or superintuitionistic logic, $M_1 \subseteq VM_L$ a selective submodel, $F_1$ the propositional frame of $M_1$, $h : F \rightarrow F_1$. Then $h$ is a natural $L$-map.

**Proof** By Lemmas 6.4.8, 6.4.10, it is sufficient to show that $h$ is selective. The monotonicity of $h$ follows from the definitions. In the modal case, to check $\langle \diamond \rangle$, suppose $\diamond_i A \in h(u)$. By selectivity of $M_1$, there exists $\Gamma \in M_1$ such that $h(u)R_{L_i} \Gamma$ and $A \in \Gamma$. Then by the lift property for $h$, there exists $v \in R_i(u)$ such that $h(v) = \Gamma$, so $A \in h(v)$.

In the intuitionistic case $\langle \forall \rangle$, $\forall$ are checked by a similar argument, which we leave to the reader.
Lemma 6.4.12

(1) Let $L$ be an $N$-modal logic, $\Gamma \in VP_L$. Then there exists a Kripke model $M$ based on a standard greedy tree $F$ such that $\nu_M(\wedge) = \Gamma$ and $M \models L$.

(2) If $L$ contains $\diamond_i \top$ for $1 \leq i \leq N$, then one can take $F = F_N T_\omega$.

(3) If $L$ is 1-modal and $L \supseteq QS4$, then the claim holds for $F = IT_\omega$.

Proof By Proposition 3.12.9, there exists a countable reliable $M_1 \subseteq VM_L$ verifying $\Gamma$. Let $F_1$ be the propositional frame of $M_1$. We may assume that $\Gamma$ is the root of $M_1$ (otherwise, replace $M_1$ with $M_1 \upharpoonright \Gamma$).

By Proposition 1.10.18, there exists a p-morphism $h$ from some greedy standard tree $F$ onto $F_1$ sending $\wedge$ to $\Gamma$. By Lemma 6.4.11, $h$ is a natural $L$-map, so we can take $M = M(h)$. Thus $\nu_M(\wedge) = h(\wedge) = \Gamma$.

In the case (2) all the formulas $\diamond_i \top$ are true in $VM_L$, hence they are true in its selective submodel $M_1$. This means that $F_1$ is serial, thus we can take $F = F_N T_\omega$, by Proposition 1.10.18.

In the case (3) we use Proposition 1.11.11 to construct a p-morphism $h : IT_\omega \longrightarrow F_1$ sending $\wedge$ to $\Gamma$.

Hence we obtain

Proposition 6.4.13 Let $F_N T_\omega$ be the set of all greedy standard subtrees of $F_N T_\omega$. Then

(1) $QK_N = ML(\kappa F_N T_\omega)$, $QD_N = ML(\kappa F_N T_\omega)$,

(2) $QK_N^= = ML^= (\kappa E(F_N T_\omega))$, $QD_N^= = ML^= (\kappa E(F_N T_\omega))$.

Now let us modify this proposition for the logics described in Theorem 6.1.29.

Proposition 6.4.14 Let $\Delta$ be a propositional one-way PTC-logic. Let $GT(\Delta)$ be the set of all greedy standard $\Delta$-trees. Then

(1) $QA = ML(\kappa GT(\Delta))$;

(2) $Q\Delta^= = ML^= (\kappa E(GT(\Delta)))$.

Proof (i) By Theorem 6.1.29, the logic $L = QA$ is Kripke complete. By Proposition 1.11.5, it is also $\Delta$-elementary. So by Proposition 3.12.8, it has the c.f.p., i.e. $L = ML(\kappa V_0(\Delta))$, where $V_0(\Delta)$ is the class of all countable $\Delta$-frames. Hence by Lemma 3.3.21, $L = ML(\kappa V_1(\Delta))$, where $V_1(\Delta)$ is the class of all countable rooted $\Delta$-frames.

Next, by Proposition 1.11.11, every frame from $V_1(\Delta)$ is a p-morphic image of a frame from $GT(\Delta)$. Therefore $ML(\kappa GT(\Delta)) \subseteq L$, by Corollary 3.3.14. The converse inclusion is a trivial consequence of the definitions.

Of course, instead of applying 3.3.14, we could apply Lemma 6.4.12.

In the case (ii) the argument is the same, based on the corresponding facts about frames with equality. ■
For logics with closed equality there is a similar result:

**Proposition 6.4.15**

1. \[ \mathbf{QK_N} + CE = \mathbf{ML}^\mathcal{K}(\mathcal{K}F_N T_\omega), \quad \mathbf{QD_N} + CE = \mathbf{ML}^\mathcal{K}(\mathcal{K}F_N T_\omega). \]

2. Let \( \Lambda \) be a propositional one-way PTC-logic.

Then \( \mathbf{QA} + CE = \mathbf{ML}^\mathcal{K}(\mathcal{K}G_T(\Lambda)). \)

**Proof** (ii) We modify the argument from the previous proof. Kripke sheaf completeness follows from Theorem 6.1.29 and Proposition 6.1.26. The logics are \( \Delta \)-elementary, since \( CE \) corresponds to a first-order property of KFEs.

As in the proof of Lemma 6.4.12, we obtain that every \( L \)-consistent theory \( \Gamma \) is satisfied at the root of a Kripke sheaf model \( M = M(h) \) based on a standard greedy \( \Delta \)-tree. Let \( \Phi \) be the Kripke sheaf of \( M \). Since \( M \models CE \), it follows that \( \Phi \models CE \). Then by Lemma 3.10.8, \( \Phi \) is isomorphic to a simple Kripke sheaf \( \Phi' \) (in fact, to apply this lemma, one should take the corresponding Horn closure). So \( \Gamma \) is satisfied in a Kripke frame from \( QG_T(\Lambda) \).

Recall (Definition 1.10.10) that the intuitionistic universal tree is \( IT_\omega := (T_\omega, \preceq) \), where

\[ \alpha \preceq \beta \iff \exists \gamma (\beta = \alpha \gamma). \]

So in the intuitionistic case we have the following analogue of 6.4.12.

**Lemma 6.4.16** Let \( L \) be a superintuitionistic logic, \( R' \) a selective relation on \( VP_L \). Then for any \( L \)-place \( \Gamma \) there exists a proper natural \( (L, R') \)-model on a standard (complete) subtree \( F \) of \( IT_\omega \) such that \( \nu_M(\lambda) = \Gamma \).

We shall often denote \( \nu_M(u) \) by \( \Gamma_u \) if there is no confusion.

It is clear that for \( v \in \beta(u) \) one can choose an \( L \)-place \( \Gamma_v \) non-equal to \( \Gamma_u \), due to the reflexivity.

Therefore there exists a natural \( (L, R') \)-model on \( IT_\omega \); in fact, one can copy \( L \)-places \( \Gamma_u \) from maximal points \( u \) of \( F \) at all points from \( IT_\omega \upharpoonright u \). This natural model is in general improper. But if for example, \( R' = R_L \), then one can directly construct a proper natural model on \( IT_\omega \); recall that every \( L \)-place \( \Gamma \) in the canonical model can be properly extended by extending its ‘domain’ \( D_\Gamma \).

Hence we obtain completeness:

**Theorem 6.4.17**

1. \[ \mathbf{QH} = \mathbf{IL}(\mathcal{K}(IT_\omega)); \]

2. \[ \mathbf{QH}^\preceq = \mathbf{IL}(\mathcal{K}E(IT_\omega)); \]

3. \[ \mathbf{QH}^{\preceq d} = \mathbf{IL}^\preceq(\mathcal{K}(IT_\omega)). \]

Now let us prove a somewhat stronger form of completeness (cf. Dragalin [1988]).
Theorem 6.4.18 Let $F$ be an effuse countable tree. Then

1. $\text{IL}(\mathcal{K}F) = \text{QH}; \text{ML}(\mathcal{K}F) = \text{QS4};$
2. $\text{IL}^=(\mathcal{K}\mathcal{E}F) = \text{QH}^=; \text{ML}^=(\mathcal{K}F) = \text{QS4}^=;$
3. $\text{IL}^=(\mathcal{K}F) = \text{QH}^d, \text{ML}(\mathcal{K}F) = \text{QS4}^d.$

Proof (1) In the modal case the argument is similar to Proposition 1.11.21. We have $F\upharpoonright u \rightarrow IT_2 \rightarrow IT_\omega$ for some $u$. Hence by Proposition 3.3.14, $\text{ML}(\mathcal{K}F \upharpoonright u) \subseteq \text{QS4}$. Thus by the generation lemma, $\text{ML}(\mathcal{K}F) \subseteq \text{ML}(\mathcal{K}F \upharpoonright u) \subseteq \text{QS4}.$

The remaining statements are left to the reader.

Corollary 6.4.19 $\text{QH}^{(=)(d)}$ is determined by the universal $n$-branching tree $IT_n$ for any $n \geq 2$.

If a relation $R'$ is selective on $UP_L$ (for an intermediate logic $L$), then similarly to 6.4.12, for any $\Gamma \in UP_L$ one can construct a proper natural $(L\forall, R')$-model on a standard greedy subtree $F$ of $IT_\omega$ and a natural $(L\forall, R')$-model on $IT_\omega$ (not necessarily proper), such that $\Gamma \land = \Gamma$. In particular, one can apply this construction to the selective relation $\leq$ on $UP_L$. However unlike the case of $VF_L$, we cannot state the existence of a proper $(L\forall, \leq)$-model on the whole $IT_\omega$. In particular, we obtain

Lemma 6.4.20 If $\text{QHP}_k^+ \subseteq L$ and $\nu$ is a proper $(L\forall, \leq)$-model on $F$, then $F$ is a frame of depth $\leq k$.

Lemma 6.4.21 If $\text{QHP}_k^+ \subseteq L$ then for any $\Gamma \in UP_L$ there exist a proper $(L\forall, \leq)$-model $\nu$ on a standard (greedy) subtree $F$ of $IT_k^\omega$ and an $(L\forall, \leq)$-model on $IT_k^\omega$ such that $\nu(\land) = \Gamma$.

Thus we obtain the following completeness result for $\text{QHP}_k^+$, $k > 0$.

Proposition 6.4.22

1. $\text{QHP}_k^+ = \text{IL}(\mathcal{K}(IT_k^\omega))$,
2. $\text{QHP}_k^+ = \text{IL}^=(\mathcal{K}\mathcal{E}(IT_k^\omega)).$

In Section 6.10 we will axiomatise the superintuitionistic predicate logic of the frames $IT_k^\omega$ of finite depth $k$ and finite branching $n$; it is just the logic of all posets of depth $\leq k$ and branching $\leq n$ (obviously, all these posets are finite).

But first we shall give some simple remarks on natural models based on posets of finite heights. The intuitionistic conditions of naturalness for this case can be slightly modified.

Lemma 6.4.23 Let $M$ be an intuitionistic $(L, R')$-model on a poset $F$ of finite depth. Then

1. $M$ is natural iff the following conditions hold (for any $u \in F$):
(\forall') \quad \text{If } (A_1 \supset A_2) \in (-\Gamma_u) \text{ and } A_1 \notin \Gamma_u, \text{ then } (A_1 \supset A_2) \notin \Gamma_u \text{ for some } v \in R^*(u);

(\forall') \quad \text{if } \forall x A(x) \in (-\Gamma_u) \text{ and } A(c) \in \Gamma_u \text{ for all } c \in D_{\Gamma_u}, \text{ then } \forall x A(x) \notin \Gamma_v \text{ for some } v \in R^*(u).

(2) \quad \text{If } M \text{ is an } (L\forall, R')-\text{model, then the condition } \forall' \text{ implies } (\forall').

\textbf{Proof}

\begin{itemize}
\item[(\forall') \Rightarrow (\forall)] \quad \text{If } A_1 \in \Gamma_v, \ A_2 \notin \Gamma_v \text{ for some } v \in R^*(u) \text{ then } (A_1 \supset A_2) \notin \Gamma_v.
\item[(\forall') \Rightarrow (\forall)] \quad \text{The proof is similar.}
\item[(\forall') \Rightarrow (\forall')] \quad \text{If } (A_1 \supset A_2) \notin \Gamma_u, \text{ then } A_1 \in \Gamma_v \text{ and } A_2 \notin \Gamma_v \text{ for a maximal } v \in R(u) \text{ such that } (A_1 \supset A_2) \notin \Gamma_v.
\item[(\forall') \& (\forall) \Rightarrow (\forall)] \quad \text{If } \forall x A(x) \notin \Gamma_u \text{ and } A(c) \in \Gamma_u \text{ for all } c \in S_u \text{ then by } (Av), \text{ for an } L\exists\forall\text{-complete } \Gamma_u, \text{ we have } \forall x (A(x) \vee C) \in \Gamma_v \text{ for some } C \notin \Gamma_u.
\text{Then } (C \supset \forall x A(x)) \notin \Gamma_u \text{ (otherwise } \forall x A(x) \in \Gamma_u) \text{ and by } (\forall'),
\text{ } (C \supset \forall x A(x)) \notin \Gamma_v \text{ for some } v \in R^*(u); \text{ hence } \forall x A(x) \notin \Gamma_v.
\end{itemize}

\hfill \blacksquare

Let us now prove two general lemmas on natural models.

\textbf{Definition 6.4.24} \quad 
A subtree \( F = (W, R_1, \ldots, R_N) \subseteq F_N T_\omega \) is said to be small if
\[ \forall u \in F \ \forall i \in I_N \ (\subset_i (u) - R_i(u)) \text{ is infinite.} \]

In the transitive (or intuitionistic) case we should take \( \beta_F(u) \) instead of \( R_i(u) \) in the above condition.

Thus every world of a small subtree \( F \) has infinitely many successors in \( F_N T_\omega \) unused in \( F \). Note that every denumerable tree, in which every point is of finite height, is isomorphic to a small subtree of \( F_N T_\omega \) (or \( IT_\omega \) in intuitionistic case).
Lemma 6.4.25

(1) Let \( L \) be an \( N \)-m.p.l. Then every \( L \)-map based on a small subtree \( F \) of \( F_{NT} \) can be extended to a natural \( L \)-map based on a (small) subtree \( F' \subseteq F_{NT} \) such that \( F' \supseteq F \).

(2) The same holds for an s.p.l. \( L \) and a small subtree \( F \) of \( IT_\omega \).

Proof

(1) By induction we construct an increasing sequence of small trees

\[
F = F_0 \subseteq F_1 \subseteq \ldots
\]

and a sequence \( h = h_0 \subseteq h_1 \subseteq \ldots \) of \( L \)-maps \( h_n : F_n \to VF_L^{(=)} \) such that

\[
F' := \bigcup_{n} F_n \text{ is small, } F_{n+1} - F_n \subseteq \bigcup_{i=1}^{N} \sqsubseteq_i (F_n) \text{ and } h_{n+1} \text{ 'cures the defects' of } h_n:
\]

\[
(*) \text{ for any } u \in F_n, \text{ if } \Diamond_i A \in h_n(u), \text{ then } \exists v \in R_{i,n+1}(u) A \in h_{n+1}(v),
\]

cf. Lemma 6.1.16. Actually we may assume that \( F_{n+1} - F_n \) consists only of the \( v \) constructed by \( (*) \). Thus the joined map \( h' := \bigcup_{n} h_n : F' \to VF_L^{(=)} \) is selective, i.e. natural, by Lemma 6.4.8. More explicitly this is done as follows. For any \( N \)-modal \( S^* \)-sentence \( \Diamond_i A \) choose the corresponding ‘Skolem function’

\[
f_{\Diamond_i A} : \xi_L^+(\Diamond_i A) \to \xi_L^+(A)
\]

such that \( \Gamma R_{L,1} F_{\Diamond_i A}(\Gamma) \) for any \( \Gamma \in \xi_L^+(\Diamond_i A) \). This function exists by the axiom of choice. Now if \( h_n \) is constructed and \( (*) \) holds for \( F_{n-1} \), then we extend \( h_n \) to \( h_{n+1} \) by putting

\[
h_{n+1}(u(2mN + i - 1)) := f_{\Diamond_i A_m}(h_n(u))
\]

whenever \( (A_m)_{m \geq 0} \) is the list of all \( N \)-modal \( S^* \)-sentences, \( u \in F_n - F_{n-1} \) and \( h_n(u) \in \text{dom}(f_{\Diamond_i A_m}) \) (i.e. \( \Diamond_i A_m \in h_n(u) \)). So it follows that \( (*) \) holds for \( F_n \) — in fact, if \( \Diamond_i A_m \in h_n(u) \), the corresponding \( v \) is \( u(2mN + i - 1) \). It also follows that \( F_{n+1} \) is small (if \( F_n \) is small). In fact, infinitely many \( \sqsubseteq_i \)-successors of \( u \in F_n - F_{n-1} \) do not appear in \( F_{n+1} \); they are of the form \( u(kN + i - 1) \) with odd \( k \).

(2) In the intuitionistic and in the modal transitive case the definition is slightly modified. In the transitive case one should take \( \beta(u) \) instead of \( \sqsubseteq_i (u) \). In the intuitionistic case the condition \( (*) \) is replaced with two (where \( \vdash \) denotes forcing in the canonical model):

\[
(**) \text{ if } h_n(u) \nvdash A \supset B, \text{ then } \exists v \in \beta(u) (h_{n+1}(v) \vdash A \land h_{n+1}(v) \nvdash B),
\]

\[
(***) \text{ if } h_n(u) \nvdash \forall x A(x), \text{ then } \exists v \in \beta(u) \exists c \in D_{h_{n+1}(v)} h_{n+1}(v) \nvdash A(c).
\]
Similarly to the modal case, we use corresponding Skolem functions $f_{A \supset B}$, $f_{\forall x A(x)}$.

$f_{A \supset B}$ is defined on the complement of $\xi^+_L(A \supset B)$, and $\Gamma R_L f_{A \supset B}(\Gamma)$, $f_{A \supset B}(\Gamma) \in \xi^+_L(A) - \xi^+_L(B)$.

Analogously, $f_{\forall x A(x)}$ is defined on $-\xi^+_L(\forall x A(x))$.

Note that in the intuitionistic case we can construct an $L\forall$-map or $L\forall^<$-map, or even $L\forall^<$-map $h'$ if the original map $h$ is of the corresponding kind. For $L\forall^<$ we should not repeat $\Gamma_u$ in future worlds and similarly for any selective relation (in the intuitionistic or in the reflexive modal case).

Lemma 6.4.25 directly implies Lemma 6.4.12, which is its particular case for a singleton frame on $\{A\}$, with $h(\lambda) = \Gamma$. A small tree $F'$ from 6.4.25 is clearly isomorphic to a greedy one (or to the whole $F_N T_\omega$, or to $IT_\omega$ in the intuitionistic case).

Actually the proof of 6.4.25 is a more explicit version of the proof of 6.4.12. In fact, the intermediate $L$-maps $h_n$ correspond to steps in calculating Skolem functions used in a well-known proof of the downward Löwenheim–Skolem theorem.

The construction in 6.4.25 extends $L$-maps by adding new points above the existing ones. Now we shall describe a construction allowing us to add new points (theories) below the existing theories. The main idea is to extend theories in a conservative way, by adding only sentences with extra constants.

Although we will further apply this construction to the linear case, we formulate it for arbitrary trees, see Lemma 6.4.30 below.

**Definition 6.4.26** The domain of an $L$-map $h$ (and of the corresponding $L$-model) is $D^+_h := \bigcup_{u \in F} D_{h(u)}$.

**Definition 6.4.27** Let $h : F \longrightarrow V F_L^{(=)}$ and $h' : F' \longrightarrow V F_L^{(=)}$ be two $L$-maps. We say that $h'$ is conservative over $h$ if for every $u \in F \cap F'$:

1. $D_{h(u)} \subseteq D_{h'(u)}$ and $D_{h'(u)} \cap D^+_h \subseteq D_{h(u)}$;
2. $h(u) \subseteq h'(u)$ (cf. 6.2.11).

We say that $h'$ is a conservative extension of $h$ if it is conservative and $F \subseteq F'$.

The condition (1) means that additional constants in $h'$-domains are 'new' for $h$.

**Lemma 6.4.28** If $h : F \longrightarrow V F_L^{(=)}$ and $h' : F' \longrightarrow V F_L^{(=)}$ are two $L$-maps on subtrees $F$, $F'$ of $F_N T_\omega$ ($IT_\omega$ in the intuitionistic case) and $h'$ is conservative over $h$, then $h'$ can be prolonged to a conservative extension $h'' : F \cup F' \longrightarrow V F_L^{(=)}$ of $h$.

Usually this lemma is applied to the case $F' \subset F$. 
Proof  Consider the case when \( F' \subsetneq F \). We construct \( h''(v) \) for \( v \in F - F' \) by induction on its length \( l(v) \). To simplify notation, we write \( \Gamma_u, \Gamma'_u, \Gamma''_u \) for \( h(u) \), \( h'(u) \), \( h''(u) \) respectively; similarly \( D_u \) abbreviates \( D_{h(u)} \), etc.

Assume that \( \Gamma''_u \) satisfying (i) and (ii) is already constructed (with \( \Gamma''_u = \Gamma'_u \) for \( u \in F \cap F' \)) and consider \( v \in R_u(u) \) (or \( v \in \beta(u) \) in the intuitionistic case).

Let us begin with the intuitionistic case. Put \((\Gamma, \Delta) := (\Gamma''_u \cup \Gamma_u, -\Gamma_u)\). Then \( D(\Gamma, \Delta) = D''_u \cup D_v \). Let us show that \((\Gamma, \Delta)\) is \( \mathcal{L} \)-consistent. In fact, suppose

\[ \vdash_{\mathcal{L}} (A'(c, c'') \land A(c, d) \supset B(c, d)), \]

where

\[ r(c) \subseteq D_v \cap D''_u = D_u \text{ (by (i) for } \Gamma''_u), \quad r(d) \subseteq D_v - D_u, \quad r(c'') \subseteq D''_u - D_u, \]

\[ A' = \bigwedge_k A'_k, \quad A = \bigwedge_i A_i, \quad B = \bigvee_j B_j, \quad A'_k \in \Gamma''_u, \quad A_i \in \Gamma_u, \quad B_j \in -\Gamma_v, \]

so \( A' \in \Gamma''_u, \quad A \in \Gamma_u, \quad B \in -\Gamma_v \). Put \( A''(c) := \exists x A'(c, x) \), where \( x \) is a distinct list corresponding to \( c'' \). Then \( \vdash_{\mathcal{L}} A''(c) \land A(c, d) \supset B(c, d) \), and

\[ A''(c) \in \Gamma''_u | D_u = \Gamma_u \subseteq \Gamma_v. \]

It follows that \( \Gamma_v \) is \( \mathcal{L} \)-inconsistent, which is a contradiction.

Now, by Lemma 6.2.6, we extend \((\Gamma, \Delta)\) to an \( \mathcal{L}_3 \)-complete \( \Gamma''_v \supseteq (\Gamma, \Delta) \); moreover, we choose \( D''_v \) such that \( D''_v \cap D''_h \subseteq D_v \cup D''_u \); then \( D''_v \cap D''_h \subseteq D_v \), since \( D''_v \cap D''_h = D_u \subseteq D_v \). Obviously, \( \Gamma''_u \subseteq \Gamma''_v \) and \( \Gamma''_v | D_v = \Gamma_v \), i.e. \( \Gamma_v \preceq \Gamma''_v \). Therefore by repeating this construction for all \( v \in (F - F') \), we obtain a conservative extension of \( F \).

Now let us consider the modal case. We put

\[ \Gamma := \square \Gamma''_u \cup \Gamma_v; \]

again \( D_\Gamma = D''_u \cup D_v \). Let us check the \( \mathcal{L} \)-consistency of \( \Gamma \). Suppose

\[ \vdash_{\mathcal{L}} \neg (A'(c, c'') \land A(c, d)), \]

where \( A' = \bigwedge_k A'_k, \quad A = \bigwedge_i A_i, \quad \square_i A'_k \in \Gamma''_u, \quad A_j \in \Gamma_v, \) so \( \square_i A' \in \Gamma''_v, \quad A \in \Gamma_v \) \((c, d, c'' \text{ are just the same as in the intuitionistic case})\). Similarly to that case, put \( A''(c) := \exists x A'(c, x) \). Then \( \vdash_{\mathcal{L}} \neg (A'' \land A) \) and \( \square A'' \in \Gamma''_v | D_u = \Gamma_u \). Since \( \Gamma_u R \Gamma \Gamma_v \) (in \( h \)), i.e. \( \square \Gamma_u \subseteq \Gamma_v \), we obtain \( A'' \subseteq \Gamma_v \). Hence \( \Gamma_v \) is \( \mathcal{L} \)-inconsistent, which is a contradiction.

Now by Lemma 6.1.9 we extend \( \Gamma \) to an \( \mathcal{L} \)-Henkin theory \( \Gamma''_v \) with a set of constants \( D''_v \) such that \( D''_v \cap D''_h \subseteq D_v \). Then \( \Gamma''_v R \Gamma \Gamma''_v \) and \( \Gamma''_v | D_v = \Gamma_v \); in fact, \( A \in -\Gamma_v \) implies \( \neg A \in \Gamma_v \subseteq \Gamma''_v \), so \( A \notin \Gamma''_v \).

**Lemma 6.4.29** Under the conditions of Lemma 6.4.28, let \( F \cup F' \) be a small subtree of \( F_N T_\omega \) (or \( IT_\omega \)). Then \( h' \) can be prolonged to a natural conservative extension \( h^* \) of \( h \) over a (small) subtree \( F^* \supseteq F \cup F' \).

**Proof**  Apply Lemma 6.4.25 to \( h'' \) constructed in Lemma 6.4.28.
6.5. **REFINED COMPLETENESS THEOREM FOR QH + KF**

**Lemma 6.4.30** Let $F$ be a subtree of $F_{NT\omega}$ (or $IT_{\omega}$). Let $h$ be an $L$-map over $F^- := F - \{\lambda\}$. Let $\Gamma$ be an $L$-place such that

1. $D_T \cap D^+_h \subseteq D_u$ for all $u \in F$ (or equivalently, for all $u \in R_i(\lambda)$, $i \in I_N$, or for all $u \in \beta(\lambda)$ resp.)

2. in the intuitionistic case: $\Gamma'(D_T \cap D^+_h) \subseteq \Gamma_u$ for all $u \in \beta(\lambda) \cap F$ (hence for all $u \in F'$);
   in the modal case: $\square_i(\Gamma'(D_T \cap D^+_h)) \subseteq \Gamma_u$ for all $u \in R_i(\lambda)$.

Then there exists a conservative extension $h''$ of $h$ to $F$ such that $h''(\lambda) = \Gamma$.

**Proof** Repeat the proof of Lemma 6.4.28 word for word. For constructing $h''(v)$ for $v \in R_i(\lambda)$ (i.e. for $u = \lambda$), we use (ii) in the intuitionistic case and (i) in the modal case. 

**Remark 6.4.31** Strictly speaking, Lemma 6.4.28 cannot be applied directly here. We cannot just prolong $h$ to $F$ by putting $\Gamma_\lambda = \Gamma'(D_T \cap D^+_h)$, since the latter does not always have the existence property. But this formal obstacle is not really essential.

### 6.5 Refined completeness theorem for QH + KF

**Definition 6.5.1** A p.o. set $F = (W, R)$ is called coatomic if it satisfies the McKinsey property:

$$\forall u \exists v (uRv \& R(v) = \{v\}),$$

i.e. every world sees a maximal world.

Recall that formula $KF$ is valid in all frames over coatomic p.o. sets.

**Definition 6.5.2** A p.o. set $F = (W, R)$ is called tree-like if it is rooted and $R^{-1}$ is non-branching.

**Definition 6.5.3** Let $IT^\#_{\omega}$ be the set of all $\omega$-paths (i.e. maximal chains) in the universal tree $IT_{\omega}$. Let us extend $IT_{\omega}$ to the tree-like p.o. set $\overline{IT}_\omega = IT_{\omega} \cup IT^\#_{\omega}$, in which $IT^\#_{\omega}$ is the set of maximal points and

$$u \leq z \iff u \subseteq z$$

for $u \in IT_{\omega}$, $z \in IT^\#_{\omega}$. A set $G \subseteq IT^\#_{\omega}$ gives rise to the subframe

$$\overline{IT}_\omega^G := \overline{IT}_\omega \mid (IT_{\omega} \cup G)$$

of $\overline{IT}_\omega$ with the set of maximal elements $G$. 
Obviously, $\mathcal{I}_\omega$ is coatomic and tree-like. $\mathcal{I}_\omega^G$ is coatomic iff the paths from $G$ cover $\mathcal{I}_\omega$, i.e.

\[(\alpha) \quad \forall u \in \mathcal{I}_\omega \exists z \in G \ u \in z.\]

It is also clear that if $\mathcal{I}_\omega^G$ is coatomic, then there exists a denumerable $G' \subseteq G$ such that $\mathcal{I}_\omega^{G'}$ is also coatomic – in fact, for each $u \in \mathcal{I}_\omega$ choose a path $z_u \in G$ containing $u$ and put $G' = \{z_u \mid u \in \mathcal{I}_\omega\}$.

**Lemma 6.5.4** Let $L$ be a superintuitionistic logic containing $KF$. Then for any $G \subseteq \mathcal{I}_\omega^*$ and for any $L$-place $\Gamma$

1. there exists an $L$-natural model $M$ on $\mathcal{I}_\omega^G$ such that $\nu_{M}(\lambda) = \Gamma$;

2. similarly, there exists an $L^{\forall}$-natural model $M$ on $\mathcal{I}_\omega^G$ with $\nu_{M}(\lambda) = \Gamma$.

**Proof** First, by Lemma 6.4.16 we find an $L$-natural model $M$ over $\mathcal{I}_\omega$. Obviously, we may assume that the set $\bigcup D_u$ is small, i.e. it is a subset of $S^*$ (cf. Section 7.1). Now for every path $z = \{u_k \mid k \in \omega\} \in G$ we put $D_z' := \bigcup D_{u_k}$ and consider the $L$-consistent theory $(\Gamma, \emptyset)$, where $\Gamma := \bigcup \Gamma_{u_k}$. By Lemma 6.3.3, this theory is $Q\text{CCL}$-consistent. So by Lemma 6.2.6, there exists a $Q\text{CCL}$-place $\Gamma_z \supseteq \Gamma$ with $D_{\Gamma_z} \supseteq D_{z'}$.

The resulting map $h : \mathcal{I}_\omega^G \to VF_L$ sending $\nu$ to $\Gamma_{\nu}$, is $L$-natural, since the conditions (\(\ast\)), (\(\forall\)) for a $Q\text{CCL}$-place $\Gamma_z$ hold trivially. It is also clear that $\Gamma_u \subseteq \Gamma_z$ whenever $u \in z$ (and $\Gamma_u \subseteq \Gamma_z$ in the proof of (2)).

Note that this construction can be applied to an arbitrary selective relation $R$ on $VF_L$ (or $UP_L$); to obtain an $(L, R)$-natural model, one should only check that $\Gamma_u R \Gamma_z$ holds for $u \in z \in \mathcal{I}_\omega^G$ in the resulting model.

**Proposition 6.5.5** $QH + KF = IL(\mathcal{I}_\omega^G) = IL(\mathcal{I}_\omega^G)$ for any coatomic poset $\mathcal{I}_\omega^G$.

One can easily obtain the corresponding completeness results for logics with equality, in the semantics $\mathcal{K}\mathcal{E}$ for $QH = KF$, and in $\mathcal{K}$ for $QH = KF$.

### 6.6 Directed frames

In this section we prove completeness results from [Corsi and Ghilardi, 1989]. We consider the following formulas:

- $ML := \forall x \lozenge \lozenge P(x) \supset \lozenge \lozenge \forall x P(x)$,
- $ML^* := \lozenge \forall x \lozenge \lozenge P(x) \supset \lozenge \lozenge \exists x P(x)$,
- $F := \lozenge \exists x P(x) \supset \lozenge \exists x \lozenge P(x)$. 
Definition 6.6.1 A propositional Kripke frame $F = (W, R)$ is called directed if the relation $R$ is reflexive, transitive and such that

$$\forall u, v \in W \exists w \in W \ (uRw \& vRw).$$

A predicate Kripke frame $F = (W, R, D)$ is called directed if its base $(W, R)$ is directed.

In this section we use specific propositional frames:

Definition 6.6.2

- $IT_\omega$ is called the subordination frame;
- $IT_\omega + 1$ ($IT_\omega$ with the added top element $\infty$) is called the subordination frame with the greatest element;
- $IT_\omega + C_\omega$ ($IT_\omega$ with the added greatest countable cluster $C_\omega$) is called the subordination frame with the greatest cluster; the elements of $C_\omega$ are denoted by $\infty_0, \infty_1, \ldots$;
- the ordinal product $IT_\omega \cdot \omega$ is called the multiple subordination frame; recall that this is the set $\omega^\infty \times \omega$ ordered by the relation

$$(\alpha, n)R(\beta, m) \text{ iff } n < m \lor (n = m \& \alpha \leq \beta).$$

Definition 6.6.3 Let $L$ be a 1-m.p.l. or s.p.l. A subordination $L$-map is a natural $L$-map $h : F \rightarrow VF_L$, where $F$ is one of the subordination frames from Definition 6.6.2, with the following properties

1. the set $D_h^+ := \bigcup_{u \in F} D_{h(u)}$ is small;
2. if $\alpha, \beta, \gamma \in \omega^\infty$ and $\alpha \sqsubset \beta, \beta \not\sqsubset \gamma$, then $(D_{h(\beta)} - D_{h(\alpha)}) \cap D_{h(\gamma)} = \emptyset$;
3. if $F = IT_\omega \cdot \omega$, then $h$ is a subordination map on each copy of $IT_\omega$:

$$\forall n \in \omega \forall \alpha, \beta, \gamma \in \omega^\infty (\alpha \sqsubset \beta \& \beta \not\sqsubset \gamma \Rightarrow (D_{h(\beta,n)} - D_{h(\alpha,n)}) \cap D_{h(\gamma,n)} = \emptyset).$$

In this case $M(h)$ is called a subordination $L$-model.

Lemma 6.6.4 Let $L \supseteq QS4$ and let $\Gamma$ be an $L$-place. Then there exists a subordination $L$-map $g : IT_\omega \rightarrow VF_L$ such that $g(\lambda) = \Gamma$.

Proof By 6.4.12, we can construct a natural $L$-map $h : IT_\omega \rightarrow VF_L$ such that $h(\lambda) = \Gamma$, so now we are going to transform it into a subordination $L$-map $g$.

To satisfy the conditions 6.6.3 (1), (2), we use renaming of constants. For this purpose we should first trace all constants appearing in $D_h^+$. Viz., for $c \in D_h^+$ we say that $h$ introduces $c$ at stage $\alpha$ if $\alpha$ is minimal in the set $\{ \beta \in$
$IT_\omega \mid c \in D_{h(\beta)}$. The same constants can be introduced at different stages, but they are all incomparable.

Now let $S^* = \{e_n \mid n \in \omega\}$ be an enumeration of our universal set of constants. We choose its small subset $S$ and present it as a two-dimensional array:

\[
S = \{c_{n,\alpha} \mid n \in \omega, \alpha \in \omega^\infty\}.
\]

Next, we define $g : IT_\omega \rightarrow VF_L$ as follows.

\[
g(\alpha) := \{A_\alpha \mid A \in h(\alpha)\},
\]

where $A_\alpha$ is obtained from $A$ by replacing all occurrences of each constant $e_n$ with $c_{n,\beta}$ such that $\beta \leq \alpha$ and $h$ introduces $e_n$ at stage $\beta$; this $\beta$ is uniquely determined by $n$ and $\alpha$.

Of course we have to ensure that $g(\alpha)$ is really an $L$-place. First note that $A$ can be also obtained from $A_\alpha$ by renaming constants. This operation respects provability, so $\vdash_L A_\alpha \Rightarrow \vdash_L A$ for any $D_{h(\alpha)}$-formula $A$; formally, one should argue by induction on the inference of $A$. Thus $\vdash\bigwedge_{i=1}^{k} (A_i)_\alpha$ implies $\vdash\bigwedge_{i=1}^{k} A_i$, i.e. the $L$-consistency of $h(\alpha)$ implies the $L$-consistency of $g(\alpha)$.

The Henkin property is almost obvious. In fact, if $\exists \alpha A_\alpha(x) = (\exists \alpha A(x))_\alpha \in \mathcal{L}(g(\alpha))$, then $\exists \alpha A(x) \in \mathcal{L}(h(\alpha))$. By the Henkin property for $h(\alpha)$, there is $n$ such that $(\exists \alpha A(x) \supset A(e_n)) \in h(\alpha)$. Therefore $(\exists \alpha A_\alpha(x) \supset A_\alpha(c_n,\alpha)) \in g(\alpha)$.

Now $g$ satisfies the condition 6.6.3 (1), since the new set of constants $S$ is small. For (2), note that if $\alpha \subseteq \beta$, then $D_{h(\beta)} - D_{h(\alpha)}$ consists exactly of the constants introduced at stage $\beta$, so all constants in $D_{g(\beta)} - D_{g(\alpha)}$ are of the form $c_{n,\beta}$. Thus if $\beta \not\subseteq \gamma$, they cannot appear in $g(\gamma)$.

**Lemma 6.6.5** Let $L \supseteq QS4.2$. If $h$ is a subordination map such that $h(\lambda) \supseteq L$, then $\bigcup_{\alpha \in \omega^\infty} \Box \neg h(\alpha)$ is $L$-consistent.

**Proof** Suppose $\bigcup_{\alpha \in \omega^\infty} \Box \neg h(\alpha)$ is $L$-inconsistent. Then for some $n$, $\bigcup_{|\alpha| \leq n} \Box \neg h(\alpha)$ is $L$-inconsistent. Take the smallest such $n$; obviously $n > 0$.

Note that $\alpha \subseteq \beta$ implies $\Box \neg h(\alpha) \subseteq \Box \neg h(\beta)$.

In fact, by monotonicity of $h$,

\[
\alpha \subseteq \beta \Rightarrow \Box \neg h(\alpha) \subseteq h(\beta).
\]

Now $A \in \Box \neg h(\alpha)$ implies $\Box A \in h(\alpha)$, hence $\Box \Box A \in h(\alpha)$ by $S4$, so $\Box A \in \Box \neg h(\alpha) \subseteq h(\beta)$, i.e. $A \in \Box \neg h(\beta)$.

Hence it follows that $\bigcup_{|\alpha| = n} \Box \neg h(\alpha)$ is $L$-inconsistent.

Now since every $\Box \neg h(\alpha)$ is $\wedge$-closed, by joining conjunctions, we obtain different sequences $\alpha_1k_1, \ldots, \alpha_mk_m$ of length $n$ and formulas $A_i \in \Box \neg h(\alpha_ik_i)$ such that $\{A_i \mid 1 \leq i \leq m\}$ is $L$-inconsistent. (Note that $\alpha_i$ are not necessarily distinct.) Every $A_i$ can be presented as $[c_i/x_i]B_i$, where $r(c_i) \subseteq D_{h(\alpha_i)}$, $x_i$ is a
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distinct list of variables and $B_i$ is a $D_{h(\alpha_i)}$-sentence. Since $h$ is a subordination map, the sets $r(c_i)$ are disjoint, so we may assume that all $x_i$ are disjoint. By Lemma 2.7.10

$$\vdash_L \neg \bigwedge_i A_i \text{ implies } \vdash_L \forall x \neg \bigwedge_i B_i,$$

where $x = x_1 \ldots x_m$.

By 2.6.15(xvi), we obtain

$$\vdash_L \exists x \bigwedge_i B_i, \text{ and thus } \vdash_L \neg \exists x \bigwedge_i B_i$$

by 2.6.15(xxvii), since $x_i$ are disjoint. Hence $\vdash_L \Box \Diamond \neg \bigwedge_i \exists x_i B_i$ ($\Box$-introduction is admissible, and $(\Box p \supset \Box \Diamond p) \in S4$), which implies

$$\vdash_L \neg \Diamond(\bigwedge_i \exists x_i B_i),$$

and thus

$$(*) \quad \vdash_L \neg \bigwedge_i \Diamond \exists x_i B_i,$$

by Lemmas 1.1.1, 1.1.3.

But $h(a\alpha_k) \models \Box A_i$, so $h(a\alpha_k) \models \exists x_i \Box B_i$, which implies $h(a\alpha_k) \models \exists \exists x_i B_i$ for all $a \in \omega$. Hence $h(\alpha) \models \Box \exists x_i B_i$.

By Lemmas 1.1.1 and soundness, this shows that $\Diamond \Box \exists x_i B_i \in \neg h(\alpha)$, so eventually $(*)$ implies the $L$-inconsistency of $\bigcup_{|\alpha| = n - 1} \Box \neg h(\alpha)$ contrary to the choice of $n$.

**Lemma 6.6.6** Let $L \supset QS4.2$. Then for every $L$-place $\Gamma$ there is an $\omega$-subordination model $M$ with $\nu_M(\lambda) = \Gamma$.

**Proof** Let us define $h$ as follows:

- By Lemma 6.6.4, there is a subordination model $M_0$ with $\nu_{M_0}(\lambda) = \Gamma$. Put $h(0, \alpha) := \nu_{M_0}(\alpha)$ for all $\alpha \in \omega^\infty$.

- By Lemmas 6.6.5 and 6.1.9, for every $n$ there is an $L$-place $\Gamma_{n+1}$ such that $D_{\Gamma_{n+1}} \supseteq \bigcup_{\alpha \in \omega^\infty} D_{h(n, \alpha)}$ and $\Gamma_{n+1} \supseteq \bigcup_{\alpha \in \omega^\infty} \Box \neg h(n, \alpha)$. By Lemma 6.6.4, there is a subordination model $M_{n+1}$ with $\nu_{M_{n+1}}(\lambda) = \Gamma_{n+1}$. Then put $h(n+1, \alpha) := \nu_{M_{n+1}}(\alpha)$ for all $\alpha \in \omega^\infty$. □
From Lemmas 6.6.4, 6.6.6 and 6.1.9 we obtain

**Theorem 6.6.7** The logic $\text{QS4.2}$ is determined by the $\omega$-subordination frame.

**Definition 6.6.8** An $L$-consistent theory $\Gamma \subseteq MFS$ is called a nearly $(L,S)$-Henkin theory if $\Gamma \vdash_L \forall x A(x)$, whenever $\Gamma \vdash_L A(c)$ for all $c \in S$.

**Lemma 6.6.9** (1) Let $L$ be an 1-m.p.l., $S$ a small set of constants. If $\Gamma$ is a nearly $(L,S)$-Henkin theory and $A$ is an $S$-sentence such that $\Gamma \cup \{A\}$ is $L$-consistent, then $\Gamma \cup \{A\}$ can be extended to an $(L,S)$-Henkin theory.

(2) Let $L \supseteq \text{QK + Ba}$. If $\Gamma$ is a nearly $(L,S)$-Henkin theory then $\Box \neg \Gamma$ is a nearly $(L,S)$-Henkin theory.

**Proof** Standard.

**Lemma 6.6.10** Let $L \supseteq \text{QS4 + ML*}$ and let $\Gamma \subseteq MFS$ be $L$-consistent and $L$-$\Box$-closed (i.e. $A \in \Gamma$ only if $\Gamma \vdash_L \Box A$). Let also $S \subseteq S'$, $|S'| = |S' - S| = \aleph_0$. Then there exists an $L$-consistent theory $\Gamma' \subseteq MFS'$ such that $\Gamma \subseteq \Gamma'$ and the following conditions hold:

(1) $\Box \neg \neg B \in \Gamma'$ or $\Box B \in \Gamma'$ for any $B \in MFS'$;

(2) if $\forall x B(x) \in MFS'$ and $\Box \neg \forall x B(x) \in \Gamma'$, then $\Box \neg \neg B(c) \in \Gamma'$ for some $c \in S'$.

**Proof** Let $\{A_i \mid i \in \omega\}$ be an enumeration of $MFS'$. We define the following sequence of theories:

- $\Gamma_0 := \Gamma$;
- $\Gamma_n$ if $\Gamma_n \cup \{\Box \neg A_n, \Box \neg B(c)\}$, where $c$ does not occur in $\Gamma_n \cup \{A_n\}$, if $\Gamma_n \cup \{\Box \neg A_n\}$ is $L$-consistent and $A_n = \forall x B(x)$;
- $\Gamma_n \cup \{\Box \neg B(c)\}$ if $\Gamma_n \cup \{\Box \neg A_n\}$ is $L$-consistent and $A_n$ is not of the form $\forall x B(x)$;
- $\Gamma_n \cup \{\Box A_n\}$ otherwise.

Each $\Gamma_n$ is clearly $L$-$\Box$-closed. We only need to show by induction that $\Gamma_n$ is $L$-consistent.

In fact, $\Gamma_0$ is $L$-consistent by assumption. For the induction step, suppose that some $\Gamma_n$ is $L$-consistent, but $\Gamma_{n+1}$ is not. Then $\Gamma_{n+1}$ is not obtained by (2). Suppose $\Gamma_{n+1}$ is obtained by (1). Then there are sentences $B_1, \ldots, B_m \in \Gamma_n$ such that

$L \vdash B_1 \land \ldots \land B_m \land \Box \neg \forall x B(x) \supset \neg \Box \neg B(c)$.

Since $c$ does not occur in $\Gamma_n$ or in $B(x)$, it follows that

$L \vdash B_1 \land \ldots \land B_m \land \Box \neg \forall x B(x) \supset \forall x \neg \Box \neg B(x)$.
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Since $L \supseteq QS4 + ML^*$, we have

$$L \vdash \Box B_1 \land \ldots \land \Box B_m \land \Box \Diamond \forall x \ B(x) \supset \Diamond \Box \forall x \ B(x),$$

hence

$$L \vdash \Box B_1 \land \ldots \land \Box B_m \supset \Diamond \Box \forall x \ B(x).$$

Since $\Gamma_n$ is $L$-$\Box$-closed, we obtain

$$\Gamma_n \vdash_L \Diamond \Box \forall x \ B(x).$$

This contradicts the $L$-consistency of $\Gamma_n \cup \{\Box \Diamond \neg A_n\}$. If $\Gamma_{n+1}$ is obtained by (3), then $\Gamma_n \cup \{\Box \Diamond \neg A_n\}$ is $L$-inconsistent, and so $\Gamma_n \vdash_L \neg \Box \Diamond \neg A_n$. If $\Gamma_{n+1}$ is $L$-inconsistent, then $\Gamma_n \vdash_L \neg \Box A_n$. By Necessitation, $\Gamma_n \vdash_L \Box \Diamond \neg A_n$, and so $\Gamma_n$ is $L$-inconsistent, contrary to hypothesis. Thus each $\Gamma_n$ is $L$-consistent.

Therefore $\Gamma' := \bigcup_{n \in \omega} \Gamma_n$ is $L$-consistent and $L$-$\Box$-closed by a standard argument. The conditions (1) and (2) hold by construction, so $\Gamma'$ is an $(L, S)$-Henkin theory.

Lemma 6.6.11 Let $L \supseteq QS4$. If $\Gamma \subseteq MF_S$ is an $L$-consistent theory satisfying conditions (1) and (2) of Lemma 6.6.10, then the following holds:

(1) $\Box \neg \Gamma$ is a nearly $(L, S)$-Henkin theory;

(2) If $\Gamma' \supseteq \Box \neg \Gamma$ is $L$-consistent then $\Box \neg \Gamma' \subseteq \Box \neg \Gamma$.

(3) If $\Gamma \vdash_L A$ implies $A \in \Gamma$, then (1) and (2) above imply 6.6.10(1) and 6.6.10(2).

Proof

(1) Suppose $\Box \neg \Gamma \not\subseteq L \forall x \ A(x)$. Then $\forall x \ A(x) \not\in \Box \neg \Gamma$, and so $\Box \forall x \ A(x) \not\in \Gamma$. By 6.6.10(1), then $\Box \Diamond \neg A(x) \in \Gamma$, and by 6.6.10(2), $\Box \Diamond \neg A(c) \in \Gamma$ for some $c$. Since $\Gamma$ is $L$-consistent, $\Gamma \not\subseteq L \Box A(c)$, and so $\Box \neg \Gamma \not\subseteq L A(c)$.

(2) Suppose $A \not\in \Box \neg \Gamma$. By 6.6.10(1) it follows that $\Box \Diamond \neg A \in \Gamma$, so $\Diamond \neg A \in \Box \neg \Gamma$ and $\Diamond \neg A \in \Gamma'$. Since $\Gamma'$ is $L$-consistent, $\Box A \not\in \Gamma'$ and so $A \not\in \Box \neg \Gamma'$.

(3) We first check 6.6.10(1). If $\Box \Diamond \neg A \not\in \Gamma$, then $\Diamond \neg A \not\in \Box \neg \Gamma$, and so $\Box \neg \Gamma \not\subseteq L \Diamond \neg A$. Hence $\Box \neg \Gamma \cup \{\Box A\}$ is consistent. From (2) it follows that $\Box \neg (\Box \neg \Gamma \cup \{\Box A\}) \subseteq \Box \neg \Gamma$ and so $A \in \Box \neg \Gamma$ and $\Box A \in \Gamma$.

Now suppose $\Gamma \vdash_L \Box \Diamond \neg \forall x \ A(x)$, but for all $c \in S$, $\Gamma \not\subseteq L \Box \Diamond \neg A(c)$. Then $\Box \Diamond \neg A(c) \not\in \Gamma$, and by 6.6.10(1), $\Box A(c) \in \Gamma$ for all $c$. Then $A(c) \in \Box \neg \Gamma$ and $\Box \neg \Gamma \vdash_L \forall x \ A(x)$. It follows that $\Gamma \vdash_L \Box \forall x \ A(x)$. This contradicts the $L$-consistency of $\Gamma$.

Lemma 6.6.12 Let $L \supseteq QS4.2 + ML^*$. Then for every $L$-place $\Gamma$ there is a subordination model $M$ with the greatest cluster such that $\nu_M(\lambda) = \Gamma$. 
Proof We construct the corresponding $h$ as follows.

Let $M_0$ be a subordination model such that $\nu_{M_0}(\lambda) = \Gamma$. Such a model exists by Lemma 6.6.4. Let $h = \nu_{M_0}$ on $\omega^\infty$. By Lemma 6.6.5, $\Gamma' := \bigcup_{\alpha \in \omega^\infty} \square^{-} h(\alpha)$ is $L$-consistent. One can easily to see that $\Gamma'$ is $L$-$\square$-closed. Let $S'' \supseteq S' = \bigcup_{\alpha \in \omega^\infty} D_{h(\alpha)}$, $|S'' - S'| = k_0$. By Lemma 6.6.10 there is an $L$-consistent set $\Gamma'' \supseteq \Gamma'$ of $S''$-sentences with properties 6.6.10(1) and 6.6.10(2). Let $A_1, \ldots$ be an enumeration of all $S''$-sentences $B$ such that $\square^{-} \Gamma'' \cup \{B\}$ is $L$-consistent.

For any $n$ we define $\Gamma_{\infty_n}$ as a Henkin $(L, S'')$-theory extending $\square^{-} \Gamma'' \cup \{A_n\}$. We have $\square^{-} \Gamma_\alpha \subseteq \Gamma_\beta$ whenever $\alpha \beta \in IT_\omega + C_\omega$. In fact, for $\alpha \in \omega^\infty$ and $n \in \omega$ we have $\square^{-} \Gamma_\alpha = \square^{-} \square^{-} \Gamma_\alpha \subseteq \square^{-} \Gamma'' \subseteq \Gamma_{\infty_n}$. For $n, m \in \omega$ we have, by Lemma 6.6.11(2), $\Gamma_{\infty_n} \subseteq \square^{-} \Gamma'' \subseteq \Gamma_{\infty_m}$. Therefore, $\alpha \beta \beta$ implies $\square^{-} \Gamma_\alpha \subseteq \Gamma_\beta$.

If $\diamond A \in \Gamma_{\infty_n}$, then $\square^{-} \Gamma'' \cup \{\diamond A\}$ is $L$-consistent. By Lemma 6.1.16, there is an $L$-place $I^*$ such that $\Gamma_{\infty_n}, R_L I^*$ and $A \in I^*$. Since we also have $\square^{-} \Gamma_{\infty_n} \subseteq I^*$, $\square^{-} \Gamma_{\infty_n} \cup \{A\}$ is $L$-consistent and $A = A_m$ for some $m$.

By applying Lemmas 6.6.4, 6.6.12, 6.1.9 we obtain

**Theorem 6.6.13** The logic $\text{QS4.2} + \text{ML}^*$ is determined by the subordination frame with the greatest cluster.

**Lemma 6.6.14** Let $L \supseteq \text{QS4.2.1} + \text{ML}^*$. Then for every $L$-place $\Gamma$ there is a subordination model $M$ with the greatest element such that with $\nu_M(\lambda) = \Gamma$.

Proof Let $h \mid \omega^\infty$, $\Gamma'$ be the same as in the proof of 6.6.12. Let us show that $\square^{-} \Gamma'$ is an $(L, S'')$-Henkin theory. Suppose that both $A \notin \square^{-} \Gamma''$ and $\neg A \notin \square^{-} \Gamma''$. Then $\square A \notin \Gamma''$ and $\square \neg A \notin \Gamma''$. By Lemma 6.6.10(1), $\square \neg A \in \Gamma''$ and $\square \neg \neg A \in \Gamma''$, in contradiction with the $L$-consistency of $\Gamma''$.

By Lemma 6.6.11(1), $\square^{-} \Gamma''$ is a nearly $(L, S'')$-Henkin theory, and so it is an $(L, S'')$-Henkin theory as well.

For any $\alpha \in \omega^\infty$ we have $\square^{-} \Gamma_\alpha = \square^{-} \square^{-} \Gamma_\alpha \subseteq \square^{-} \Gamma''$ and $D_{\Gamma_\alpha} \subseteq S''$. Let $\diamond A \in \square^{-} \Gamma''$. If $A \notin \square^{-} \Gamma''$, then $\neg A \in \square^{-} \Gamma''$ and so $\square \neg A \in \Gamma''$ and $\square \neg A \in \square^{-} \Gamma''$ contradicting the $L$-consistency of $\square^{-} \Gamma''$.

By applying Lemmas 6.6.4, 6.6.14, 6.1.9 we obtain

**Theorem 6.6.15** The logic $\text{QS4.2.1} + \text{ML}^*$ is determined by the subordination frame with the greatest element.

Now let us consider the intuitionistic case. Analogously to Lemma 6.6.4 we have

**Lemma 6.6.16** Let $L \supseteq \text{QI}$ and let $\Gamma$ be an $L$-place. Then there exists a subordination map $g : F \rightarrow VFL$ such that $g(\lambda) = \Gamma$.

Proof Same as Lemma 6.6.4, using Lemma 6.4.16 instead of Lemma 6.4.12. We need to check the existence property instead of the Henkin property, but this is trivial.
We also obtain an analogue to Lemma 6.6.6:

**Lemma 6.6.17** Let \( L \supseteq \mathsf{QH} + J \) and let \( h \) be a subordination map such that \( L \supseteq h(\lambda) \). Then \( \bigcup_{\alpha \in IT^\omega} h(\alpha) \) is \( L \)-consistent.

**Proof** The argument is basically the same as in the proof of 6.6.17. By monotonicity, \( \alpha \leq \beta \Rightarrow h(\alpha) \leq h(\beta) \). So the \( L \)-inconsistency of \( \bigcup_{\alpha \in IT^\omega} h(\alpha) \) implies the \( L \)-inconsistency of \( \bigcup_{|\alpha| \leq n} h(\alpha) \) for some \( n \). Consider the least such \( n \).

As in 6.6.5, we obtain \( A_i \in h(\alpha_i k_i) \) with different \( \alpha_i k_i \) of length \( n \) such that \( \{ A_{ij} \mid 1 \leq i, j \leq m \} \) is consistent. Next, we choose maximal generators \( B_i \) of \( A_i \), so for distinct \( c_i \) and \( x_i A_i = [c_i/x_i]B_i \).

Hence \( L \vdash \forall x_i \neg \bigwedge_i B_i \), which implies \( L \vdash \neg \exists x_i B_i \) by 6.6.10 (xvi, xxvii), and next

\[
\tag{*} L \vdash \neg \bigwedge_i \neg \exists x_i B_i,
\]

since \( \mathsf{QH} \vdash p \supset \neg \neg p \). On the other hand, \( h(\alpha_i k_i) \vdash \exists x_i B_i \), thus \( h(\alpha_i) \not\models \neg \exists x_i B_i \), which implies

\[
\tag{**} h(\alpha_i) \models \neg \neg \exists x_i B_i,
\]

since \( h(\alpha_i) \models \neg \exists x_i B_i \lor \neg \neg \exists x_i B_i \) by AJ. Now \((*) , (**) \) imply the \( L \)-inconsistency of \( \bigcup_{|\alpha|=n-1} h(\alpha) \).

**Corollary 6.6.18** Let \( L \supseteq \mathsf{QH} + J \) and let \( h \) be a subordination map such that \( L \supseteq h(\lambda) \). Then \( \bigcup_{\alpha \in T} h(\alpha) \) is \( L \)-consistent for any standard subtree \( T \) of \( IT^\omega \).

**Proof** Obvious.

**Lemma 6.6.19** Let \( L \supseteq \mathsf{QH} + J \) and let \( \Gamma \) be an \( L \)-place. Then there exists an \( \omega \)-subordination model \( M \) such that \( \nu_M(\lambda) = \Gamma \).

**Proof** Same as Lemma 6.6.6, replacing modal lemmas with their analogues.

From Lemmas 6.6.16, 6.6.19 and 6.2.6 we obtain

**Theorem 6.6.20** The logic \( \mathsf{QH} + J \) is determined by the \( \omega \)-subordination frame.

**Lemma 6.6.21** Let \( L \supseteq \mathsf{QH} + KF + J \) and let \( \Gamma \) be an \( L \)-place. Then there exists an subordination map with the greatest element \( h : IT^\omega \longrightarrow VF_L \) such that \( h(\lambda) = \Gamma \).
Proof Let \( g \) be the subordination map given by Lemma 6.6.16. By Lemmas 6.6.17, 6.2.6 and 6.3.4 there exists a \( \text{QLC}_3 \)-complete \( \forall \)-place \( \Gamma' \supseteq \bigcup_{\alpha \in IT^\omega} g(\alpha) \).

Let \( h(\alpha) = g(\alpha) \) for all \( \alpha \in IT^\omega \), \( h(\infty) = \Gamma' \).

Since \( \Gamma' \) is \( \leq \)-maximal and \( \leq \) is selective on \( UPL \), we observe that \( A \in \Gamma' \) and \( B \not\in \Gamma' \) whenever \( (A \supset B) \not\in \Gamma' \) and \( \forall x A(x) \in \Gamma' \) whenever \( A(c) \in \Gamma' \) for all \( c \in D_{\Gamma'} \). Therefore \( h \) is natural.

Hence we obtain

**Theorem 6.6.22** *The logic \( \text{QH} + \text{KF} + J \) is determined by the subordination frame with the greatest element.*

### 6.7 Logics of linear frames

In this section we prove strong Kripke-completeness for the logic \( \text{QLC} \) w.r.t. denumerable linear Kripke frames, or moreover, w.r.t. Kripke frames over the set of nonnegative rationals \( \mathbb{Q}^+ \).

The first proof of this result found by Corsi [Corsi, 1992] used the 'method of finite diagrams', which is a modification of the original S. Kripke’s method of semantic tableaux for the linear case, and it was rather laborious and complicated. Another proof using linear natural models was given in [Skvortsov, 2005]. Now we present a new, essentially simplified version of the latter proof basing on Lemma 6.4.28.

Let \( \mathcal{C}_{\text{lin}} \) be the class of all linearly ordered sets, \( \mathcal{C}_{\text{lin}}^{\omega} \) the class of all denumerable sets from \( \mathcal{C}_{\text{lin}} \).

Recall that every set from \( \mathcal{C}_{\text{lin}}^{\omega} \) is embeddable in \( \mathbb{Q} \).

**Theorem 6.7.1**

1. \( \text{QLC} = \text{IL}(\mathcal{K}(\mathcal{C}_{\text{lin}})) = \text{IL}(\mathcal{K}(\mathcal{C}_{\text{lin}}^{\omega})) \);
2. \( \text{QLC}^\omega = \text{IL}^\omega(\mathcal{KE}(\mathcal{C}_{\text{lin}})) = \text{IL}^\omega(\mathcal{KE}(\mathcal{C}_{\text{lin}}^{\omega})) \);
3. \( \text{QLC}^{\omega d} = \text{IL}^\omega(\mathcal{K}(\mathcal{C}_{\text{lin}})) = \text{IL}(\mathcal{K}(\mathcal{C}_{\text{lin}}^{\omega})) \).

Moreover, strong completeness holds in all these cases.

**Lemma 6.7.2** \(^2\) *If \( F \) is a countable linearly ordered set, then \( \mathbb{Q} \rightarrow F \).*

**Theorem 6.7.3**

1. \( \text{QLC} = \text{IL}(\mathcal{K}(\mathbb{Q}^+)) = \text{IL}(\mathcal{K}(\mathbb{Q})) \);
2. \( \text{QLC}^\omega + \text{IL}^\omega(\mathcal{KE}(\mathbb{Q}^+)) = \text{IL}^\omega(\mathcal{KE}(\mathbb{Q})) \);
3. \( \text{QLC}^{\omega d} = \text{IL}^\omega(\mathcal{KE}(\mathbb{Q}^+)) = \text{IL}(\mathcal{KE}(\mathbb{Q})) \).

\(^2\)[Takano, 1987]
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**Proof** From the previous lemma it follows that $Q_+ \Rightarrow F$ for any rooted $F \in C_{\text{lin}}'$. Then apply Theorem 6.7.1 and Proposition 3.3.14.

Soundness $QLC \subseteq \mathbf{IL}(K(C_{\text{lin}}))$ follows from 3.2.34.

Let us prove strong completeness. Let $L \supseteq QLC$ be a superintuitionistic predicate logic.

**Definition 6.7.4** A finite $L$-chain is an $L$-map over a finite linearly ordered set $W = \{0, 1, \ldots, k\}$, $0 < 1 < \ldots < k$, i.e. a sequence of $L$-places $M = (\Gamma_0, \Gamma_1, \ldots, \Gamma_k)$ such that $\Gamma_i \subseteq \Gamma_{i+1}$ for $i < n$. Recall that $D^+_M := \bigcup_{i} D_{\Gamma_i} = D_{\Gamma_m}$. 

**Comment** We shall construct a natural $L$-model step-by-step; $L$-chains will be finite fragments of this model at different stages of the construction. As usual, if say, $(A \supset B) \not\in \Gamma_j$, then we have to insert a new point above $j$, i.e. a new theory $\Gamma'$ extending $\Gamma_j$ such that $A \not\in \Gamma'$, $B \not\in \Gamma'$. But in general we cannot put this $\Gamma'$ above the whole $n$-chain, because it may happen that $(A \supset B) \in \Gamma_{j+1}$. Then we have to insert a new point between $j$ and $(j+1)$, and after extending the theory $(\Gamma_j \cup \{A\}, \{B\})$ to $\Gamma'$ by Lemma 6.2.6, we should extend $\Gamma_{j+1}$ to a new theory $\Gamma'_{j+2}$ in a conservative way, etc. and thus obtain a conservative extension $(\Gamma'_{i+1} \mid i > j)$ of $(\Gamma_i \mid i > j)$, which together with the 'old' $(\Gamma_i \mid i \leq j)$ and the 'new' $\Gamma'$ makes a $(k+2)$-element $L$-chain extending the original $(k+1)$-element $L$-chain. To make this, we apply Lemma 6.4.28 (its condition is satisfied due to the linearity axiom).

**Lemma 6.7.5** Let $M = (\Gamma_0, \Gamma_1, \ldots, \Gamma_k)$ be a finite $L$-chain and assume that for a certain $j < k$

- $(1)$ $(A_1 \supset A_2) \in \Gamma_{j+1} - \Gamma_j$
- or

- $(2)$ $\forall x A(x) \in \Gamma_{j+1} - \Gamma_j$.

Then there exists a finite $L$-chain $M' = (\Gamma_0, \ldots, \Gamma_j, \Gamma'_{j+1}, \Gamma'_{j+2}, \ldots, \Gamma'_{k+1})$ such that $D^+_{\Gamma'_{j+1}} \cap D^+_M = D_{\Gamma_j}$, $M'|\{0, \ldots, k+1\} - \{j+1\}$ is a conservative extension of $M$ and respectively:

- $(1') A_1 \in \Gamma'_{j+1}$, $A_2 \not\in \Gamma'_{j+1}$ in case $(1)$,
- $(2') A(c) \not\in \Gamma'_{j+1}$ for some $c \in D_{\Gamma'_{j+1}} - D_{\Gamma_j}$ in case $(2)$.

**Comment** Between $\Gamma_j$ and $\Gamma_{j+1}$ we insert a new theory $\Gamma'_{j+1}$ satisfying the standard falsity condition for $(A \supset B)$ or $\forall y A(y)$. This position is exact, for it is impossible to put $\Gamma'_{j+1}$ after $\Gamma_{j+1}$ and it is not necessary to put it before $\Gamma_j$. Along with putting $\Gamma'_{j+1}$ before $\Gamma_{j+1}$, we extend the members of the former subchain $\Gamma_{j+1}, \ldots, \Gamma_k$ to $\Gamma'_{j+2}, \ldots, \Gamma'_{k+1}$ (by applying Lemma 6.4.28). Of course, we may not preserve the conservativity, if we use (say, in $\Gamma'_{j+1} - \Gamma_j$ or in $\Gamma'_{j+2} - \Gamma_{j+1}$, etc.) the constants that are already present in $\Gamma_{j+1}, \Gamma_{j+2}$, etc. Thus we should add only new constants.
Proof Consider the following $L$-consistent theory $(\Gamma, \Delta)$ in the case (1) or (2) respectively:

(1) $(\Gamma, \Delta) := (\Gamma_j \cup \{A_1\}, \{A_2\})$,

(2) $(\Gamma, \Delta) := (\Gamma_j, \{A(c)\})$, with $c \not\in D^+_M$.

By Lemma 6.2.6, there exists an $L$-place $\Gamma'_{j+1}$ such that $(\Gamma, \Delta) \preceq \Gamma'_{j+1}$ and $D_{\Gamma'_{j+1}} \cap D^+_M = D_{\Gamma_j}$ (i.e. we add only new constants). Then $\Gamma_j \subseteq \Gamma'_{j+1}$ and the condition (1') or (2') (respectively) holds.

If $j = k$, the construction terminates. So consider the case $j < k$. Put $B_0 := (A_1 \supset A_2)$ or $B_0 := \forall x A(x)$ in case (1) or (2) respectively; then $B_0 \in (\Gamma_{j+1}\big|D_{\Gamma_j}) - \Gamma'_{j+1}$.

Let us show that $\Gamma'_{j+1}\big|D_{\Gamma_j} \subseteq \Gamma_{j+1}$. In fact, let $B \in \Gamma_{j+1}\big|D_{\Gamma_j}$. By LC, $(B \supset B_0) \lor (B_0 \supset B) \in \Gamma_j$. Also $(B \supset B_0) \not\in \Gamma_j$, since otherwise $B \in \Gamma'_{j+1}$ implies $B_0 \in \Gamma'_{j+1}$. Thus $(B \supset B) \in \Gamma_j \subseteq \Gamma_{j+1}$, and so $B \in \Gamma_{j+1}$, since $B_0 \in \Gamma_{j+1}$.

Now Lemma 6.4.28 yields a conservative extension $(\Gamma'_{j+2}, \ldots, \Gamma_{k+1})$ of $(\Gamma_{j+1}, \ldots, \Gamma_k)$ such that $\Gamma'_{j+1} \subseteq \Gamma'_{j+2}$. This completes the construction; obviously $M'|W$ is a conservative extension of $M$.

Proposition 6.7.6 Let $\Gamma$ be an $L$-place. Then there exists a natural $L$-model $M$ over a subframe (not necessarily generated) $F \subseteq Q_+$ with the root 0 such that $\nu_M(0) = \Gamma$.

Proof Rather straightforward. Consider an enumeration $B_0, B_1, B_2, \ldots$ of the set $IF(S^*)$ of $S^*$-sentences (where $S^*$ is the universal set of constants as usual), in which every sentence occurs infinitely many times. We construct an exhausting sequence of finite subsets of $Q_+$: $\{\} = W_0 \subseteq W_1 \subseteq \ldots$ and a sequence of finite $L$-chains $h_n : M_n \rightarrow VF_L$ over $W_n$ (for $n \in \omega$) such that $h_n(0) = \Gamma$ (for all $n$) and every restriction $M_{n+1}|W_n$ is a conservative extensions of $M_n$.

The inductive step is as follows. Assume that $M_n$ is already constructed, with $W_n = \{u_0, \ldots, u_k\}$, $h_n(u_i) = \Gamma_{u_i}$, and $B_n$ has the form $(A_1 \supset A_2)$ or $\forall x A(x)$, so that $B_n \in (-\Gamma_{u_i})$ for some $i \leq k$. Then we choose the greatest $j \leq k$ such that $B_n \not\subseteq \Gamma_{u_j}$ and construct an $L$-chain $M_{n+1}$ over $W_{n+1} = \{u_0, \ldots, u_j, v, u_{j+1}, \ldots, u_n\}$, where $u_j < v < u_{j+1}$, according to Lemma 6.7.5; so $M_{n+1}|W_n$ is a conservative extension of $M_n$.

Now put $W := \bigcup W_n$; $M := \bigcup M_n$, with $h(u) = h_n(u)$ whenever $u \in W_n$.

Thus $h(0) = \Gamma$ and every $M|W_n$ is a conservative extension of $M_n$. By the construction it is clear that the $L$-map $h$ is natural. In fact, if $(A_1 \supset A_2) \in -h(u)$ for some $u \in W$, then $(A_1 \supset A_2) \in -h_n(u)$ for some $n$ such that $u \in W_n$ and $B_n = (A_1 \supset A_2)$ (recall that $(A_1 \supset A_2)$ occurs infinitely many times in our enumeration). Let $W_n = \{u_0, \ldots, u_k\}$, $u = u_i$. Then $W_{n+1} = \{u_0, \ldots, u_j, v, u_{j+1}, \ldots, u_k\}$ for some $j \geq i$, and $A_1 \in h_{n+1}(v), A_2 \not\in h_{n+1}(v)$. So $A_1 \in h(v), A_2 \not\in h(v)$ (by conservativity) and $v > u_j \geq u_i = u$ in $W$.

The case $\forall x A(x) \in -h(u)$ is quite similar.
Corollary 6.7.7 QLC is strongly Kripke complete.

Remark 6.7.8 The completeness proof given in [Skvortsov, 2005] follows the same scheme, but instead of Lemma 6.4.28, it uses the notion of a 'prechain' under a finite L-chain $M = (\Gamma_{u_1}, \ldots, \Gamma_{u_k})$ over $W = \{u_1, \ldots, u_k\}$. Such a prechain has the form $((\Gamma, \Delta), \Gamma_{u_1}, \ldots, \Gamma_{u_k})$ where $(\Gamma, \Delta)$ is an L-consistent theory. It should satisfy a special condition, when it can be extended to a finite L-chain $M' = (\Gamma'_{u_0}, \Gamma'_{u_1}, \ldots, \Gamma'_{u_k})$ such that $(\Gamma, \Delta) \preceq \Gamma'_{u_0}$ and $M'|W$ is a conservative extension of $M$. But this notion makes sense only for L-chains $M$, where $\Gamma_{u_i} \ldots \Gamma_{u_k}$ (i.e. all extensions $\Gamma_{u_i} \ldots \Gamma_{u_k}$ are proper). So for constructing a natural $L^\infty$-model (cf. the proof of Proposition 6.7.6) one has to take care of properness of all inclusions between L-places. But Lemma 6.4.28 allows us to avoid these complications in the above proof.

Corollary 6.7.9 Let $F^Q := (Q_+, D^Q)$ be a Kripke frame over $Q_+$, in which $D^Q_u := \omega \times \{v \in Q_+ \mid v \leq u\}$. Then $IL(F^Q) = \text{QLC}$. Note that here $|D_u - \bigcup_{v < u} D_v| = \aleph_0$ for every $u \in Q_+$.

Proof Let $F' = (Q_+, D')$ be a Kripke frame with denumerable domains. Then we can construct a p-morphism $f = (f_0, f_1) : F^Q \to F'$. Namely, we put $f_0 := \text{id}_{Q_+}$, and for any $u \in Q_+$ fix a surjective map $f_u : \omega \times \{u\} \to D'_u$. Then we define $f_u : D'_u \to D'_u$ such that $f_u(\omega \times \{v\}) = f'_v$ for any $v \leq u$.

Obviously, $(f_0, f_1)$, where $f_1 = (f_u \mid u \in Q_+)$, is a p-morphism. $\blacksquare$

From Chapter 3 it follows that $\text{QLC}^w$ is characterised by all KFEs over $F^Q = (Q_+, D^Q)$. On the other hand, $\text{QLC}^d \neq \text{IL}^d(F^Q)$. Moreover, there does not exist a single KFE $F$ over $Q_+$ such that $\text{IL}^w(F) = \text{QLC}^w$, or a Kripke frame $F$ over $Q_+$ such that $\text{IL}^d(F) = \text{QLC}^d$. In fact, one cannot refute both formulas $\neg\forall x(y = x)$ and $\neg\forall x(y = x)$, which are obviously not in $\text{QLC}^d$, in the same KFE over $Q_+$ (e.g. since $Q_+ \vdash J$).

Now let us consider the modal case. Let us prove the result on strong Kripke completeness of QS4.3 from [Corsi, 1989]. Our proof follows the same lines as 6.7.1. Finite L-chains are defined as in 6.7.4, i.e. $0 < 1 \leq \cdots < k$ corresponds to $\Gamma_0R_L\Gamma_1R_L\cdots R_L\Gamma_k$ (with perhaps $\Gamma_{j+1}R_L\Gamma_j$ for some $j$). Lemma 6.7.5 is replaced with the following.

Lemma 6.7.10 Let $M = (\Gamma_0, \Gamma_1, \ldots, \Gamma_k)$ be a finite L-chain such that for a certain $j \leq k$ $\Diamond A \in \Gamma_j - \Gamma_{j+1}$.

Then there exists a finite L-chain $M' = (\Gamma_0, \ldots, \Gamma_j, \Gamma_{j+1}, \Gamma_{j+2}, \ldots, \Gamma_{k+1})$ such that $D'_{\Gamma_{j+1}} \cap D_M^+ = D'_{\Gamma_j}$, $M'|\{0, \ldots, k + 1\} - \{j + 1\}$ is a conservative extension of $M'$ and $A \in \Gamma'_{j+1}$.
Note that in this case $\neg(\Gamma_{j+1} R L \Gamma_j)$, since $\Gamma_j \not\models \lozenge A$, while $\Gamma_{j+1} \not\models \lozenge A$.

**Proof** There exists an $L$-place $\Gamma_{j+1}'$, such that $\square \neg \Gamma_j \cup \{A\} \subseteq \Gamma_{j+1}'$ and $D_{\Gamma_{j+1}'} \cap D_M^+ = D_{\Gamma_j}$ (we add only new constants); so $\Gamma_j R L \Gamma_{j+1}'$ and $A \in \Gamma_{j+1}'$ as required. Now we can apply Lemma 6.4.28 for extending $(\Gamma_{j+1}, \ldots, \Gamma_k)$ to $(\Gamma_{j+2}', \ldots, \Gamma_{k+1}')$ in a conservative way so that $\Gamma_{j+1} R L \Gamma_{j+2}$; it is sufficient to check that

$$\square \neg (\Gamma_{j+1}' | D_{\Gamma_j}) \subseteq \Gamma_{j+1}; \quad (***)$$

recall that $D_{\Gamma_{j+1}'} \cap D_M^+ = D_{\Gamma_j}$.

In fact, let $\square B \in \Gamma_{j+1}'$, $B \in L(D_{\Gamma_j})$. Note that $\square (\square B \supset \neg A) \lor \square (\neg \square A \supset B) \in \Gamma_{j+1}'$, since $\text{QS4.3} \subseteq \Gamma_j$. Also $\square (\square B \supset \neg A) \not\models \Gamma_j$, since otherwise $(\square B \supset \neg A) \in \Gamma_{j+1}'$ (remember that $\Gamma_j R L \Gamma_{j+1}'$), and so $\square B \in \Gamma_{j+1}'$ would imply $\neg A \in \Gamma_{j+1}'$. Hence $\square (\square \neg A \supset B) \in \Gamma_j$, so $(\square \neg A \supset B) \in \Gamma_{j+1}$, since $\Gamma_j R L \Gamma_{j+1}$. Now $B \in \Gamma_{j+1}$ follows from $\square \neg A \in \Gamma_{j+1}$.

**Proposition 6.7.11** $\text{QS4.3}$ is strongly Kripke complete.

**Proof** Repeat the proof of 6.7.7 with the inductive step for $B_n = \lozenge A$. Note that $i \leq j$ & $\lozenge A \in \Gamma_j \Rightarrow \lozenge A \in \Gamma_i$. So we choose the largest $j$ such that $\lozenge A \in \Gamma_j$ and then apply 6.7.10.

Now we obtain full analogues of the intuitionistic completeness results:

**Theorem 6.7.12**

1. $\text{QS4.3} = \text{ML}(\mathcal{K}(C_{\text{lin}})) = \text{ML}(\mathcal{K}(C_{\text{lin}}^\omega))$;
2. $\text{QS4.3}^\omega = \text{ML}^\omega(\mathcal{K}\mathcal{E}(C_{\text{lin}})) = \text{ML}^\omega(\mathcal{K}\mathcal{E}(C_{\text{lin}}^\omega))$;
3. $\text{QS4.3}^{\omega d} = \text{ML}^\omega(\mathcal{K}(C_{\text{lin}})) = \text{ML}(\mathcal{K}(C_{\text{lin}}^\omega))$.

Again strong completeness holds in all these cases.

**Corollary 6.7.13**

1. $\text{QS4.3} = \text{ML}(\mathcal{K}(Q_+)) = \text{ML}(\mathcal{K}(Q))$;
2. $\text{QS4.3}^\omega = \text{ML}^\omega(\mathcal{K}\mathcal{E}(Q_+)) = \text{IL}^\omega(\mathcal{K}\mathcal{E}(Q))$;
3. $\text{QS4.3}^{\omega d} = \text{ML}^\omega(\mathcal{K}\mathcal{E}(Q_+)) = \text{ML}^\omega(\mathcal{K}\mathcal{E}(Q))$.

### 6.8 Properties of $\Delta$-operation

In this section we use completeness to prove syntactic properties of the $\Delta$-operation introduced in Section 2.13. First let us introduce some notation. For a poset $F$, let $F_-$ be its restriction to the set of all nonminimal elements (of course, it may happen that $F_- = F$.) Then by induction we define:

$$F_{-0} := F, \quad F_{-(n+1)} := (F_{-n})_-.$$
If $F$ is an intuitionistic Kripke frame or a KFE (or a Kripke sheaf) over $F$, let $F_-$ be the restriction $F|F_-; F_{-n} := F|F_{-n}$.

**Proposition 6.8.1** For any intuitionistic formula $A$ and $k \geq 0$

$$F \models \delta_k A \text{ iff } F_- \models A.$$ 

**Proof** We write $A$ as $A(x)$ for $r(x) = FV(A)$.

(If.) Let us consider Kripke frames or KFEs; the case of sheaves is similar. Suppose

$$F \not\models \delta_k,PA(x) = \forall y (P(y) \lor (P(y) \supset A(x)))$$

for disjoint $x, y$. Then there exist a Kripke model $M$ over $F$, a world $w$, and tuples $a, b$ in $D_w$ such that $M, w \not\models P(a), P(a) \supset A(b)$.

So there is $v \in R(w)$ (where $R$ is the accessibility relation in $F$) such that $M, v \models P(a)$ and $M, v \not\models A(b)$. Then $w \neq v$, so $v$ is not minimal, i.e. $v \in F_-$. Let $M_-$ be the restriction $M|F_-; since it is a generated submodel, we have $M_-, v \models A(b)$ by 3.3.18. Hence $F_- \not\models A$.

(Only if.) Suppose $F_- \not\models A(x)$. Then by 3.3.18 there is a model $M$ over $F$ such that $M, v \not\models A(a)$ for some $v \in F_-$ and a tuple $a$ in $D_v$. Then consider a model $M'$ over $F$ differing from $M$ only on $P$, i.e. such that for any $w, Q \neq P$ and a tuple of individuals $b$ of the corresponding length

$$M', w \models Q(b) \text{ iff } M, w \models Q(b),$$

and also such that

$$M', w \models P(c) \text{ iff } w \neq u_0$$

for any $c \in D^k_w$, where $u_0$ is the root of $F$. Now by induction it easily follows that

$$M', w \models B \text{ iff } M, w \models B$$

for any world $w$ and $D_w$-sentence $B$ that does not contain $P$. Thus

$$M', v \not\models A(a); M', v \models P(c),$$

and on the other hand, $M', u_0 \not\models P(c)$. Hence

$$M', u_0 \not\models P(c) \lor (P(c) \supset A(a)),$$

and therefore $M', u_0 \not\models \neg \delta_k A$ implying $F \not\models \delta_k A$. 

So the validity of $\delta A$ at some world means that 'A will be true tomorrow'. All $\delta_k A$ are $KE$-indistinguishable from $\delta A$; this also follows from their deductive equivalence:

$$QH + \delta_k A = QH + \delta A,$$

cf. Proposition 2.13.22.

**Corollary 6.8.2** Let $F$ be a rooted predicate Kripke frame (or a KFE, or a Kripke sheaf).
(1) \( F \vdash \delta_k A \) iff \( \forall v \in F_- \ F \downarrow v \vdash A \).

(2) \( \Delta L \subseteq \mathbb{IL}^{(=)}(F) \) iff \( L \subseteq \mathbb{IL}^{(=)}(F_-) \).

(3) \( \Delta^n L \subseteq \mathbb{IL}^{(=)}(F) \) iff \( L \subseteq \mathbb{IL}^{(=)}(F_{-n}) \), for \( n \in \omega \).

**Proof** (1) By 3.3.21

\[ F_- \vdash A \text{ iff } \forall v \in F_- \ F_- \downarrow v \vdash A. \]

It remains to note that \( F_- \uparrow v = F \uparrow v \).

(2) Readily follows from 6.8.1, the definition of \( \Delta L \) and soundness.

(3) Follows from (2) by induction on \( n \), since \( F_{-(n+1)} = (F_{-n})_- \). 

**Lemma 6.8.3** Let \( F = (F, D) \) be an intuitionistic Kripke frame, in which the intersection of all individual domains is non-empty, and let \( a_0 \in \bigcap_{u \in F} D_u \). Next, let \( M \) be an intuitionistic Kripke model over \( F \), \( L \) an s.p.l. such that \( M \vdash \bar{L} \). Consider an intuitionistic Kripke model \( M' = \delta M \) obtained by adding a new root \( u_0 \) with the domain \( \{a_0\} \) below \( M \), such that \( M' \uparrow F = M \) and \( M', v \vdash p \text{ iff } v = u_0 \). Then \( M' \vdash \overline{\text{Sub}(\delta L)}. \)

**Proof** If \( A = p \vee (p \supset B) \in \delta \bar{L} \) (where \( B \in \bar{L} \), and \( p \) does not occur in \( B \)), then every strict substitution instance of \( A \) has the form \( C \vee (C \supset B') \) for a sentence \( C \) and \( B' \in \overline{\text{Sub}(L)} \subseteq \bar{L} \). So let us show

\[ M', u_0 \vdash C \vee (C \supset B'). \]

In fact, otherwise \( M', u_0 \nvdash C, C \supset B' \), and then \( M', v \nvdash B' \) for some \( v \in M \). Since \( M \) is a generated submodel, it follows that \( M, v \nvdash B' \) contradicting \( M \vdash \bar{L}. \)

**Lemma 6.8.4** For any intuitionistic formula \( A \) and an s.p.l. \( L \)

\[ \Delta L \vdash \delta A \text{ iff } L \vdash A. \]

**Proof** Although this lemma is syntactic, we know only a model-theoretic proof.

(If.) Obvious, by the definition of \( \Delta L. \)

(Only if.) Again we write \( A \) as \( A(x) \). Suppose \( L \nvdash A(x) \), or equivalently, \( \bar{L} \nvdash A \times (x) \). Then by the canonical model theorem and the generation lemma 3.3.12, the there exists a Kripke model \( M \) over \( F \) with a root \( v_0 \) such that \( M, v_0 \vdash \bar{L} \) and \( M, v_0 \nvdash A(a) \) for a tuple \( a \) in \( D_{a_0} \). Consider the model \( M' = \delta M \) defined in Lemma 6.8.3. By that lemma we have

\[ M', u_0 \vdash \overline{\text{Sub}(\delta \bar{L})}. \]

Since \( M \) and \( M' \) coincide on \( F \), an inductive argument as in the proof of 6.8.1 shows that

\[ M', w \vdash B \iff M, w \vdash B \]
for any \( w \in M \) and a \( D_w \)-sentence \( B \) that does not contain \( p \). Hence \( M', v_0 \not\models A(a) \), and thus \( M', v_0 \not\models \delta A(a) \).

Now put \( \Gamma := \text{Sub}(\delta L) \). By truth-preservation \( M', v_0 \models \Gamma \) implies \( M' \models \Gamma \); hence by Lemma 3.2.31 applied to \( \Gamma \) and \( L = \text{QH} \), we obtain

\[
\text{Sub}(\delta L) \not\models_{\text{QH}} \delta A(x),
\]

i.e.

\[
\Delta L \not\models_{\text{QH}} \delta A(x)
\]

by the deduction theorem.

**Proposition 6.8.5** For any s.p.l. \( L_1, L_2 \)

1. \( L_1 \subseteq L_2 \) iff \( \Delta L_1 \subseteq \Delta L_2 \);
2. \( L_1 = L_2 \) iff \( \Delta L_1 = \Delta L_2 \).

**Proof**

(i) 'Only if' is trivial. For 'if' note that \( A \in L_1 - L_2 \) implies \( \delta A \in \Delta L_1 - \Delta L_2 \) by 6.8.4.

(ii) Follows from (i).

So as in the propositional case, \( \Delta \) is monotonic.

Let us now prove an analogue to Lemma 1.16.10.

**Lemma 6.8.6** For s.p.l.s \( L_1 \) and \( L_2 \) and sentences \( A_1, A_2 \), if \( A_1 \in (L_1 - L_2) \) and \( A_2 \in (L_2 - L_1) \), then \( (\delta A_1 \lor \delta A_2) \in (\Delta L_1 \cap \Delta L_2 - \Delta(L_1 \cap L_2)) \).

**Proof** Since \( \Delta L_1 \vdash \delta A_1 \) and \( \Delta L_2 \vdash \delta A_2 \), it follows that

\[
\Delta L_1 \cap \Delta L_2 \vdash \delta A_1 \lor \delta A_2.
\]

Next, \( L_2 \not\models A_1 \) implies \( L_2 \not\models p \supset A_1 \) (cf. the proof of Lemma 2.9.4). Similarly \( L_1 \not\models p \supset A_2 \). So the theories \( (L_2 \cup \{p\}, \{A_1\}) \) and \( (L_1 \cup \{p\}, \{A_2\}) \) are \( \text{QH} \)-consistent, and by strong completeness there exist Kripke models \( M_1, M_2 \) with roots \( u_1, u_2 \) such that \( M_1, u_1 \models (L_2 \cup \{p\}, \{A_1\}) \) and \( M_2, u_2 \models (L_1 \cup \{p\}, \{A_2\}) \). Obviously we may assume that there is a common individual \( a_0 \) in the domains of the worlds \( u_1, u_2 \).

Then consider the model \( M' := \delta(M_1 \cup M_2) \) described in Lemma 6.8.3. Since \( M_i \models L_i \), it follows that \( M_1 \cup M_2 \models L_1 \cap L_2 \). Hence

\[
M', u_0 \models \text{Sub}(\delta(L_1 \cap L_2))
\]

by Lemma 6.8.3. We also have \( M_i, u_i \not\models p \supset A_i \), hence \( M', u_0 \not\models p \supset A_i \). Since also \( M', u_0 \not\models p \), we obtain \( M', u_0 \not\models \delta A_1 \lor \delta A_2 \). Therefore \( \Delta(L_1 \cap L_2) \not\models \delta A_1 \lor \delta A_2 \) by Lemma 3.2.31.

Let us mention the following analogue of Proposition 1.6.11.

**Proposition 6.8.7** Let \( L_1 \) and \( L_2 \) be superintuitionistic predicate logics.

1. \( \Delta(L_1 + L_2) = \Delta L_1 + \Delta L_2 \);
2. \( \Delta(L_1 \cap L_2) = \Delta L_1 \cap \Delta L_2 \) iff \( L_1 \) and \( L_2 \) are comparable by inclusion.
6.9 $\Delta$-operation preserves completeness

Definition 6.9.1 Let $F = (W, R)$ be a non-empty poset, $\max F$ the set of its maximal points $\vec{G} := (G_w \mid w \in \max F)$ a family of rooted posets. Then $F + \vec{G} := F + \bigcup_{w \in \max F} G_w$ is a poset obtained by putting disjoint isomorphic copies of $G_w$ above the corresponding points $w \in \max F$. Speaking precisely, if

\[ W' := W_+ \cup \{(v_w, w) \mid w \in \max F\} \]

for $w \in \max F$, are disjoint $\vec{R}$-stable subsets in $F + \vec{G}$, and $(F + \vec{G}) \upharpoonright V'_w \cong G_w$. If $v_w$ is the root of $G_w$, then

\[ W' := W_+ \cup \{(v_w, w) \mid w \in \max F\} \]
is an $\overline{R}^{-1}$-stable subset of $F + \overline{G}$, $(F + \overline{G})|W' \cong F$. It is also clear that

$$\overline{W} = W' \cup \bigcup_{w \in \text{max } F} V'_w$$

and $W' \cap V'_w = \{(v_w, w)\}$ for $w \in \text{max } F$.

**Definition 6.9.2** Let $F = (F, D)$ be a predicate Kripke frame over a poset $F$ and for $w \in \text{max } F$ let $G_w = (G_w, D_w)$ be Kripke frames over rooted posets $G_w$ (with roots $v_w$) such that $D_w = D'_w(v_w)$. Then we define a Kripke frame $F + \overline{G} := (F + \overline{G}, \overline{D})$, in which $\overline{D}(w) = D(w)$ for $w \in W$ and $\overline{D}(v, w) = D'_w(v)$ for $v \in G_w$.

**Definition 6.9.3** Similarly for Kripke sheaves $F = (F, p, D)$ and $G_w = (G_w, p'_w, D'_w)$, with roots $v_w$, where $w \in \text{max } F$, and $D_w = D'_w(v_w)$, we define a Kripke sheaf $F + \overline{G} := (F + \overline{G}, \overline{p}, \overline{D})$, in which $\overline{p}(u, (v, w)) = p(u, w) \circ p'_w(v_w, v)$ for $u \in F^-$, $w \in R(u) \cap \text{max } F$, $v \in G_w$; in all other cases $\overline{p}$ is inherited from $F$ or $G_w$ in the natural way.

**Definition 6.9.4** We can also define a KFE $F' = F + \overline{G}$ for KFEs $F = (F, D, \succ)$ and $G_w = (G_w, D'_w, \succ_w)$, $w \in \text{max } F$ such that $D(w) = D'_w(v_w)$ and $\succ_w = (\succ'_w)_{v_w}$ for $w \in \text{max } F$. The details are left to the reader.

**Definition 6.9.5** Let $C_0$ and $C$ be classes of rooted predicate Kripke frames.

Then $C_0 + C$ denotes the class of all Kripke frames of the form $F + \overline{G}$, where $F \in C_0$ and $\overline{G} = (G_w | w \in \text{max } F)$ is a family of frames from $C$. Similarly we define a class $C_0 + C$ of KFEs or Kripke sheaves for classes $C_0$ and $C$ of rooted KFEs or Kripke sheaves respectively.

**Definition 6.9.6** A tree of finite height is called uniform if all its maximal points are of the same height.

Recall that $IT^k_n$ (for $k > 0$, $n \leq \omega$) is the tree of $n$-sequences of length $< k$. Obviously all these trees are uniform.

**Lemma 6.9.7** Let $F$ be a uniform tree of height $k$, $\overline{F} = F + \overline{G}$. Then $\overline{F}_{-k} \cong \bigsqcup \overline{G}$. Similarly, if $F$ is a predicate frame over $F$ and $G_w$ are predicate frames over $G_w$, then

$$(F + \overline{G})_{-k} \cong \bigsqcup \overline{G}.$$

**Proof** Obvious. The levels $l < k$ in $\overline{F}$ and $F$ are the same, and the level $k$ in $\overline{F}$ consists of the roots of posets $G_w$ for $w \in \text{max } F$. 

$\overline{D}'_w$ denotes the domain function in $G_w$, so we denote the domain at a world $u$ in $G_w$ by $D'_w(u)$. $D(u)$ (respectively, $\overline{D}(u)$) is the domain at $u$ in $F$ (respectively, in $\overline{F}$).
Proposition 6.9.8  (1) Let $L$ be an s.p.l., $C$ a class of rooted predicate Kripke frames, and let $F_0$ be a uniform tree of height $k$, $C_0 = K(F_0)$. Then

$$\Delta^k L \subseteq \text{IL}(=)(C_0 + C) \iff L \subseteq \text{IL}(=)(C).$$

(2) Similarly for KFEs or Kripke sheaves, with $C_0 = K\mathcal{E}(F_0)$.

Note that if $k = 0$; then $F$ is a singleton and $C_0 + C$ is equivalent to $C$.

Proof (If.) Let $\vec{G} = (G_w \mid w \in \max F_0)$ be a family of frames from $C$. Then by 3.5.24

$$\text{IL}(\bigcup \vec{G}) = \bigcap_{w \in \max F} \text{IL}(G_w) \supseteq \text{IL}(C).$$

By Lemma 6.9.7, for a frame $F_0$ over $F_0$

$$\bigcup \vec{G} \cong (F_0 + \vec{G})_{-k},$$

so $L \subseteq \text{IL}(C)$ implies $L \subseteq \text{IL}((F_0 + \vec{G})_{-k})$, which is equivalent to

$$\Delta^k L \subseteq \text{IL}(F_0 + \vec{G})$$

by 6.8.2. Since $\vec{G}$ is arbitrary, this implies

$$\Delta^k L \subseteq \text{IL}(C_0 + C).$$

(Only if.) Assuming $\Delta^k L \subseteq \text{IL}(C_0 + C)$, let us show that $L \subseteq \text{IL}(F)$ for any $F \in C$. Given $F$ with a domain $V$ at the root, consider the constant family

$$\vec{G} := (F \mid v \in \max F_0).$$

Then there exists a frame $F_0$ over $F_0$ such that copies of $F$ can be stuck to its maximal points. For example, we can put $F_0 = F_0 \odot V$, the frame with the constant domain $V$.

Then

$$(F_0 + \vec{G})_{-k} \cong \bigcup_{v \in \max F_0} F,$$

so

$$\text{IL}((F_0 + \vec{G})_{-k}) = \bigcap_{v \in \max F_0} \text{IL}(F) = \text{IL}(F).$$

By our assumption

$$\Delta^k L \subseteq \text{IL}(C_0 + C) \subseteq \text{IL}(F_0 + \vec{G}),$$

hence by 6.8.2, $L \subseteq \text{IL}((F_0 + \vec{G})_{-k}) = \text{IL}(F)$. \qed
Proposition 6.9.9 Let $L$ be an intermediate predicate logic (with or without equality) strongly complete w.r.t. a class $C$ of Kripke frames or KFEs. Then $\Delta^k L$ is strongly complete w.r.t. $C_0 + C$, where $C_0 = \mathcal{K}(IT_{\omega}^{k+1})$ (or $\mathcal{K}\mathcal{E}(IT_{\omega}^{k+1})$ respectively).

Hence $\Delta^h L$ is Kripke-complete or Kripke sheaf complete w.r.t. $C_0 + C$. Note that we cannot state Kripke completeness of $\Delta^h L$ for a (strongly) Kripke complete logic $L$ with equality, because in this case $C_0$ is still $\mathcal{K}\mathcal{E}(IT_{\omega}^{k+1})$.

Proof Soundness follows from 6.9.8.

Now for a $\Delta^k(L)\forall$-place $\Gamma_0$, let us construct a proper $(\Delta^h L, \forall \leq)$-map $(\Gamma_u \mid u \in IT_{\omega}^{k+1})$, beginning with $\Gamma_\lambda = \Gamma_0$.

(**) if $h_j(u) \not\models A \supset B$, then $\exists v \in \beta(u) \cap F_{j+1} (h_{j+1}(v) \models A \& h_{j+1}(v) \not\models B)$,

(****) if $h_j(u) \not\models \forall x A(x)$, then $\exists v \in \beta(u) \cap F_{j+1} \exists c \in D_{h_{j+1}(v)} h_{j+1}(v) \not\models A(c)$.

After we reach a point $w$ of level $k$ in $IT_{\omega}^{k+1}$, we obtain a $(k+1)$-element sequence $\Gamma_\lambda < \ldots < \Gamma_w$, so $\Delta^h L \subseteq \Gamma_\lambda$ implies $L \subset \Gamma_w$ by Proposition 6.9.8.

Now we can use the strong completeness of $L$ and obtain a natural model with our $\Gamma_w$ at the root of a frame from $C$. So we obtain a natural model over a frame from $C_0 + C$. Clearly, this construction gives us a natural model over a frame from $G_0^C$. Naturally, for a logic with equality, we construct places $\Gamma_w$ for $u \in IT_{\omega}^{k+1}$ in the language with equality and hence we obtain a natural KFE.

There exists one special case. Namely, if for some point $u \in IT_{\omega}^k$ there do not exist places $> \Gamma_u$, i.e. $\Gamma_u$ is maximal in $U \leq F\Delta^k(L)$, then by Lemma 7.3.2(i) $\Gamma_u$ is QCL\forall-place. In this case the naturalness conditions hold with $\Gamma_v = \Gamma_u$ and we can put $\Gamma_v = \Gamma_u$ for all $v \in (IT_{\omega}^{k+1} \uparrow u)$.

And again for points $w$ of level $k$ we can apply the strong completeness of $L$, because $\Gamma_w = \Gamma_u$ is an $L\forall$-place since $L \subseteq QCL$.

Note that the condition $L \subseteq QCL^{(=)}$ is necessary in proposition 6.8.7. Namely:

Proposition 6.9.10 Let $L$ be a superintuitionistic predicate logic, $L \not\subseteq QCL^{(=)}$, and $h > 0$. Then $\Delta^h L \neq IL^{(=)}(C_0 + C)$ for any class $C \subseteq \mathcal{K}\mathcal{E}$ and any $C_0 \subseteq \mathcal{K}\mathcal{E}(IT_{\omega}^{h+1})$.

Proof Consider a sentence $A \in L - QCL^{(=)}$ and suppose $\Delta^h L = IL^{(=)}(C_0 + C)$. Then $L \subseteq IL^{(=)}(C)$, by Corollary 6.8.6, so $A \in IL^{(=)}(C)$ and hence obviously $\neg \neg A \in IL^{(=)}(C_0 + C)$. On the other hand, $\neg \neg A \not\in \Delta^h L$ since $\Delta^h L \subseteq \Delta L \subseteq QCL^{(=)}$.

Now let us reformulate the previous completeness result (Proposition 6.8.7) for the tree $T^k_n$ with the uniform finite branching $n > 0$ instead of $T^k_\omega$. 

Proposition 6.9.11 Let \( L \) be an intermediate predicate logic strongly complete \( w.r.t. \) a class \( C \) of Kripke frames. Let \( k \leq 0, \ n > 0 \), let \( L = \Delta^h L + \Theta \) for a set \( \Theta \) of sentences \( \forall \)-perfect in \( \Delta^h L \). Let \( C_0 = \mathcal{K}(IT^{k+1}_n) \). Let \( L' = \Delta^k L + B_n[\Theta] \), then \( L' \) is strongly complete \( w.r.t. \) \( C + C_0 \).

Recall that \( \Delta^h L \subseteq L \).

**Proof** Soundness follows from Lemma 7.8.5.

Now, let \( \Gamma \) be a \( L\forall \) place. Again we construct a \( L\forall \leq \) map over \( IT^{k+1}_n(\Gamma_u | \in IT^{k+1}_n) \) in such a way that \( \Gamma_v = \Gamma_0 \) and \( L \subseteq \Gamma_v \) for all \( v \in \text{max}(IT^{k+1}_n) \) and the naturalness condition (\( \forall' \)) for all \( v \in IT^k \) with \( v \in \beta(u) \) holds. Then for any \( v \in \text{max}(IT^{k+1}_n) \) we apply the strong completeness for \( L \) and find a natural \( L \)-model on a frame \( F_v \in C \) with \( \Gamma_v \) in its root; these models give us a natural model over \( F + \langle F_v | v \in \text{max}(IT^{k+1}_n) \rangle \), where \( F \in C_0 \).

Let us describe the inductive step. Let \( \Gamma_u \) for \( u \in IT^k_n \) be already constructed. If \( L = \Delta^k L + \Theta \subseteq \Gamma_u \), then we put \( \Gamma_v = \Gamma_u \) for all \( v \in (IT^{k+1}_n \uparrow u) \). And if \( \Delta^k L + \Theta \not\subseteq \Gamma_u \), then we apply Lemma 7.8.8 to obtain \( L' \forall \)-places \( \Gamma_v > \Gamma_u \) for \( n \) points \( v \in \beta(u) \). Note that a characteristic formula \( E \) for \( \Gamma \) exists by Lemma 7.4.21. In fact, here \( \Delta^{l+1} L \subseteq \Gamma \) and \( \Delta^L \not\subseteq \Gamma \). Finally, if \( n' = 0 \) (in applying of Lemma 7.8.8), then \( \Gamma_u \) is a \( QCL \)-place, and again we put \( \Gamma_v = \Gamma_u \) for all \( (v \in (IT^{k+1}_n \uparrow u) \); here we use that \( L \subseteq QCL \), cf. the concluding argument in the proof of 7.10.8.

A similar statement holds for \( KFE \). Note that the completeness for \( LP^{+}_{k+1} + Br_n = IL(IT^{k+1}_n) \) and for \( LP^+_{k+1} + Br_n = IL(IT^{k+1}_n) \) are the particular cases of these statements for \( L = QLC \).

### 6.10 Trees of bounded branching and depth

As we shall see in Volume 2, the s.p.l. \( IL(\mathcal{K}(\text{Fin})) \) determined by all finite posets is not recursively axiomatisable. Still we can explicitly describe 'approximations' of this logic of bounded height and width or branching.

Recall (from Chapter 1) that the propositional axiom

\[
F_n := \bigvee_{i=0}^{n} \left( p_i \supset \bigvee_{j \neq i} p_j \right)
\]

characterises posets of width \( \leq n \). Similarly, the axiom

\[
Br_n := \bigwedge_{i=0}^{n} \left( p_i \supset \bigvee_{j \neq i} p_j \supset \bigvee_{j \neq i} p_j \right) \supset \bigvee_{i=0}^{n} p_i
\]

or its equivalent version:

\[
Br'_n := \bigwedge_{i=0}^{n} (q \supset p_i \supset q) \wedge \bigwedge_{i<j} (q \supset p_i \vee p_j) \supset q
\]
identifies posets of branching \( \leq n \) among posets of finite depth. We also know that \( \mathbb{H} + Br_n \) is the propositional logic of finite \( n \)-ary trees and \( \mathbb{QH} + Br_1 = \mathbb{QH} + W_1 \).

In this section we study the propositional axiom \( Br_n \) in predicate logics. Now it is not sufficient for axiomatising Kripke frames over finite \( n \)-ary trees, as we shall see in Volume 2. But still it works properly above \( QHP^+_k \), i.e. for posets of bounded height.

So consider classes \( B^k_n := B_n \cap P^+_k \) of posets of depth \( \leq h \) and branching \( \leq n \) (for \( n, k > 0 \)) and also classes \( W^k_n := W_n \cap P^+_k \) of posets of depth \( \leq k \) and width \( \leq n \). Actually the predicate logics are determined by rooted posets, so we can deal only with the corresponding classes \( B^{k\dagger}_n \) and \( W^{k\dagger}_n \). Up to isomorphism, each of these classes contains finitely many finite posets. Therefore by 3.2.20, the logics \( IL(K(B^k_n)) \) and \( L(K(W^k_n)) \) are uniformly recursively axiomatisable. Note that \( W^{k\dagger}_n \subseteq B^{k\dagger}_n \subseteq W^{k\dagger}_{n-1} \). Also note that every finite poset belongs to \( W^k_n \) (hence to \( B^k_n \)) for some \( n, k \); thus \( \text{Fin}^\dagger = \bigcup_{n,k>0} W^k_n = \bigcup_{n,k>0} B^{k\dagger}_n \). It follows that the logic of finite predicate Kripke frames \( IL(K(\text{Fin})) \) is in \( \Pi^2_2 \). Now we shall find finite axiomatisations for the logics \( IL(K(B^k_n)) \) and thus obtain a simpler and more natural \( \Pi^2_2 \)-presentation for \( IL(K(\text{Fin})) \).

First we need some lemmas.

**Lemma 6.10.1** Let \( \Gamma \) be an \( L \)-place, \( Br_n \in L \), and assume that there exists a characteristic sentence \( E \) for \( \Gamma \) such that the theories \( (\Gamma \cup \{E\}, \Delta_i) \) are \( L \)-consistent for \( i = 0, 1, \ldots, n \). Then \( (\Gamma \cup \{E\}, \Delta_i \cup \Delta_j) \) is \( L \)-consistent for some \( i \neq j \).

**Proof** Suppose \( (\Gamma \cup \{E\}, \Delta_i \cup \Delta_j) \) is \( L \)-inconsistent for any \( i \neq j \).

Then for any pair \((i, j)\) with \( i \neq j \), we have

\[
\Gamma, E \vdash_L \bigvee \Delta^1_{ij} \vee \bigvee \Delta^2_{ji}
\]

for some finite \( \Delta^1_{ij} \subseteq \Delta_i \), \( \Delta^2_{ji} \subseteq \Delta_j \).

Hence by the Deduction theorem,

\[
\Gamma \vdash_L E \supset C_i \vee C_j,
\]

where

\[
\Delta^0_i := \bigcup_{j \neq i} (\Delta^1_{ij} \cup \Delta^2_{ji}), \quad C_i := \bigvee \Delta^0_i.
\]

On the other hand, \( \vdash_L E \wedge (E \supset C_i) \supset C_i \) and \( (\Gamma \cup \{E\}, \Delta_i) \) is \( L \)-consistent, so \( (E \supset C_i) \notin \Gamma \).

Since \( ((E \supset C_i) \vee (E \supset C_i \supset E)) \in \Gamma \) (by the property of characteristic formula, see 6.3.10), it follows that \( (E \supset C_i \supset E) \in \Gamma \). But

\[
\vdash_L \bigwedge_{i=0}^n (E \supset C_i \supset E) \wedge \bigwedge_{i<j} (E \supset C_i \vee C_j) \supset E,
\]
since $Br'_n \in L$. Since $\Gamma$ is an $L$-place, we obtain $E \in \Gamma$, which contradicts Definition 6.3.10.

Lemma 6.10.2 (Main lemma on $n$-branching) Let $\Gamma$ be an $L$-place, $Br_n \in L$, and let $E$ be a characteristic formula for $\Gamma$. Assume that $D_\Gamma = S \subseteq S'$, $|S' - S| = |S| = \aleph_0$. Then there exist $L$-places $\Gamma_1, \ldots, \Gamma_m$, $m \leq n$ such that

(a) $\Gamma \cup \{E\} \subseteq \Gamma_i$ for $i = 1, \ldots, m$ (so $\Gamma R_L \Gamma_i$);
(b) if $(B \supset C) \in (-\Gamma)$ and $B \not\in \Gamma$, then $(B \supset C) \not\in \Gamma_i$ for some $i$.

Proof Let $\{(B \supset C) \in (-\Gamma) \mid B \not\in \Gamma\} = \{A_k \mid k \in \omega\}$. We define a family of theories $\Delta_k \subseteq L(\Gamma)$ for $k \in \omega$, $i > 0$ such that

(0) $\forall i, k \Delta_k^i \subseteq \Delta_{k+1}^i$,
and for any $k$ there exists $n_k \leq n$ with the following properties

(1) $\Delta_k^i \neq \emptyset$ iff $1 \leq i \leq n_k$,
(2) $(\Gamma \cup \{E\}, \Delta_k^i)$ is $L$-consistent for $1 \leq i \leq n_k$,
(3) $(\Gamma \cup \{E\}, \Delta_k^i \cup \Delta_j^i)$ is $L$-inconsistent for any different $i, j \leq n_k$.

Hence by Lemma 6.10.1, $n_k \leq n$.

To begin the construction, put $\Delta_0^i := \emptyset$ for all $i > 0$ (i.e. $n_0 = 0$). $\Delta_k^i$ is constructed as follows. Consider $A_k = B \supset C$. There are two cases.

(I) If $(\Gamma \cup \{E\}, \Delta_k^i \cup \{B \supset C\})$ is $L$-consistent for some $i \leq n_k$, then take the minimal $i$ with this property and put

$\Delta_{k+1}^i := \Delta_k^i \cup \{B \supset C\}$, $\Delta_{k+1}^j := \Delta_k^j$ for all $j \neq i$ (thus $n_{k+1} = n_k$).

(II) If for any $i \leq n_k$, $(\Gamma \cup \{E\}, \Delta_k^i \cup \{B \supset C\})$ is $L$-inconsistent, then put

$\Delta_{k+1}^{n+1} := \{B \supset C\}$ and $\Delta_{k+1}^i := \Delta_k^i$ for all $i \leq n_k$ (thus $n_{k+1} = n_k + 1$).

The properties (1), (3) hold trivially by definition, the same with (2) in case (I). To check (2) in case (II), we have to show that $(\Gamma \cup \{E\}, \{B \supset C\})$ is $L$-consistent, i.e. $(E \supset, B \supset C) \not\in \Gamma$.

In fact, $B \not\in \Gamma$ and $B \lor (B \supset E) \in \Gamma$, since $E$ is a characteristic formula, thus $(B \supset E) \in \Gamma$. So $(E \supset, B \supset C) \in \Gamma$ implies $(B \supset, B \supset C) \in \Gamma$, which contradicts $(B \supset C) \not\in \Gamma$. Therefore $(E \supset, B \supset C) \not\in \Gamma$.

Next, put $m := \max\{n_k \mid k \in \omega\}$ (thus $m \leq n$) and

$\Delta_i := \bigcup_{k \in \omega} \Delta_k^i$

for $1 \leq i \leq m$. Since the sequence $(\Delta_k^i)_{k \in \omega}$ is increasing, (2) implies the $L$-consistency of every $(\Gamma \cup \{E\}, \Delta_i)$. So there exist $LV$-places $\Gamma_i \succ (\Gamma \cup \{E\}, \Delta_i)$.

Thus condition (a) holds. (b) holds by our construction, since $B \supset C = A_k$ for some $k$, whenever $B \not\in \Gamma$; so $C \in \Delta_i$. ■
Lemma 6.10.3 Assume that $\text{QHP}_k^+ + Br_n \subseteq L$. Then for any $L$-place $\Gamma$ there exists a proper natural $(LV, \leq)$-model $N = (\Gamma_u \mid u \in F)$ based on a greedy standard subtree of $IT^k_n$ such that $\Gamma_\lambda = \Gamma$ and a natural $(LV, \leq)$-model on $IT^k_n$ (not necessarily proper).

Proof We construct $\Gamma_u$ using Lemma 6.10.2, which gives us the condition of naturalness from 6.4.9. If $0 < m < n$, we repeat some theories; if $m = 0$ then the construction terminates at this point. Recall that characteristic formulas used in Lemma 6.10.2 always exist by 6.3.12. The construction terminates at some height $\leq h$, due to Lemma 6.3.7. Finally we repeat the theories from the maximal points of the resulting subtree at all leaves of $IT^k_n$.

Proposition 6.10.4 $IL(\mathcal{K}(B^k_n)) = IL(\mathcal{K}(IT^k_n)) = \text{QHP}_k^+ + Br_n$ for any $n, k > 0$, where $IT^k_n$ is the $n$-ary tree of height $k$.

Proof Readily follows from 6.10.3.

In particular, for $n = 1$ we have:

Corollary 6.10.5 $\text{QHP}_k^+ + \text{LC}$ is determined by an $k$-element chain.

Corollary 6.10.6

$$IL(\mathcal{K}(<\text{Fin}>) = \bigcup_{n,k}(\text{QHP}_k^+ + Br_n) = \bigcap_{n,k}(\text{QHP}_k^+ + W_n).$$

Proof In fact, $H + W_n \vdash Br_n$ and $H + AP_k \land Br_n \vdash W_m$ for some $m$ (e.g. due to Kripke completeness). Thus

$$\bigcap_{n,k}(\text{QHP}_k^+ + Br_n) = \bigcap_{n,k}(\text{QHP}_k^+ + W_n).$$

On the other hand, the logics $\text{QHP}_k^+ + W_n$ for $k \geq 3$, $n \geq 2$ are Kripke incomplete [Skvortsov, 2006]; the proof will be given in Volume 2. Our conjecture (based on some results from Volume 2) is that all the corresponding logics $IL(\mathcal{K}(W^h_n))$ are not finitely axiomatisable.

6.11 Logics of uniform trees

As we shall see in Volume 2, generalised versions of formulas $Br_n$ allow us to axiomatise the intermediate predicate logic of all finite trees explicitly. Recall that the logic determined by an arbitrary finite poset is recursively axiomatisable (Chapter 3), but the proof of this fact provides only an implicit axiomatisation.

In this section we make a step towards the logic of all finite trees clarifying main ideas of the whole construction; here we describe an explicit axiomatisation of an s.p.l. determined by an arbitrary ‘uniform’ tree of finite depth.
Definition 6.11.1 A tree $F$ is called (levelwise) uniform if all its cones $F \uparrow u$ with roots $u$ of the same height are isomorphic.

A levelwise uniform tree $F$ of depth $h+1$ gives rise to an $h$-sequence $t = (t_i | i < h)$ of non-null cardinals $t_i$; $t_i$ is the branching $|\beta(u)|$ for all points $u \in F$ of height $i < h$ (obviously, the branching at a maximal point is 0).

The other way round, every $h$-sequence $t$ of cardinals corresponds to a unique tree $T_t$ of depth $(h+1)$ and of branching $t_i$ at points of height $i$; this tree is clearly levelwise uniform. Obviously $T_t$ is finite (respectively, countable) iff every $t_i$ is finite (respectively, countable). The corresponding $h$-sequences are called $h^f$- (respectively, $h^c$-)sequences.

Every denumerable uniform tree can be represented as a subtree of $IT_\omega$:

$$T_t = \{ \alpha \in \omega^* \mid |\alpha| \leq h, \forall i \alpha_i < t_i \}.$$  

Every $h$-sequence $t$ of cardinals gives rise to the truncated $h^{\leq \omega}$-sequence $t^+$, in which every infinite $t_i$ is replaced with $\omega$.

We shall construct explicit axiomatisations for all denumerable levelwise uniform trees and show that

$$IL(\mathcal{K}(T_t)) = IL(\mathcal{K}(T_{t^+}))$$

for any uncountable $T_t$. Note that for $h = 0$ there is the empty sequence corresponding to a one-element tree $T_k$ of depth 1. A constant sequence $t = (n, \ldots, n)$ of length $h \geq 0$ corresponds to a uniform tree $T_t = IT_n^{k+1}$ considered in Section 6.10 or to $IT_\omega^{k+1}$ if $n = \omega$.

Definition 6.11.2 Let $A$ be a predicate formula and let $q, p_0, p_1, \ldots, p_n$ be different proposition letters non-occurring in $A$, $n > 0$. We put

$$p_i^- := \bigvee \{ p_j \mid 0 \leq j \leq n, j \neq i \}$$

and define the formulas

$$Br_n^A := \bigwedge_{i=0}^n \left( (p_i \supset p_i^- \supset p_i^-) \land \left( A \supset \bigvee_{i=0}^n p_i \right) \supset \bigvee_{i=0}^n p_i \right),$$

$$Br'_n^A := \bigwedge_{i=0}^n \left( (q \supset p_i \supset q) \land \bigwedge_{j < i} (q \supset p_i \lor p_j) \land (A \supset q) \supset q \right).$$

We will also use the notation $Br_n[A], Br_n'[A]$ for $Br_n^A, Br_n'^A$.

Obviously, $Br_n^A$ and $Br_n'^A$ are equivalent to $\left( A \supset \bigvee_{i=0}^n p_i \right) \supset Br_n$ and $(A \supset q) \supset Br'_n$ respectively. Thus $Br_n^+$ and $Br_n'^+$ are equivalent to $Br_n$ and $Br'_n$.

Now let us show that both axioms are actually deductively equivalent (cf. Lemma 2 for $Br_n$ and $Br'_n$):

Lemma 6.11.3 $QH + Br_n^A = QH + Br_n'^A$ (for $n > 0$).
6.11. LOGICS OF UNIFORM TREES

Proof (2) Let $p := p_1 \ldots p_n$, $C := C_1 \ldots C_n$, where $C_i := \bigwedge_{j \neq i} p_j$ and let us show that

\[(1) \quad \text{QH} \vdash [C/p] Br_n^A \supset Br_n^A.\]

Let $B$ be the premise of $Br_n^A$, $C_i := \bigvee_{j \neq i} C_j$,

\[E := \bigwedge_{i=0}^n (C_i \supset C_i^- \supset C_i^-) \land \left( A \supset \bigvee_{i=0}^n C_i \right) \]

the premise of $[C/p] Br_n^A$.

It suffices to check that

\[(2) \quad B \vdash_{\text{QH}} E \land \left( \bigvee_{i=0}^n C_i \supset q \right); \]

hence we can obtain

\[B, [C/p] Br_n^A = \left( E \supset \bigvee_{i=0}^n C_i \right) \vdash_{\text{QH}} \bigvee_{i=0}^n C_i, \bigvee_{i=0}^n C_i \supset q, \]

and thus

\[[C/p] Br_n^A, B \vdash_{\text{QH}} q, \]

which implies (1).

For the proof of (2) first note that

\[(3) \quad \bigwedge_{i<j} (q \supset p_i \lor p_j) \vdash_{\text{H}} q \supset \bigvee_{i=0}^n C_i. \]

To show this, we argue in H. Assume $q$ and $\bigwedge_{i<j} (q \supset p_i \lor p_j)$. Then $\bigwedge_{i<j} (p_i \lor p_j)$, which is equivalent to $\bigvee_{i=0}^n C_i$ by distributivity - in fact, this should be a disjunction of conjunctions, in which every conjunction contains either $p_i$ or $p_j$ from each distinct pair $(i, j)$; so there is at most one $p_i$ missing from that conjunction.

Thus

\[(4) \quad B \vdash_{\text{QH}} q \supset C_i \lor C_i^- . \]

We also have

\[(5) \quad \text{H} \vdash C_i^- \supset p_i, \]

since $p_i$ is present in every $C_j$ for $j \neq i$. Hence

\[(6) \quad C_i \supset C_i^- \vdash_{\text{H}} C_i \supset p_i \]
by (5),

(7) \[ B, C_i \supset C_i^\perp \vdash_{\text{QH}} q \supset C_i^\perp \]

by (4),

(8) \[ B, C_i \supset C_i^\perp \vdash_{\text{QH}} q \supset p_i \]

by (7), (5),

(9) \[ B, C_i \supset C_i^\perp \vdash_{\text{QH}} q \]

by (8), since \((q \supset p_i) \supset q\) is a conjunct in \(B\),

(10) \[ B, C_i \supset C_i^\perp \vdash_{\text{QH}} C_i^\perp \]

by (9), (7). Thus

(11) \[ B \vdash_{\text{QH}} C_i \supset C_i^\perp \supset C_i^\perp . \]

Next,

(12) \[ B \vdash_{\text{QH}} A \supset \bigvee_{i=0}^{n} C_i, \]

by (3) and since \(A \supset q\) is also a conjunct in \(B\).

Next, we obviously have

\[ C_i \vdash_{\text{H}} p_j, \]

for \(j \neq i\), hence

\[ C_i \vdash_{\text{H}} q \supset p_j, \]

and consequently

(13) \[ B, C_i \vdash_{\text{QH}} q, \]

since \(B \vdash_{\text{QH}} (q \supset p_j) \supset q\). Thus

(14) \[ B \vdash_{\text{QH}} C_i \supset q. \]

by (13). Eventually (2) follows from (11), (12), and (14).

\((\subseteq)\) Similarly,

(15) \[ [D, p_1^\perp, \ldots, p_n^\perp / q, p_1, \ldots, p_n] B_{r_n^A}^n \vdash_{\text{QH}} B r_n^A, \]

where \( D := \bigvee_{i=0}^n p_i. \)

In fact, \( \vdash_{\text{H}} D \equiv p_i \lor p_i^\perp \), hence

\[ \vdash_{\text{H}} (D \supset p_i^\perp) \equiv (p_i \supset p_i^\perp), \]
and thus
\[ p_i \supset p_i^- \supset p_i^- \vdash \text{H} \supset p_i^- \supset p_i^- . \]
Since \( \vdash \text{H} \supset p_i^- \supset D \), this implies
\[ \vdash \text{H} \supset p_i^- \supset D \supset p_i^- \supset D . \]

It also obvious that
\[ \vdash \text{H} \supset D \equiv p_i^- \lor p_j^- \]
for \( i < j \). So by (16), (17) the premise of \( Br_n^A \):
\[ \bigwedge_{i=0}^{n} (p_i \supset p_i^- \supset p_i^-) \land (A \supset D) \]
implies the premise of \([D, p_1^-, \ldots, p_n^-/q, p_1, \ldots, p_n] Br_n^A : \]
\[ \bigwedge_{i=0}^{n} (D \supset p_i^- \supset D) \land \bigwedge_{i<j} (D \supset p_i^- \lor p_j^-) \land (A \supset D). \]

Since the conclusions of \([D, p_1^-, \ldots, p_n^-/q, p_1, \ldots, p_n] Br_n^A \) and \( Br_n^A \) coincide, the claim (15) follows. \( \blacksquare \)

**Lemma 6.11.4** The following formulas are \( \text{QH} \)-theorems:

1. \( A \supset Br_n^A, \ A \supset Br_n'. \)
2. \( (A_1 \supset A_2) \supset (Br_n^A \supset Br_n^A), \ (A_1 \supset A_2) \supset (Br_n'A \supset Br_n'A). \)
3. \( \bigwedge_{i=1}^{r} Br_n^A_i \supset Br_n \left[ \bigwedge_{i=1}^{r} A_i \right], \ \bigwedge_{i=1}^{r} Br_n'A_i \supset Br_n' \left[ \bigwedge_{i=1}^{r} A_i \right]. \)
4. \( Br_n \supset Br_n^A, \ Br_n' \supset Br_n'. \)

**Proof**

1. A trivial exercise.
2. Note that \( A_1 \supset A_2, A_2 \supset D \vdash A_1 \supset D. \)
3. By Lemma 1.1.2(5),
\[ \text{QH} \vdash \bigwedge_{i=1}^{r} (A_i \supset q) \supset q = \left( \left( \bigwedge_{i=1}^{r} A_i \supset q \right) \supset q \right). \]
Since \( Br_n'A \) is equivalent to \( B \supset (A \supset q \supset q) \), where \( B \) does not depend on \( A \) and \( q \), the claim follows. For \( Br_n'A \) replace \( q \) with \( \bigvee_{i} p_i. \)
4. Follows from (2), since \( \vdash \text{QH} \perp \supset A. \) \( \blacksquare \)
For a set of formulas $\Theta$ we define the sets

$$Br^\Theta_n := Br_n[\Theta] := \{ Br^A_n \mid A \in \Theta \}, Br^\Theta_n := Br^A_n[\Theta] := \{ Br^A_n \mid A \in \Theta \}. $$

Then

$$QH + Br^\Theta_n = QH + Br^\Theta_n \subseteq QH + \Theta$$

by 6.11.3 and 6.11.4 (1).

We also put

$$Br_\omega := Br^A_\omega := \top, Br^\Theta_\omega := Br'^\Theta_\omega := \emptyset.$$ 

Lemma 6.11.5 Let $\Theta_1, \Theta_2$ be sets of sentences and let $L$ be an s.p.l., $n > 0$.

(1) If $L + \Theta_1 \subseteq L + \Theta_2$, then

(a) $L + Br_n[\Theta_1] \subseteq L + Br_n[\Theta_2]$ provided $\Theta_2$ is $\forall$-perfect in $L$;

(b) $L + Br_n[\Theta_1'] \subseteq L + Br_n[\Theta_2']$ for any $\Theta_1, \Theta_2$.

(2) If $L + \Theta_1 = L + \Theta_2$, then

(a) $L + Br_n[\Theta_1] = L + Br_n[\Theta_2]$ provided $\Theta_1, \Theta_2$ are $\forall$-perfect in $L$;

(b) $L + Br_n[\Theta_1'] = L + Br_n[\Theta_2']$.

Proof Cf. Lemma 2.8.7 (on $\delta$-operation). Recall that $\Theta^V$ is $\forall$-perfect in $QH$ for any $\Theta$. }

Lemma 6.11.6 Let $F$ be an intuitionistic Kripke frame of finite depth, $0 < n < \omega$. Then

$$Br^A_n \in IL(F) \text{ iff } \forall u \in F \ (|\beta(u)| > n \Rightarrow F \uparrow u \not\models A).$$

So $Br^A_n$ states that 'the branching is $\leq n$ until $A$ becomes true'.

Proof Let $\leq$ be the accessibility relation in $F$.

(If.) Suppose $Br^A_n \notin IL(F)$, so due to the finiteness of height, there exists a Kripke model over $F$ and $u \in F$ of maximal height refuting $Br^A_n$, i.e.

$$u \not\models (q \supset p_i) \supset q, u \not\models (A \supset q), u \models q \supset p_i \lor p_j \text{ for } i \neq j, u \not\models q, \forall v > u \not\models q.$$ 

Then $u \not\models q \supset p_i$, so there exist a minimal $v_i > u$ such that $v_i \not\models q$, $v_i \not\models p_i$, and obviously $v_i \in \beta(u)$. Since $u \not\models q \supset p_i \lor p_j$, it follows that $v_i \not\models p_j$ for all $j \neq i$. So all $v_i$ differ, and thus $|\beta(u)| > n$.

Since $u \not\models q$, $u \models A \supset q$, it follows that $u \not\models A$, therefore $F \uparrow u \not\models A$.

(Only if.) Suppose $F \uparrow u \not\models A$ and there exist different $v_0, \ldots, v_n \in \beta(u)$. Consider an intuitionistic model over $F$ such that $u \not\models A$ and

$$v \not\models q \iff v \leq u,$$

$$v \not\models p_i \iff v \leq v_i.$$
(Such a model exists, since \( p_i, q \) do not occur in \( A \).) Then \( u \not\models q \) and \( u \models A \supset q \), since \( u \not\models A \). Also \( u \models q \supset p_i \lor p_j \) for \( i \neq j \), since 

\[ v \supset u \Rightarrow v \not\subseteq u_i \lor v \not\subseteq u_j. \]

Finally note that \( v \not\models q \supset p_i \); thus \( u \not\models q \supset u \) and \( u \models (q \supset p_i) \supset q \). Hence 

\( F \not\models Br_n^{IA} \)

Remark 6.11.7 Note that \( QH + Br_n^{EM} = QH + Br_n \), and so 

\( QH + Br_n^{EM} = QH + Br_n \)

(recall that \( EM = p \lor \neg p \)). In fact, \( Br_n^{EM} \models (EM \supset q) \supset Br_n \), whence we deduce \( (q \lor \neg q \supset q) \supset Br_n \). The latter is equivalent to \( \models (\neg q \supset q) \supset Br_n \), which readily implies \( Br_n \), since \( H \models ((q \supset p_i) \supset q) \supset (\neg q \supset q) \).

From the semantical point of view, the statement we have proved means that if the branching is \( \leq n \) in all nonmaximal points then it is \( \leq n \) everywhere. However the equality of logics does not directly follow from this semantical argument, once we have not yet proved the completeness of \( QH + Br_n^{EM} \).

Now let us prove a natural analogue of the main lemma on \( n \)-branching, 6.10.2, for \( Br_n^{O} \). We begin with an analogue of 6.10.1.

Lemma 6.11.8 Let \( L \) be an s.p.l., \( \Theta \) a set of sentences that are \( \forall \)-perfect in \( L \). Assume that \( Br_n^{O} \subseteq L, \Gamma \in VP_L \) and \( L + \Theta \not\subseteq \Gamma \). Also let \( E \) be a characteristic formula for \( \Gamma \), and assume that \( (\Gamma \cup \{ E \}, \Delta_i) \) are \( L \)-consistent theories in the language \( L(\Gamma) \) for \( i = 0, 1, \ldots , n \).

Then \( (\Gamma \cup \{ E \}, \Delta_i \cup \Delta_j) \) is \( L \)-consistent for some \( i \neq j \).

Proof Since \( L + \Theta \not\subseteq \Gamma \), there exists \( A \in \text{Sub}(\Theta) \) such that \( A \in (\neg \Gamma) \). In fact, if \( B \in L + \Theta - \Gamma \), then by \( \forall \)-perfection, there exist \( A_1, \ldots , A_k \in \text{Sub}(\Theta) \) such that \( \models_L A_1 \land \ldots \land A_k \supset B \); by Lemma 2.7.11 we may assume that \( A_1, \ldots , A_k \in L(\Gamma) \), since \( D_\Gamma \) is denumerable and the parameters of \( A_i \) can be replaced with constants from \( D_\Gamma \). Hence \( A_i \in (\neg \Gamma) \) for some \( i \).

Now we proceed as in the proof of Lemma 6.10.1 and find \( C_i \) (for \( 0 \leq i \leq n \)) such that \( ((E \supset C_i) \supset E) \in \Gamma \) and \( (E \supset C_i \lor C_j) \in \Gamma \) whenever \( i < j \).

Finally, since \( Br_n^{O} \subseteq L \), 

\[ \models_L \bigwedge_{i=0}^{n} ((E \supset C_i) \supset E) \land \bigwedge_{i<j} (E \supset C_i \lor C_j) \land (A \supset E) \supset E, \]

So by \( L \)-consistency, this formula is in \( \Gamma \). Hence \( (A \supset E) \not\in \Gamma \), which contradicts the property of characteristic formula \( E \). 

Lemma 6.11.9 Let \( L \) be an s.p.l., \( \Theta \) be a set of sentences that are \( \forall \)-perfect in \( L, \Gamma \in VP_L \). Also assume that \( L + \Theta \not\subseteq \Gamma \) and \( Br_n^{O} \subseteq L \) for a finite \( n \). Let \( E \) be a characteristic formula for \( \Gamma \).

Then there exist \( \Gamma_1, \ldots , \Gamma_m \in VP_L \) (where \( m \leq n \)) such that
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(1) \( D_{\Gamma_1} = \cdots = D_{\Gamma_m} \);

(2) \( \Gamma \cup \{ E \} \subseteq \Gamma_i \) for \( i = 1, \ldots, m \) (and thus, \( \Gamma < \Gamma_i \));

(3) if \( (A_1 \supset A_2) \in (-\Gamma) \) and \( A_1 \not\in \Gamma \), then \( (A_1 \supset A_2) \not\in \Gamma_i \) for some \( i \).

Proof Along the same lines as 6.10.2, using Lemma 6.11.8 instead of 6.10.1.

Definition 6.11.10 Let \( t = (t_i \mid i < h) \) be an \( h^+ \)-sequence, \( h > 0 \). Put

\[
C^+(t) := \{ h + 1 \} \cup \{ i < h \mid i \neq 0, t_{i-1} < t_i \}, \\
C(t) := \{ 0, h + 1 \} \cup \{ i < h \mid i \neq 0, t_{i-1} \neq t_i \}.
\]

The levels from \( C^+(t) \) and \( C(t) \) are respectively called \(+\)-critical and critical for \( t \).

For \( i \in C(t) \), \( i < h \), put

\[
i^* := \min \{ j > i \mid j \in C(t) \}, \\
i^+ := \begin{cases} h + 1 & \text{if } \forall j > i\ t_j \leq t_i, \\ \min \{ j > i \mid t_j > t_i \} & \text{otherwise}. \end{cases}
\]

Obviously, \( i < i^* \leq i^+ \) and \( i^+ \in C^+(t) \), although \( i^+ \) may be non-equal to \( \min \{ j > i \mid j \in C^+(t) \} \) (the next element of \( C^+(t) \)).

It is also clear that for any \( u \in T_t \) of level \( i < h \), the tree \( T_t \uparrow u \) is of depth \( h_i := h + 1 - i \), and \( T_t \uparrow u \cong T_t^i := T_{t^i} \), where \( t^i := (t_i, t_{i+1}, \ldots, t_{h-1}) \).

Lemma 6.11.11 Let \( t = (t_i \mid i < h) \) be an \( h^+ \)-sequence, and let \( i \in C(t) \), \( 0 \leq i < h \). Assume that the logics are strongly complete w.r.t. the classes of frames \( K(T^i_t) \) and \( K(T^{i^+}_t) \) respectively. Let \( \Theta_{i^+} \) be a set of sentences that are \( \forall \)-perfect in \( \text{QHP}^+_{k+1} \), and assume that \( \text{QHP}^+_{k+1} + \Theta_{i^+} = L_{i^+} \).

Then the logic

\[
L_i := \Delta^{i^* - i}(L_{i^*}) + Br_{t_i}[\Theta_{i^+}]
\]

is strongly complete w.r.t. \( K(T^i_t) \), and therefore \( L_i = \text{IL}(K(T^i_t)) \).

We may also put

\[
L_{i^*} := \text{QH} + \bot \text{ iff } i^* = k + 1
\]

and similarly,

\[
\Theta_{i^+} := \{ \bot \} \text{ iff } i^+ = k + 1.
\]

So \( T^{k+1}_t \) is an 'empty tree' and its logic is inconsistent. Also note that \( \text{QHP}^+_{k+1 - i} \subseteq L_i \), since \( \text{LP}^+_{i-1} \subseteq L_{i^*} \), so \( \text{QHP}^+_{k+1 - i} = \Delta^{i^* - i}(\text{LP}^+_{k+1 - i}) \subseteq \Delta^{i^* - i}(L_{i^*}) \subseteq L_i \).
Proof Fix a point \( u_0 \in T_t \) of height \( i \), so \( T_{t}^{i} = T_t \uparrow u_0 \).

First let us show that \( T_t^{i} \) is an \( L_i \)-frame. In fact, \( \Delta^{i^*-i}(L_{i^*}) \subseteq IL(T_t^{i}) \), since \( L_{i^*} \) is valid at all points of height \( \geq i^* - i \) in \( T_t^{i} \), i.e. at levels \( \geq i^* \) in \( T_t^{i} \). Also \( Br_{t_{i}}^{\uparrow} \in IL(T_{t}^{i}) \) for any \( A \in \Theta_{i^*} \) by Lemma 6.11.6, since \( A \in IL(T_{t}^{i}) = IL(T_t \uparrow v) \) for any \( v \) of level \( j \geq i^* \) and the branching of \( T_t \) is \( \leq t_i \) at any point of level \( i \leq j < i^* \) (to minimise calculations, we consider levels in the tree \( T_t \) not in \( T_t^{i} \)).

Now we prove the strong completeness. Let \( R \) be the accessibility relation in \( T_t \). Consider an \( L_i \)-place \( \Gamma \). We put \( \Gamma_{u_0} := \Gamma \) for root \( u_0 \) of \( T_t^{i} \) and construct (inductively) \( L_i \)-places \( \Gamma_u \) for points \( u \in T_t^{i} \) of levels from \( i \) till \( i^* \). During our construction we take care of the naturalness property \((\exists)\)' from Lemma 6.4.23 and the property of \( \forall \leq \)-models: \( \Gamma_u \leq \Gamma_v \) if \( uRv \) in \( T_t^{i} \).

Let us begin with the 'trivial' case: \( L_i + \Theta_{i^*} \subseteq \Gamma_{u_0} \). Then we put \( \Gamma_v := \Gamma_{u_0} \) for all \( v \in R(u_0) \) of levels \( < i^* \). And for any \( w \) of level \( i^* \) we obtain a natural model over \( T_t^{i} \) if \( w = T_{t}^{i*} \) with \( \Gamma_w \leq \Gamma_v \) by the strong completeness of \( L_{i^*} \); in fact, \( \Gamma_u \) is an \( L_i \)-place, since \( L_{i^*} = QHP_{k+1}^{+} + \Theta_{i^*} \subseteq L_i + \Theta_{i^*} \subseteq \Gamma_{u_0} \). Note that the naturalness condition \((\exists)\)' holds for all points \( v \) of levels \( j < i^* \); in fact, if \( (A_1 \supset A_2) \in (-\Gamma_v) \), then \( (A_1 \supset A_2) \notin \Gamma_w \) for any \( w \in R(v) \) of level \( i^* \).

Now consider the 'nontrivial' case \( L_i + \Theta_{i^*} \not\subseteq \Gamma_{u_0} \). Then if \( t_i < \omega \), we apply Lemma 6.11.9 (for \( n = t_i \), \( \Theta = \Theta_{i^*} \)) and find \( L_i \)-places \( \Gamma_v \) for \( v \in \beta(u_0) \) such that \( \Gamma_{u_0} < \Gamma_v \) for all \( v \) and the naturalness condition \((\exists)\)' for \( \Gamma_{u_0} \) holds, cf. the proof of 6.10.3 (recall that a characteristic formula \( E \) exists by Proposition 6.3.12, since \( LP_{i^*+1}^{+} \subseteq L_i \)).

Then we repeat the same construction for points \( v \in \beta(u_0) \) and so on, till level \( i^* \); note that \( t_j = t_i \) for all levels \( j \) such that \( i \leq j < i^* \). If for a certain \( v \), \( L_i + \Theta_{i^*} \subseteq \Gamma_v \) and Lemma 6.11.9 is inapplicable, then we proceed as in the 'trivial' case, and repeat this \( L_i \)-place \( \Gamma_v \) at all points above \( v \) till level \( i^* \). If we successfully arrive at a point \( w \) of level \( i^* \), then we obtain a sequence of \( L_i \)-places \( \Gamma_u < \ldots < \Gamma_w \). By Lemma 6.3.1 we can conclude that \( L_{i^*} \subseteq \Gamma_w \) (i.e. \( \Gamma_w \) is an \( L_{i^*} \)-place), since \( \Gamma_{u_0} \) is a \( \Delta^{i^*-i}(L_{i^*}) \)-place. Now we apply the strong completeness of \( L_{i^*} \) and obtain a natural model over \( T_t^{i} \).

There also exists a degenerate case, when at point \( u_0 \) (or at some intermediate point) Lemma 6.11.9 should be applied to \( m = 0 \), so it does not produce any \( L_i \)-place \( \Gamma_v \) for \( v \in \beta(u_0) \). But in this case the naturalness condition \((\exists)\)' trivially holds for \( \Gamma_{u_0} \), so we can repeat the place \( \Gamma_{u_0} \) at every \( v \in T_t \uparrow u_0 \) and thus obtain a required natural model.

Finally let us mention the case \( t_i = \omega \). Now if \( (A_1 \supset A_2) \in (-\Gamma_{u_0}) \) and \( A_1 \notin \Gamma_{u_0} \), we construct a \( L_i \)-place \( \Gamma_v \) such that \( \Gamma_{u_0} \cup \{A_1\} \subseteq \Gamma_v \), \( A_2 \notin \Gamma_v \) (thus \( (A_1 \supset A_2) \notin \Gamma_v \) and \( \Gamma_{u_0} < \Gamma_v \)), cf. the proof of Lemma 6.2.13(2). So we obtain infinitely many \( L_i \)-places \( \Gamma_v \) for \( v \in \beta(u_0) \). We apply this construction to all points from \( \beta(u_0) \) and further on, till level \( i^* \). Eventually we obtain \( L_{i^*} \)-places \( \Gamma_w \) for all points \( w \) of level \( i^* \) and use the strong completeness of \( L_{i^*} \) as in the previous case.

It is clear that Lemma 6.11.11 yields a \( \forall \)-perfect axiomatisation of \( IL(K(T_t^{i})) \)
for any $h^t$-sequence $t$ and $i \in C(t)$; in particular, for $\text{IL}(\mathcal{K}(T_t^i)) = \text{IL}(\mathcal{K}(T_{0}^i))$.

**Proposition 6.11.12** Let $t = (t_i \mid i < h)$ be an $h^t$-sequence. Consider the sequence $\Theta_i$ for $i \in C(t)$ constructed by induction on $h + 1 - i$:

\[ \Theta_{h+1} := \{ \bot \}, \]
\[ \Theta_i := \delta_{i-1}^{0} (\Theta_{i+1}) \cup (Br_{i+1}[\Theta_{i+1}])^{\vee} \quad \text{for} \quad i < h. \]

Then all $\Theta_i$ are $\forall$-perfect in $\mathbf{LP}_{h+1}^{+}$,

\[ \text{IL}(\mathcal{K}(T_t^i)) = \mathbf{LP}_{h+1}^{+} + \Theta_i, \]

and these logics are strongly Kripke-complete w.r.t. $\mathcal{K}(T_t^i)$. In particular,

\[ \text{IL}(\mathcal{K}(T_t)) = \mathbf{LP}_{h+1}^{+} + \Theta_0 \]

is strongly Kripke-complete w.r.t. $\mathcal{K}(T_t)$.

**Proof** Every $\Theta_i$ is $\forall$-perfect in $\mathbf{LP}_{h+1}^{+}$ and

\[ \mathbf{LP}_{h+1}^{+} + \delta_{i-1}^{0} (\Theta_{i+1}) = \mathbf{LP}_{h+1}^{+} + \Delta^{i-1} (\mathbf{LP}_{h+1}^{+} + \Theta_{i+1}) \]

by Proposition 2.13.33. Recall that $\mathbf{LP}_{h+1}^{+} \subseteq \text{IL}(\mathcal{K}(T_t)) \subseteq \text{IL}(\mathcal{K}(T_{0}^i))$ and also that every set of the form $\Theta^{\vee}$ is $\forall$-perfect in $\mathbf{QH}$ and $\forall$-perfection is $\cup$-additive. So we can apply Lemma 6.11.11 at the induction step $i$. ■

Note that $AP_{h+1-i} = \delta_{h+1-i}^{0} \bot \in \Theta_i$ for any $i$, since $\bot \in \Theta_{h+1}$ and $\delta_{i-1}^{0} (\delta_{h+1-i}^{0} \bot) = \delta_{h+1-i}^{0} \bot$.

Recall that $\mathbf{LP}_{h+1}^{+} + AP_{h+1-i} = \mathbf{LP}_{h+1-i}$ by Proposition 2.13.31, so $\mathbf{LP}_{h+1-i} \subseteq \mathbf{LP}_{h+1}^{+} + \Theta_i$ for all $i \in C(t)$.

Also note that for $h = 0$ and the empty $h$-sequence $t$ we have $C(t) = \{0, 1\}$, $\Theta_0 = \{ \bot \}$, $\Theta_0 = \{ \delta \bot \} = \{ AP_1 \}$, and this axiom generates classical logic $\mathbf{QCL} = \text{IL}(\mathcal{K}(T_k))$.

For a constant sequence $t = (n, \ldots, n)$ of length $k > 0$ (where $0 < n \leq \omega$) we have $C(t) = \{ 0, k+1 \}$, $\Theta_k+1 = \{ \bot \}$. If $n$ is finite, then $\Theta_0 = \{ \delta_{k+1}^{0} \bot \} \cup \{ Br_n \} = \{ AP_{k+1}, Br_n \}$, so we obtain the axiomatisation $\mathbf{LP}_{k+1}^{+} + Br_n$ for the uniform tree $T_n^{k+1}$ described in Section 6.10.

If $n = \omega$, then $\Theta_0 = \{ AP_{k+1} \}$, so we obtain a standard axiomatisation $\mathbf{LP}_{k+1}^{+}$ of $\mathbf{IT}_{\omega+1}$; recall that $Br_{\omega}^{\vee} = \emptyset$.

In some cases the axiomatisation described in Proposition 6.11.12 can be simplified. Note that by Lemmas 2.9.2 and 6.11.5, $\delta_{k}^{0}$ and $Br_n[(\ldots)]^{\vee}$ preserve deductive equivalence, so every $\Theta_i$ can be replaced with its simpler deductive equivalent. E.g. if $A \in \Theta_i$, and $A \in \Theta_i$ for some $i < h$, then we can eliminate $\delta_{i-1}^{0} A$ from $\delta_{i-1}^{0} (\Theta_i^t)$, as this formula follows from $A$.

Proposition 6.11.12 can be strengthened as follows.
6.11. LOGICS OF UNIFORM TREES

Proposition 6.11.13 Let $t = (t_i \mid i < h)$ be an $h^i$-sequence, $L_0$ an s.p.l. such that $L_{h+1}^+ \subseteq L_0 \subseteq \mathcal{L}(K(T_h))$. For each $i \in C(t)$ let $\Theta_i$ be a set of sentences, which is $\forall$-perfect in $L_0$ whenever $i \in C^+(t)$. Assume that $L_0 + \Theta_{h+1} = \mathcal{Q}H + \bot$, and for any $i < h$ in $C(t)$

$$L_0 + \Theta_i = L_0 + \Delta^{i^*} - i(L_0 + \Theta_{i^*}) + Br_{t_i}[\Theta_{i^*}].$$

Then $L_0 + \Theta_i = \mathcal{IL}(K(T^*_h))$ for all $i \in C(t)$; in particular,

$$L_0 + \Theta_0 = \mathcal{IL}(K(T_h)).$$

Proof Again we proceed by induction on $h + 1 - i$, and the argument at the induction step fully repeats the proof of Proposition 6.11.12, cf. the proof of Lemma 6.11.11, which is actually a particular case of this construction for $L_0 = L_{h+1}^+$. 

So Proposition 6.11.12 yields an infinite axiomatisation for any nonconstant $t$, since any set of the form $(Br_{t_i})^\forall$ is infinite, while Proposition 6.11.13 allows us to find a finite axiom system for the case of increasing $t$.

Viz., consider an increasing $h^i$-sequence $t = (t_i \mid i < h)$, i.e. $t_0 \leq t_1 \leq \ldots \leq t_{h-1}$, with the critical levels $0 = i_0 < i_1 < \ldots < i_k < h$ and put $i_{k+1} := h + 1$, so all these levels are $+$-critical except for 0. Let $t'$ be the corresponding subsequence: $t'_j := t_{i_j}$ for $0 \leq j \leq k + 1$, so

$$t_i = \begin{cases} t'_j & \text{if } i \in [i_j, i_{j+1}], j < k, \\
 t'_k & \text{if } i_k \leq i < h.
\end{cases}$$

Obviously,

$$i^* = i^* = \min\{j > i \mid t_j > t_i\} = i_{j+1}$$

for $i = i_j$, $0 \leq j \leq k$.

Then we define the finitely axiomatisable logics

$$L(t) := L_{h+1}^+ + Br_{t'_k} + \{Br_{t'_j}[AP_{h+1-i_{j+1}}] \mid j < k\}.$$

Note that if $t'_k = \omega$, the axiom $Br_{t'_k} = \top$ is redundant.

Theorem 6.11.14 $\mathcal{IL}(K(T_h)) = L(t)$ for any increasing $h^i$-sequence $t$.

Proof We apply Proposition 6.11.13 to $L_0 = L(t)$ and $\Theta_i = \{AP_{h+1-i}\}$ for each $i \in C(t) = \{i_0, \ldots, i_k, i_{k+1}\}$ (where $i_0 = 0$, $i_{k+1} = h + 1$). Obviously $L(t) \subseteq \mathcal{IL}(K(T_h))$ by Lemma 6.11.6, since the branching is $\leq t'_j$ at all points of levels $< i_{j+1}$, and $AP_{h+1-i_{j+1}} \in \mathcal{IL}(K(T^*_h))$ for $i \geq i_{j+1}$. The sets $\Theta_i$ are $\forall$-perfect in $L_0$ by 2.8.13(2).

For $i = i_j \in C(t)$, $j < k$ we have

$$L_0 + \Delta^{i^*} - i(L_0 + \Theta_{i^*}) = L_0 + \Delta^{i_{j+1} - i_j}(L_0 + AP_{h+1-i_{j+1}})$$

$$= L_0 + \delta^{i_{j+1} - i_j}AP_{h+1-i_{j+1}} = L_0 + AP_{h+1-i_j}.$$
and $Br_{t_i}[\Theta_{i^+,j}] = Br_{t_j^+}(AP_{h+1-i_{j+1}}) \in L_0$. Thus

$$L_0 + \Theta_i = L_0 + \Delta^{i^+-i}(L_0 + \Theta_{i^+}) + Br_{t_i}[\Theta_{i^+}] .$$

Finally by 6.11.13, $\text{IL}(K(T_i)) = L_0 + \Theta_0 = L_0 + AP_{h+1} = L(t)$, since $AP_{h+1} \in L(t)$.

On the other hand, for all non-increasing $h^t$-sequences $t$ the corresponding logics $\text{IL}(K(T_t))$ are not finitely axiomatisable [ ... ]; a proof will be given in Volume 2.

Let us also present a simplified infinite axiomatisation for $\text{IL}(K(T_t))$ for a decreasing $t$.

**Proposition 6.11.15** Let $t = (t_i \mid i < h)$ be a decreasing $h^t$-sequence: $t_0 \geq t_1 \geq ... \geq t_{h-1}$; then

$$\text{IL}(K(T_t)) = LP_{h+1} + Br_{t_0} + \bigcup \{\delta_i^0(\{Br_{t_i}\}^\vee) \mid 0 < i < h, \ t_i \neq t_{i-1}\} .$$

**Proof** Readily follows from Proposition 6.11.13. Note that here $C^+(t) = \{h+1\}$, $i^+ = h+1$ for any $i < h$, $Br_{t_i}[\Theta_{h+1}] = \{Br_{t_i}^{+1}\}$ and $Br_{t_i}^{+1}$ is $H$-equivalent to $Br_{t_i}$.

Finally we obtain axiomatisations for uncountable levelwise uniform trees.

**Theorem 6.11.16** Let $t$ be an $h$-sequence with some $t_i > \omega$ (i.e. $T_t$ is uncountable). Then $\text{IL}(K(T_t)) = \text{IL}(K(T_{t^+}))$.

**Proof** Obviously $T_t \rightarrow T_{t^\omega}$, so by Proposition 3.3.14, $\text{IL}(K(T_t)) \subseteq \text{IL}(K(T_{t^\omega}))$.

On the other hand, $\text{IL}(K(T_t)) \supseteq \text{IL}(K(T_{t^\omega}))$, since all the axioms used in the axiomatisation $\Theta_0$ for $T_{t^\omega}$ (Proposition 6.11.12), are also valid in $T_t$.

More precisely, we can show (by induction on $h+1-i$) that $\Theta_i \subseteq \text{IL}(K(T_{t^\omega}))$ for points $u$ of level $i$, for any $i \in C(t^\omega)$. The argument is almost obvious, cf. soundness proof for Lemma 6.11.11. In fact, recall that $Br_{\omega}^0 = \emptyset$, so the levels $i$ with $t_i = \omega$ do not bring anything new to the axiomatisation $\Theta_i$, and thus to $\Theta_j$ for $j \leq i$.

Therefore we really have an explicit recursive axiomatisation for an arbitrary levelwise uniform tree of any cardinality.

For an $h$-sequence of cardinals $t = (t_i \mid i < h)$ let us also consider a poset $T_t + 1$ obtained by adding the top element. Then the following holds.

**Proposition 6.11.17** Let $t = (t_i \mid i < h)$ be an $h^t$-sequence. Then $\text{IL}(K(T_t + 1)) = QHP_{h+2} + J + \Theta_0$, where the sets $\Theta_i$ for $i \in C[t]$ are constructed just as in Proposition 6.11.12, but with a difference at the beginning

$$\hat{\Theta}_{h+1} := \{AP_1\} ,$$
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\[ \hat{\Theta}_i := \delta^0_{i^*} (\hat{\Theta}_{i^*}) \cup (Br_{t_i} [\hat{\Theta}_{i^*}])^\forall \text{ for } i < h. \]

All of them are \( \forall \)-perfect in \( \text{QHP}^+_{h+1} \).

This logic is strongly complete (w.r.t. \( K(T_t + 1) \)).

We can also obtain a finite axiomatisation of \( \text{IL}(K(T_t + 1)) \) for an increasing \( t \) (cf. Theorem 6.11.14) and a simplified infinite axiomatisation for a decreasing \( t \) (cf. Proposition 6.11.15). We also have

\[ \text{IL}(K(T_t + 1)) = \text{IL}(K(T_{t^*} + 1)) \]

for an uncountable \( t \) (cf. Theorem 6.11.16).

One can also consider logics with equality and obtain the following consequence from the results of Section 3.9:

**Proposition 6.11.18** Let \( t \) be an \( h \)-sequence \((h \geq 0)\). Then

\[ \text{IL}^= (K(E(T_t))) = (\text{IL}(K(T_t)))^=, \]

\[ \text{IL}^= (K(T_t)) = (\text{IL}(K(T_t)))^=, \]

and similarly for \( T_t + 1 \). The corresponding logics are strongly complete w.r.t. these classes of Kripke frames or sheaves.

\footnote{So to say, the depths increase by 1.}
Chapter 7

Kripke completeness for constant domains

In this chapter we prove completeness results in Kripke semantics with constant domains for modal logics containing Barcan axioms and superintuitionistic logics containing $CD$.

7.1 Modal canonical models with constant domains

Let us first describe canonical models with constant domains for modal logics. Since the Barcan formula is not valid in frames with varying domains, it is clear that every m.p.l. containing $Ba$ is not $V$-canonical. Thus canonical models should be restricted to $L$-places having a fixed denumerable domain $S_0$.

Here is a precise definition.

Definition 7.1.1 Let $L$ be an $N$-modal logic containing $Ba_i$ for $1 \leq i \leq N$. Let $CP_L$ be the set of all $L$-places (from $VPL$) with the set of constants $S_0$. The canonical frame and the canonical model with a constant domain for $L$ are defined as $CF_L := VF_L|CP_L$, $CM_L := VM_L|CP_L$.

For the sake of simplicity, the notation of relations and the domain function in $CF_L$ is the same as in $VF_L$. So in particular, $D_L$ is constant: $D_L(\Gamma) = S_0$.

It is clear that mentioning the superset $S^*$ is not necessary. Thus we can consider $CP_L$ as consisting of $(L, S_0)$-Henkin theories rather than $L$-places.

Now we have to show that the restricted relations $R_{Li}$ are selective on $CP_L$. Unfortunately, a direct analogue of Lemma 6.1.9 does not hold for the case of constant domains, because one cannot extend an $L$-consistent theory $\Gamma \subseteq MF_{S_0}^{(m)}$ to an $(L, S_0)$-Henkin theory $\Gamma$ without adding new constants. But the following lemma suggests the way out.

1The corresponding counterexample for the intuitionistic case was constructed by Ghilardi.
Lemma 7.1.2 Let \( L \) be an \( N \)-modal predicate logic, \( L \vdash \text{Ba}_i \) for every \( i \leq N \). Let \( \Gamma \in CP_L, A \in MF^{(=)}_N(S_0) \) and \( \diamond_i A \in \Gamma \) for some \( i \). Then there exists \( \Gamma' \in CP_L \) such that \( \square_i \Gamma \subseteq \Gamma' \) and \( A \in \Gamma' \).

This lemma means that the relations \( R_{Li} \) on \( CP_L \) are selective, cf. Definition 6.1.31 (recall that \( \Gamma R_{Li} \Gamma' \iff \square_i \Gamma \subseteq \Gamma' \iff \diamond_i \Gamma' \subseteq \Gamma \), cf. Definition 6.1.14).

Proof Let us enumerate the set of modal \( S_0 \)-sentences of the form \( \exists x A(x) \), viz. \( \{ \exists x A_k(x) \mid k > 0 \} \). Then we construct a sequence of finite subsets \( (\Gamma_k)_{k \in \omega} \) of \( MF^{(=)}_{S_0} \) such that for any \( k \in \omega \)

\[
\diamond_i \left( \bigwedge \Gamma_k \right) \in \Gamma. \tag{\diamond} \]

We put \( \Gamma_0 := \{ A \} \) and \( \Gamma_k := \Gamma_{k-1} \cup \{ \exists x A_k(x) \supset A_k(c) \} \) for some \( c \in S_0 \) (say, for the first one in an enumeration of \( S_0 \)), such that \( (\diamond) \) holds. Let us show the existence of this \( c \).

In fact, by the inductive hypothesis, \( \diamond_i (\bigwedge \Gamma_{k-1}) \in \Gamma \). We also have

\[
L \vdash \Box i \exists y (\exists x A_k(x) \supset A_k(y))
\]

(say, for a new variable \( y \)). Thus

\[
\diamond_i \exists y \left( \bigwedge \Gamma_{k-1} \land (\exists x A_k(x) \supset A_k(y)) \right) \in \Gamma.
\]

Now by the Barcan axiom,

\[
\exists y \diamond_i \left( \bigwedge \Gamma_{k-1} \land (\exists x A_k(x) \supset A_k(y)) \right) \in \Gamma,
\]

and thus by property (vi) of Henkin theories

\[
\diamond_i \left( \bigwedge \Gamma_{k-1} \land (\exists x A_k(x) \supset A_k(c)) \right) \in \Gamma \text{ for some } c \in S_0.
\]

Let \( \Gamma_\omega := \bigcup_{k \in \omega} \Gamma_k \). Then \( \square \neg \Gamma \cup \Gamma_\omega \) is \( L \)-consistent. In fact, suppose

\[
\square \neg \Gamma \vdash_L \neg \left( \bigwedge \Gamma_k \right) \text{ for some } k. \quad \text{Then } \Gamma \vdash_L \Box i \neg \left( \bigwedge \Gamma_k \right), \text{ i.e. } \neg \diamond_i (\bigwedge \Gamma_k) \in \Gamma. \text{ This contradicts } (\diamond).
\]

Finally by Lemma 6.1.5, \( \square \neg \Gamma \cup \Gamma_\omega \) can be extended to an \( (L, S_0) \)-complete theory \( \Gamma' \). Clearly \( \Gamma' \) is an \( (L, S_0) \)-Henkin theory, and \( \square i \Gamma \cup \{ A \} \subseteq \Gamma' \).

Remark 7.1.3 The above lemma implies that every \( L \)-consistent theory of the form \( \square \neg \Gamma \cup \Gamma_0 \) with finite \( \Gamma_0 \), can be extended to an \( (L, S_0) \)-Henkin theory. In fact, 7.1.2 is applicable, since in this case \( \diamond_i (\bigwedge \Gamma_0) \in \Gamma \) (otherwise \( \neg \diamond_i (\bigwedge \Gamma_0) \in \Gamma \), which implies \( \Box i (\bigwedge \Gamma_0) \in \Gamma \), i.e. \( \neg (\bigwedge \Gamma_0) \in \square \neg \Gamma \)).

This argument fails if \( \Gamma_0 \) is infinite. In fact, if \( \diamond_i (\bigwedge \Gamma_0) \in \Gamma \) for any finite subset \( \Gamma_0' \) of \( \Gamma_0 \) and \( \exists y A(y) \) has Henkin property (w.r.t. \( \Gamma \)) for any \( \Gamma_0' \), then there exist constants \( c' \in S_0 \) satisfying \( \bigwedge \Gamma_0' \land A(c') \) for all finite \( \Gamma_0' \subseteq \Gamma_0 \), but we cannot state the existence of a single constant \( c \) for all these \( \Gamma_0 \), and thus for the whole infinite set \( \Gamma_0 \).
7.2 INTUITIONISTIC CANONICAL MODELS

Now since the relations $R_L^i$ are selective on $CP_L$, we readily obtain the main property of the canonical model with a constant domain (cf. Lemma 6.1.21):

**Theorem 7.1.4** For any $\Gamma \in CP_L$ and $A \in MF_N^{(=)}(S_0)$,

$$CM_L, \Gamma \models A \iff A \in \Gamma.$$  

**Corollary 7.1.5** For a logic $L$ with Barcan axioms, for any formula $A$,

$$CM_L \models A \iff L \vdash A.$$  

**Proof** (Cf. 6.1.22). We can consider only the case when $A$ is a sentence. 'Only if' follows from 6.1.3 and Theorem 7.1.4. To check 'if', suppose $A \not\in L$; then by 6.1.10, we have $A \not\in \Gamma$ for some $(L, S_0)$-Henkin theory $\Gamma$; hence $CM_L, \Gamma \not\models A$, by Theorem 7.1.4.

**Definition 7.1.6** A m.p.l.(=) is called $C$-canonical if $CF_L \models L$.

**Corollary 7.1.7** Every $C$-canonical m.p.l. is strongly $CK$-complete. Every $C$-canonical m.p.l.= is strongly $CKE$-complete.

**Proof** Similar to Corollary 6.1.24.

For an $N$-m.p.l.(=) $A$ put $\Delta C := \Delta + \bigwedge_{i=1}^{N} Ba_i$.

The next result is an analogue of 6.1.29 for constant domains; the proof is very similar.

**Theorem 7.1.8** Let $\Lambda$ be a propositional PTC-logic. Then the logics $QLC$, $QAC$ are $C$-canonical.

**Proof** The argument for closed axioms is trivial, cf. 6.1.29.

If $A = \bigodot_\beta \Box_\kappa P \supset \Box_\gamma P \in L = Q\Lambda^{(=)} + Ba$, then $R_L^{L_\beta} \circ R_L^{L_\gamma} \subseteq R_L^K$. In fact, suppose $\Gamma R_L^{L_\beta} \Delta_1, \Gamma R_L^{L_\alpha} \Delta_2, \Box_k B \in \Gamma$. Then $\bigodot_\beta \Box_k B \in \Gamma$ (note that $B \in L(\Delta_1) = L(\Gamma)$, since the domain is constant), and thus from $\bigodot_\beta \Box_k B \supset \Box_\gamma B \in L \subseteq \Gamma$ it follows that $\Box_\gamma B \in \Gamma$. Therefore $B \in \Delta_2$, since $\Gamma R_L^{L_\gamma} \Delta_2$. So we obtain $\Delta_1 R_k \Delta_2$, and eventually $CF_L \models A$.

This theorem is actually a particular case of a more general completeness result by Tanaka-Ono, which will be proved in Section 7.4.

## 7.2 Intuitionistic canonical models with constant domains

Let us now consider intuitionistic models.

Let $L$ be a superintuitionistic predicate logic (with or without equality) containing the formula $CD$. Then $L$ is definitely not $V$-canonical, and we have to extract a subframe of $VF_L$ with a fixed denumerable domain $S_0$. 

**Definition 7.2.1** Let $\textit{CP}_L$ be the set of all $(\exists \forall, S_0)$-complete intuitionistic theories; they are called CDL-places.

Recall that (Section 7.2) for $L$ containing $CD$, $\exists \forall$-completeness is equivalent to the property

\[(Ac) \quad \forall x A(x) \in \Gamma \text{ iff for all } c \in S_0 \ A(c) \in \Gamma.\]

This reflects the property of forcing in intuitionistic Kripke models with constant domains:

\[u \models \forall x A(x) \text{ iff for all } c \in D(u) \ u \models A(c).\]

As in the modal case, we omit $S_0$ from our notation.

**Definition 7.2.2** The canonical frame and the canonical model with a constant domain for a superintuitionistic logic $L$ containing $CD$ are (respectively)

\[\textit{CFL} := V\textit{FL}|\textit{CP}_L \text{ and } \textit{CML} := V\textit{ML}|\textit{CF}_L.\]

Again we use the same notation of relations and domains as in $V\textit{FL}$; so $(D_L(\Gamma) = S_0)$ and $\Gamma R_L \Gamma' \text{ iff } \Gamma \subseteq \Gamma'$.

Note that $R_L$ coincides with $\leq$ on $\textit{CP}_L$, since every proper extension of $\Gamma$ contains additional $S_0$-formulas (see Definition 6.2.11).

Now let us show that the relation $R_L$ is selective on $\textit{CP}_L$. The property 6.2.12($\forall$) follows readily from $(Ac)$; so it remains to check 6.2.12($\exists$).

**Lemma 7.2.3** Let $L$ be a superintuitionistic predicate logic containing $CD$, $\Gamma \in \textit{CP}_L$, and let $(\Gamma \cup \Gamma_0, \Delta_0)$ be an $L$-consistent theory with finite $\Gamma_0, \Delta_0$. Then there exists an $\exists \forall$-complete theory $\Gamma' \supseteq (\Gamma \cup \Gamma_0, \Delta_0)$ (i.e. $\Gamma \cup \Gamma_0 \subseteq \Gamma'$, $\Delta_0 \cap \Gamma' = \emptyset$).

**Proof** Similar to Lemma 6.2.6. We enumerate $IF^{(=)}(S_0)$ as $\{B_k \mid k \in \omega\}$ and construct an increasing sequence of finite theories $(\Gamma_0, \Delta_0) \subseteq \ldots \subseteq (\Gamma_k, \Delta_k) \subseteq \ldots$ such that $(\Gamma \cup \Gamma_k, \Delta_k)$ is consistent for any $k \in \omega$. The construction is as follows.

1. If $B_k$ is not of the form $\exists x A(x)$ or $\forall x A(x)$, we define

\[(\Gamma_{k+1}, \Delta_{k+1}) := \begin{cases} (\Gamma_k \cup \{B_k\}, \Delta_k) & \text{if } (\Gamma \cup \Gamma_k \cup \{B_k\}, \Delta_k) \text{ is } L-\text{consistent}, \\ (\Gamma_k, \Delta_k \cup \{B_k\}) & \text{otherwise.} \end{cases}\]

Then the consistency of $(\Gamma \cup \Gamma_k, \Delta_k)$ implies the consistency of $(\Gamma \cup \Gamma_{k+1}, \Delta_{k+1})$; this is proved as in Lemma 6.2.6.

2. If $B_k = \exists x A(x)$ and $(\Gamma \cup \Gamma_k \cup \{B_k\}, \Delta_k)$ is $L$-consistent, then

\[\Gamma \nvdash_L \bigwedge_k \exists x A(x) \supset \bigvee_k \Delta_k,\]

and thus

\[\forall x \left( \bigwedge_k \Gamma_k \wedge A(x) \supset \bigvee_k \Delta_k \right) \not\in \Gamma.\]
Hence by (Ac),
\[(\bigwedge \Gamma_k \land A(c) \supset \bigvee \Delta_k) \notin \Gamma \text{ for some } c \in S_0,\]
and so
\[\Gamma \not\vdash \bigwedge \Gamma_k \land A(c) \supset \bigvee \Delta_k,\]
by Lemma 6.2.5(v). Thus
\[\Gamma \not\vdash \bigwedge \Gamma_k \land \exists x A(x) \land A(c) \supset \bigvee \Delta_k,\]
i.e. the theory \((\Gamma \cup \Gamma_{k+1}, \Delta_{k+1})\), where
\[\Gamma_{k+1} := \Gamma_k \cup \{\exists x A(x), A(c)\}, \quad \Delta_{k+1} := \Delta_k,\]
is \(L\)-consistent.
Otherwise, if \((\Gamma \cup \Gamma_k \cup \{B_k\}, \Delta_k)\) is \(L\)-inconsistent, we put
\[(\Gamma_{k+1}, \Delta_{k+1}) := (\Gamma_k, \Delta_k \cup \{B_k\});\]
then as in (1), we obtain that \((\Gamma \cup \Gamma_{k+1}, \Delta_{k+1})\) is consistent.
(3) If \(B_k = \forall x A(x)\) and
\[\Gamma \not\vdash_L \bigwedge \Gamma_k \supset \bigvee \Delta_k \lor \forall x A(x),\]
then by axiom \(CD\),
\[\forall x(\bigwedge \Gamma_k \supset \bigvee \Delta_k \lor A(x)) \notin \Gamma.\]
Hence
\[(\bigwedge \Gamma_k \supset \bigvee \Delta_k \lor A(c)) \notin \Gamma\]
for some \(c \in S_0\). Take this \(c\) and define
\[(\Gamma_{k+1}, \Delta_{k+1}) := (\Gamma_k, \Delta_k \cup \{\forall x A(x), A(c)\});\]
then \((\Gamma \cup \Gamma_{k+1}, \Delta_{k+1})\) is \(L\)-consistent.
Otherwise, if
\[\Gamma \vdash_L \bigwedge \Gamma_k \supset \bigvee \Delta_k \lor \forall x A(x),\]
we define
\[(\Gamma_{k+1}, \Delta_{k+1}) := (\Gamma_k \cup \forall x A(x), \Delta_k);\]
then \((\Gamma \cup \Gamma_{k+1}, \Delta_{k+1})\) is again \(L\)-consistent.
Eventually, the theory \(\Gamma' := \Gamma \cup \bigcup_{k \in \omega} \Gamma_k\) (or \(\bigcup_{k \in \omega} \Gamma_k\) itself) is \(L\exists V\)-complete, and \((\Gamma \cup \Gamma_0, \Delta_0) \preceq \Gamma'\) as required. \(\blacksquare\)

Note that only property (Ac) of the original theory \(\Gamma\) is essential in the above proof, and the existential property is not used.
Lemma 7.2.4 Suppose \( L \vdash CD \), \( \Gamma \in CP_L \), \( (A_1 \supset A_2) \notin \Gamma \). Then there exists \( \Gamma' \in CP_L \) such that \( \Gamma \subseteq \Gamma' \) (i.e. \( \Gamma R_L \Gamma' \)), \( A_1 \in \Gamma' \), \( A_2 \notin \Gamma' \). In other words, the relation \( R_L \) on \( CP_L \) satisfies 6.2.12 (\( \supset \)) and is selective.

Proof If \( (A_1 \supset A_2) \notin \Gamma \), then the theory \( (\Gamma \cup \{A_1\}, \{A_2\}) \) is \( L \)-consistent, and so we can apply Lemma 7.2.3.

Remark 7.2.5 Note that in Lemma 7.2.3 the sets \( \Gamma_0, \Delta_0 \) are finite, and thus it is actually equivalent to Corollary 7.2.4. If \( \Gamma_0 \cup \Delta_0 \) is infinite, one cannot always find an \( L\exists\forall \)-complete extension for an \( L \)-consistent theory \( (\Gamma \cup \Gamma_0, \Delta_0) \) (as we have mentioned, this was shown by Ghilardi).

From Corollary 7.2.4 and the conditions 6.2.12(\( \supset \)), (\( \forall \)) we obtain the main property of the canonical model:

Theorem 7.2.6 For any \( \Gamma \in CP_L \) and \( A \in IF(=)(S_0) \)
\[ CM_L, \Gamma \vdash A \] iff \( A \in \Gamma \).

Corollary 7.2.7 For a logic \( L \) containing \( CD \), for any formula \( A \)
\[ CM_L \vdash A \] iff \( L \vdash A \).

Proof By Lemma 6.2.6, for an \( L \)-unprovable \( S_0 \)-sentence \( A \) there exists an \( L\exists\forall \)-complete theory \( \Gamma \in CP_L \) such that \( A \notin \Gamma \). Thus \( CM_L, \Gamma \nvdash A \), by Theorem 7.2.6.

Now we can repeat Definition 7.1.6 for the intuitionistic case.

Definition 7.2.8 An s.p.l (=) \( L \) is called C-canonical if \( CF_L \equiv L \).

From Corollary 7.2.7 we obtain

Proposition 7.2.9 Every C-canonical s.p.l. containing \( CD \) is Kripke-complete.

7.3 Some examples of C-canonical logics

Unlike the case of varying domains, there are quite a few natural examples of C-canonicity; some of them were first found by H. Ono.

Let \( QHC := QH + CD \), \( QHCK := QHC + KF \). Sometimes we also use the notation \( QAC \), \( QACK \) for an arbitrary intermediate propositional logic \( A \).

Theorem 7.3.1 The logics \( QHC^{=} \), \( QHC \), \( QHC^{=}^\dagger \), \( QHC^{=}^s \) are C-canonical.

Proof Similar to Proposition 7.2.15.

Proposition 7.3.2 Classical logic \( QCL \) is C-canonical.
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Proof The frame $CF_{QcL}$ is discrete, i.e. $\Gamma R_{QcL}^G$ iff $\Gamma = \Gamma'$, cf. the proof of Lemma 6.3.4(i) (recall that $R_L = \subseteq$ in $CFL$).

C-canonicity also holds for many logics of finite depth.

Lemma 7.3.3 Let $k > 0$, and let $L$ be a superintuitionistic logic.

(1) If $CD \land AP_k \in L$, then $CFL$ is of depth $\leq k$.

(2) If $L = QHC + AP_k$, then $CFL$ is of depth $k$.

Proof

(1) Suppose $\Gamma_k \subseteq \Gamma_{k-1} \subseteq \ldots \subseteq \Gamma_1 \subseteq \Gamma_0$ in $CFL$, and let $A_i \in (\Gamma_{i-1} - \Gamma_i)$ for $1 \leq i \leq k$. Then by induction on $i$ it follows that

$$B_i := A_i \lor (A_i \supset A_{i-1} \lor (A_{i-1} \supset \cdots \supset A_2 \lor (A_2 \supset A_1 \lor \neg A_1) \ldots )) \notin \Gamma_i$$

(for $1 \leq i \leq k$). In fact, $B_{i+1} = A_{i+1} \lor (A_{i+1} \supset B_i)$ for $0 \leq i \leq k-1$ (where $B_0 = \perp$). Then $B_0 \notin \Gamma_0$, by consistency. Suppose $B_i \notin \Gamma_i$, but $B_{i+1} \in \Gamma_{i+1}$. Then $A_{i+1} \in \Gamma_{i+1}$ or $(A_{i+1} \supset B_i) \in \Gamma_{i+1}$. The first option contradicts the choice of $A_{i+1}$. In the second case we have $(A_{i+1} \supset B_i) \in \Gamma_{i+1} \subseteq \Gamma_i$. Since $A_{i+1} \in \Gamma_i$, we obtain $B_i \in \Gamma_i$, which contradicts the inductive hypothesis.

Therefore $B_k \notin \Gamma_k$, while $B_k$ is a substitution instance of $AP_k$. This contradicts Corollary 7.2.7.

(2) Recall that $L \not\models AP_{k-1} = q_k-1 \lor (q_k-1 \supset \cdots \supset (q_2 \lor (q_2 \supset q_1 \lor \neg q_1)))$ (since $AP_{k-1}$ is refuted in a $k$-element chain with a constant domain). By Corollary 7.2.7 there exists $\Gamma_{k-1} \in CPL$ such that $\Gamma_{k-1} \not\models AP_{k-1}$. Then $\Gamma_{k-1} \not\models q_{k-1}$, $\Gamma_{k-1} \not\models q_{k-1} \supset AP_{k-1}$, and so there exists $\Gamma_{k-2} \supset \Gamma_{k-1}$ such that $\Gamma_{k-2} \models q_{k-1}$, $\Gamma_{k-2} \not\models AP_{k-2}$. By induction we obtain a chain $\Gamma_{k-1} \subseteq \Gamma_{k-2} \subseteq \cdots \subseteq \Gamma_1 \subseteq \Gamma_0$ such that $\Gamma_i \not\models AP_i$, $\Gamma_i \not\models q_i$, $\Gamma_{i-1} \models q_i$ for $i > 0$.

Remark 7.3.4 Note that the argument used in the above proof of (1) fails for the case of varying domains, because the formula $A_i$ can contain additional constants not occurring in $\Gamma_i$, thus we may not find an appropriate substitution instance of $AP_k$ refuted in $\Gamma_k$. This observation somewhat explains why propositional axioms cannot describe the finite depth in this case and we need predicate axioms taking new individuals into account.

Lemma 7.3.5 Let $L$ be a superintuitionistic logic containing $Z$. Then every cone in $CFL$ is linearly ordered, i.e. for any $\Gamma, \Gamma', \Gamma'' \in CPL$, $\Gamma \subseteq \Gamma'$ and $\Gamma \subseteq \Gamma''$ imply that $\Gamma'$ and $\Gamma''$ are $\subseteq$-comparable.

Proof Suppose $A_1 \in (\Gamma' - \Gamma''')$ and $A_2 \in (\Gamma'' - \Gamma')$. Then $\Gamma \not\models A_1 \supset A_2$, $A_2 \supset A_1$, and thus $\Gamma \not\models (A_1 \supset A_2) \lor (A_2 \supset A_1)$. This contradicts $CM_L \models L$. 
Proposition 7.3.6 The following superintuitionistic predicate logics are C-canonical:

1. \( QLC + CD = QH + CD \land AZ \);
2. \( QH + CD \land AP_k \);
3. \( QH + CD \land AZ \land AP_k \) for \( k > 0 \).

They are respectively determined by the following classes of Kripke frames with constant domains:

1. linear Kripke frames;
2. Kripke frames of depth \( k \);
3. \( k \)-element chains.

These logics are also determined by countable frames of the corresponding kind.

(1) was proved in Ono [Ono, 1983], [Minari, 1983].

Remark 7.3.7 The logic \( QLC + CD \) was also characterised algebraically in [Horn, 1969]. Note that it is also equivalent to the logic IF introduced in [Takeuti and Titani, 1984].

Remark 7.3.8 Note that canonical and quasi-canonical frames with varying domains (for any logic) obviously do not validate AZ. In fact, two extensions of an \( L \)-place with incomparable sets of new constants are always \( R_L \)-incomparable. Thus the logic QLC and all its extensions are not \( V_-, U_-, \) or \( U^\leq \)-canonical. Still QLC is Kripke complete, as we showed in Section 6.7.

Now consider the logics

\[ QHCK := QHC + KF, \quad QS4CK^+ := QS4C + MK^+ \]

where

\[ MK^+ := \Box \forall x (P(x) \supset \Box P(x)) \].

Note that QS4CK\(^+\) contains S4.1.

Lemma 7.3.9 (1) Let \( L \) be a superintuitionistic logic containing Kuroda formula \( KF \). Then its C-canonical frame \( CF_L \) is coatomic.

Moreover, for any CDL-place \( \Gamma \in CP_L \) there exists a CDQCL-place \( \Gamma' \supseteq \Gamma \).

(2) Let \( L \) be an m.p.l. containing QS4CK\(^+\). Then \( CF_L \) is coatomic.

Note that QCL-complete theories are maximal in \( CF_L \) (cf. Lemma 6.3.4(i)).
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Proof (1) For $\Gamma \in CPL$, we construct a sequence $\Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ in $CPL$ as follows

Let $A_0, A_1 \ldots$ be an enumeration of $IF$. For a CDL-place $\Theta$ a formula $A$ is said to be critical if $CM_{\Gamma}, \Theta \not\vdash \neg \forall (A \lor \neg A)$.

Since $L \vdash KF$, we have $\Gamma_L \vdash \neg \forall (A \lor \neg A)$, which follows from $\Gamma_{QCL} \vdash \neg \forall (A \lor \neg A)$ by the Glivenko theorem. Hence $\Theta \not\vdash \neg \forall (A \lor \neg A)$ for any $\Theta$ and $A$.

Now assume that $\Gamma_n$ is constructed and $A_n$ is critical for $\Gamma_n$. Since $\Gamma_n \not\vdash \neg \forall (A \lor \neg A)$, there exists $\Gamma_{n+1} \supseteq \Gamma_n$ such that $\Gamma_{n+1} \vdash \forall (A \lor \neg A)$. If $A_n$ is not critical for $\Gamma_n$, we put $\Gamma_{n+1} := \Gamma_n$.

This procedure makes all formulas noncritical — $A_n$ is always noncritical for $\Gamma_{n+1}$.

Eventually we obtain a theory $\Gamma' := \bigcup_{n} \Gamma_n$. It is clear that $(\Gamma', -\Gamma')$ is $L$-consistent, otherwise $\Gamma_n \vdash \forall (A \lor \neg A)$ for some $n$ and finite $\Delta \subseteq -\Gamma' \subseteq -\Gamma_n$ — but then $(\Gamma_n, -\Gamma_n)$ is $L$-inconsistent. Since every $\Gamma_n$ has the (EP), so does $\Gamma'$.

So $\Gamma'$ is an $L$-place, and moreover, $QCL \subseteq \Gamma'$. In fact, by the deduction theorem, $B \in QCL$ implies $\vdash B$; hence there exist formulas $A_0, A_1, \ldots$ such that $\forall (A_0 \lor \neg A_0), \ldots, \forall (A_n \lor \neg A_n) \vdash B$; therefore $\Gamma' \vdash B$, whence $B \in \Gamma'$.

By 6.3.3, $\Gamma'$ is a $QCL$-place, and therefore an $L\forall$-place (by 6.3.2), and it is maximal in $CF_L$.

(2) Similar to (1). Again starting from $\Gamma \in CPL$, we construct a sequence $\Gamma = \Gamma_0 R_L \Gamma_1 \ldots$ in $CF_L$.

Let $MF_1$ be the set of all $S_0$-formulas with at most one parameter, and let $A_0, A_1 \ldots$ be an enumeration of $MF_1$. Now we call a formula $A$ critical for $\Theta \in CPL$ if $CM_{\Gamma}, \Theta \not\vdash \Box (A \supset \Box A)$.

Suppose $\Gamma_n$ is constructed and $A_n$ is critical for $\Gamma_n$. Since $L \supseteq QS4CK^+$, we have $\Gamma_n \vdash \Box \Box (A_n \supset \Box A_n)$, so there exists $\Gamma_{n+1} \in R_L(\Gamma_n)$ such that $\Gamma_{n+1} \vdash \Box (A_n \supset \Box A_n)$. If $A_n$ is not critical for $\Gamma_n$, we put $\Gamma_{n+1} := \Gamma_n$.

Then we claim that the theory $\Gamma' := \bigcup_{n} \Box \neg \Gamma_n$ is $L$-consistent. In fact, otherwise $\bigcup_{n=1}^{k} \Box \neg \Gamma_n$ is $L$-inconsistent for some finite $k$.

But this is impossible, since $\Gamma_n R_L \Gamma_{k+1}$ for $n \leq k+1$, and thus $\bigcup_{n=1}^{k} \Box \neg \Gamma_n \subseteq \Gamma_{k+1}$.

Now by the Lindenbaum lemma, there exists an $L$-complete $\Delta \supseteq \Gamma'$. It follows that $\Delta$ is a Henkin theory. In fact, consider an arbitrary formula $A_n(y)$ of the form $\exists x B(x) \supset B(y)$, where $FV(B(x)) \subseteq \{x\}$ as usual. By our construction and reflexivity, $\Gamma_{n+1} \vdash \forall y (A_n(y) \supset \Box A_n(y))$. Since $\Gamma_{n+1}$ is a Henkin theory, there exists $c \in S_0$ such that $\Gamma_{n+1} \vdash A_n(c)$. We also have $\Gamma_{n+1} \vdash A_n(c) \supset \Box A_n(c)$, so $\Gamma_{n+1} \vdash \Box A_n(c)$. Hence $A_n(c) \in \Box \neg \Gamma_{n+1} \subseteq \Delta$. Therefore $\Delta \in CPL$.

It remains to note that $\Delta$ is maximal — assuming $\Delta R_L \Delta'$, let us show that $\Delta = \Delta$ (or $\Delta \subseteq \Delta'$). Suppose $A = A_n \in \Delta$ (i.e. $\Delta \supseteq A_n$) for an $S_0$-sentence
A_n. By construction \( \Gamma_n R \Delta \), so \( \Delta \vdash \forall (A_n \supset \Box A_n) \), which is the same as \( \Delta \vdash A_n \supset \Box A_n \), and thus \( \Delta \vdash \Box A_n \), which implies \( A_n \in \Delta' \).

Remark 7.3.10 Note that the proof in case (1) almost repeats the proof of Lemma 7.2.3, when we construct a \( QCL^{(\vDash)} \)-complete extension \( \Gamma' \) of \( \Gamma \). But in our case we cannot directly apply Lemma 7.2.3 to the logic \( QCL^{(\vDash)} \) and empty \( \Gamma_0, \Delta_0 \), because the basic theory \( \Gamma \) is not in general \( QCL^{(\vDash)} \)-complete. Nor we can apply Lemma 7.2.3 to an \( L_3 \)-complete \( \Gamma' \) and \( A \).

Proposition 7.3.11 The logics \( QHC^{k} \) and \( QLC^{k} \) are \( C \)-canonical. The former is determined by Kripke frames with constant domains and the McKinsey property, and the latter by chains with the greatest element and constant domains and also by countable chains with the greatest element and constant domains.

Recall that \( QH + CD \wedge AP_k \vdash KF \) for any \( k \), since a poset of a finite depth is clearly coatomic.

Remark 7.3.12 Note that the logic of the weak excluded middle with constant domains \( QH + CD \wedge AJ \) is not \( C \)-canonical. Recall that a Kripke frame validates \( AJ \) iff it is directed, i.e. \( \forall v_1 \forall v_2 \exists w (v_1 Rw \& v_2 Rw) \). This property fails for the canonical frame, and our 'extension lemma' 8.2.3 does not allow us to extend the \( L \)-consistent theory \( \Gamma' \cup \Gamma'' \) (where \( \Gamma \subseteq \Gamma' \) and \( \Gamma \subseteq \Gamma'' \)) to an \( L^{(\vDash)} \)-complete theory with the same set of constants \( S_0 \). The logic \( QH + CD \wedge AJ \) is actually Kripke-incomplete [Shehtman and Skvortsov, 1990; Ghilardi, 1989], see also Volume 2. But there is no difficulties with the logic \( QH + CD \wedge AJ \wedge KF \), as we shall see now.

Lemma 7.3.13 Let \( L \) be a superintuitionistic logic containing \( CD \wedge AJ \wedge KF \). Then for any \( \Gamma \in CPL \) the set \( \{ \Delta \in CPL \mid \Gamma \subseteq \Delta \} \) has the greatest element.

Proof By 7.3.9, there exists a CDQCL-place \( \Delta \supseteq \Gamma \). Suppose \( \Delta \) is not the greatest in \( CF_L \mid \Gamma \), then there exists another

CDQCL-place \( \Delta_1 \supseteq \Gamma \), and thus \( \Delta \not\subseteq \Delta_1 \), since \( \Delta \) is maximal. So there exists \( A \in IF(S_0) \) such that \( \Delta \Vdash A \), \( \Delta_1 \not\Vdash A \); hence \( \Delta_1 \not\Vdash \neg A \) by maximality of \( \Delta_1 \). Thus \( \Gamma \not\Vdash \neg A \vee \neg \neg A \), which contradicts \( \vdash_L \neg A \vee \neg \neg A \).

Proposition 7.3.14 The logics \( QHC + AJ \wedge KF \) and \( QHC + CD \wedge AJ \wedge AP_k \) for \( k > 0 \) are \( C \)-canonical. The former is determined by Kripke frames with constant domains and top elements, and the latter, by the same kind of frames of depth \( k \).

Definition 7.3.15 A subordination \( L \)-map with constant domains is a natural \( L \)-map from a subordination frame to \( CF_L \).

Lemma 7.3.16 Let \( L \supseteq QH + CD \) and let \( \Gamma \) be an \( L^{(\forall)} \)-place. Then there exists a subordination \( L \)-map with constant domains \( g : ITmw \rightarrow CF_L \) such that \( g(\lambda) = \Gamma \).
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Proof From Lemma 7.2.4 and an analogue to Lemma 6.4.16.

Lemma 7.3.17 Let $L \supseteq \mathbf{QH} + \mathbf{CD} + \mathbf{KF} + \mathbf{J}$ and let $\Gamma$ be an $\mathcal{L}^\mathcal{Q}$-place. Then there exists a subordination $L$-map with constant domains $h : IT_\omega + 1 \rightarrow CF_L$ such that $h(\lambda) = \Gamma$.

Proof Let $g$ be the subordination map given by Lemma 7.3.16. Let $\Gamma' = \{A \mid \neg \neg A \in \Gamma\}$.

Suppose $A \in g(\alpha)$ for some $\alpha \in IT^\omega$. Then $\vdash_L \neg A \lor \neg \neg A$ by $AJ$, $\Gamma \not\vdash \neg A$, therefore $\Gamma \vdash \neg \neg A$ and $A \in \Gamma'$.

Moreover, $\Gamma'$ is a $\mathbf{QCLV}$-place. In fact, it is clearly $L$-consistent and closed under deduction in $L$. Since for every $D_L$-sentence $A$, $\vdash_L \neg \neg A \lor \neg \neg \neg A$, $\Gamma$ contains either $\neg \neg A$ or $\neg \neg \neg A$, so $\Gamma'$ contains either $\neg \neg A$ or $\neg \neg \neg A$.

To check the existence property, suppose $A(c) \notin \Gamma'$ for all $c$. Then by $AJ$ we deduce that $\neg A(c) \in \Gamma$ for all $c$ and so $\forall x \neg A(x) \in \Gamma \subseteq \Gamma'$. Since $\vdash_{\mathbf{QH}} \forall x \neg A(x) \supset \exists x \neg A(x)$, we have $\neg \exists x A(x) \in \Gamma'$ and so $\exists x A(x) \notin \Gamma'$.

To show the coexistence property, suppose $\forall x A(x) \notin \Gamma'$. Then $\neg \forall x A(x) \in \Gamma'$. But $\mathbf{QH} + \mathbf{KF} \vdash \neg (\forall x A(x) \supset \exists x \neg A(x))$, hence $(\forall x A(x) \supset \exists x \neg A(x)) \in \Gamma'$ and $\exists x \neg A(x) \in \Gamma'$. By existence property, $\neg A(c) \in \Gamma'$ for some $c$ and so $A(c) \notin \Gamma'$. Since $\Gamma'$ is maximal, we see that $A \in \Gamma'$ and $B \notin \Gamma'$ whenever $(A \supset B) \notin \Gamma'$. Therefore $h$ extending $g$ by $h(\infty) := \Gamma'$ is natural.

From Lemmas 7.2.3 and 7.3.17, we obtain

Theorem 7.3.18 The logic $\mathbf{QH} + \mathbf{CD} + \mathbf{KF} + \mathbf{AJ}$ is determined by the subordination frame with the greatest element and constant domains.

7.4 Predicate versions of subframe and tabular logics

The results on canonicity of some families of predicate logics presented in this and the next section are due to H. Ono, T. Shimura and Y. Tanaka. But the original proofs are essentially simplified, thanks to canonical models and some results on propositional logics.

Definition 7.4.1 Let $L$ be a modal or superintuitionistic predicate logic with constant domains. The canonical general frame of $L$ is

$$C\Phi_L := ((CF_L)_\pi, \mathcal{W}_L),$$

where $\mathcal{W}_L$ is $\{\theta_L(A) \mid A \in MF_N(S_0)\}$ in the modal case or $\{\theta_L^I(A) \mid A \in IF(S_0)\}$ in the intuitionistic case.

Lemma 7.4.2 $C\Phi_L$ is well-defined.
Proof Similar to 1.6.7. Note that $\mathcal{W}_L$ is a subalgebra of $MA((CL)_\pi)$, since by definition
\[
\theta_L(\neg A) = -\theta_L(A), \quad \theta_L(A \land B) = \theta_L(A) \cap \theta_L(B), \quad \theta_L(\Box_i A) = \Box_i \theta_L(A).
\]

Lemma 7.4.3 $C\Phi_L$ is refined.

Proof It is again similar to the propositional logic. $C\Phi_L$ is differentiated, since $\Gamma \not\models \Delta$ implies $\Gamma \not\subseteq \Delta$ (or vice versa), and thus by Theorem 7.1.4 (or 7.2.6 in the intuitionistic case), $\Gamma \in \theta_L(A)$, $\Delta \not\in \theta_L(A)$ for some $A$.

Tightness in the modal case
\[
\forall A (\Gamma \models \Box_i A \Rightarrow \Delta \models A) \Rightarrow \Gamma R_{L,i} \Delta
\]
follows from the definition of $R_{L,i}$ and 7.1.4. In the intuitionistic case this is written as
\[
\forall A (\Gamma \models A \Rightarrow \Delta \models A) \Rightarrow \Gamma \subseteq \Delta
\]
and follows from 7.2.6.

Remark 7.4.4 $C\Phi_L$ is not descriptive, as S. Ghilardi noticed.

Proposition 7.4.5 For any m.p.l. or s.p.l. $L$, $C\Phi_L$ validates $L_\pi$.

Proof Again we consider only the modal case. We can argue as in the proof of 1.7.5. Viz., consider a formula $A \in L_\pi[k]$ and a valuation $\psi$ in $C\Phi_L$. Choose $B_i$ such that $\psi(p_i) = \theta_L(B_i)$, and let $S := [B_1, \ldots, B_k/p_1, \ldots, p_k]$. Then $SA \in L$, so $\psi(A) = \theta_L(SA) = CP_L$ by 1.2.9 and the canonical model theorem 7.1.4.

Proposition 7.4.6 If a propositional modal or intermediate logic $A$ is $r$-persistent, then the predicate logic $QAC$ is $C$-canonical.

Proof Let $L = QAC$. By 7.4.5, $C\Phi_L \models A$. By 7.4.3, this frame is refined, so by $r$-persistence, we obtain $CL \models A$. Since $CL$ has constant domains, it also validates the Barcan formula, and thus the whole $L$.

Hence we obtain the main result from [Tanaka and Ono, 1999]:

Theorem 7.4.7 (Tanaka-Ono) For any universal propositional modal logic $A$, the logic $QAC$ is $C$-canonical. [and so, strongly Kripke complete]

Proof By Theorem 1.12.8, $A$ is $r$-persistent, so $QAC$ is $C$-canonical by Proposition 7.4.6.

This allows us to construct many examples of canonical predicate logics. In almost the same way we prove the result from [Shimura, 1993]:
Theorem 7.4.8 (Shimura) If \( \Lambda \) is a subframe intermediate logic, then \( QAC \) is \( C \)-canonical.

**Proof** By Theorem 1.12.27, \( \Lambda \) is \( r \)-persistent. So we can apply Proposition 7.4.6.

The next result for the intuitionistic case was proved in [Ono, 1983]; the modal version is probably new.

Theorem 7.4.9 If a propositional modal or intermediate logic \( \Lambda \) is tabular, then \( QAC \) is \( C \)-canonical.

**Proof** By Proposition 1.14.7, \( \Lambda \) is \( r \)-persistent. So we can apply 7.4.6.

### 7.5 Predicate versions of cofinal subframe logics

In this section we prove completeness results from [Shimura, 2001].

Now we use an auxiliary 'pre-canonical' model dual to the Lindenbaum algebra of \( S_0 \)-sentences.

**Definition 7.5.1** Let \( L \) be an \( N \)-m.p.l. with constant domains, \( C^+P_L \) the set of all \( L \)-complete theories with the set of constants \( S_0 \). We define the following propositional frames and models:

- the pre-canonical frame of \( L \) is \( C^+F_L := (C^+P_L, R_1, \ldots, R_N) \), where \( R_i \) is defined as in the canonical model:

\[
\Gamma R_i \Delta \iff \forall A \in MF_N(S_0) (\square_i A \in \Gamma \Rightarrow A \in \Delta);
\]

- the pre-canonical model of \( L \) is \( C^+M_L := (C^+F_L, \theta^+_L) \), where

\[
\theta^+_L(p_i) := \{ \Gamma \in C^+P_L | p_i \in \Gamma \};
\]

- the pre-canonical general frame of \( L \) is

\[
C^+\Phi_L := (C^+F_L, \{|A| | A \in MF_N(S_0)|),
\]

where \( |A| := \{ \Gamma \in C^+P_L | A \in \Gamma \} \).

**Lemma 7.5.2** \( C^+\Phi_L \) is well-defined.

**Proof** We have to show that the sets \( |A| \) constitute a subalgebra of \( MA(C^+F_L) \). This follows from the equalities:

\[
|\neg A| = -|A|, \ |A \land B| = |A| \cap |B|, \ |\square_i A| = \square_i |A|.
\]

The first two are just reformulations of 6.1.2(iii) and 6.1.3(i). For the third, note that the inclusion \( |\square_i A| \leq \square_i |A| \) readily follows from the definition of \( R_i \).
The converse $|\square_i A| \supseteq |\square_i| A|$ is proved as in propositional logic and similarly to 6.1.16. In fact, suppose $\square_i A \notin \Gamma$. Then the theory

$$\Delta_0 := \square^{-1}_i \Gamma \cup \{\neg A\}$$

is $L$-consistent. For otherwise we have $\square^{-1}_i \Gamma \vdash_L A$, hence $\Gamma \vdash_L \square_i A$ by Lemma 2., which contradicts $\square_i A \notin \Gamma$.

Now by the Lindenbaum lemma 6.1.5 there exists an $L$-complete $\Delta \supseteq \Delta_0$. So $\Delta \in |\neg A| = |\neg A|$, and also $\Gamma R_i \Delta$. Thus $\Gamma \notin \square_i |A|$. ■

As complete theories may be not Henkin, the canonical model theorem for this model holds only for propositional formulas; this explains the name ‘pre-canonical’.

**Proposition 7.5.3** Let $L$ be an $N$-m.p.l. with constant domains.

1. For any $N$-modal propositional formula $A$, for any $L$-complete $\Gamma$

$$C^+ M_L, \Gamma \vdash A \iff A \in \Gamma.$$

2. For any $N$-modal propositional formula $A$,

$$C^+ M_L \vdash A \iff A \in L.$$

3. $\text{ML}(C^+ \Phi_L) = L_\pi$.

**Proof**

(1) Standard, by induction on the length of $A$, using properties of complete theories 6.1.3. We also need an analogue of 6.1.16 for complete theories; its proof uses 6.1.5.

(2) ‘If’ follows from (1), since $L$-complete theories contain $L$. For ‘only if’ use the Lindenbaum lemma 6.1.5.

(3) $(\subseteq)$ If $A \notin L_\pi$, then by (2), $C^+ M_L \not\vdash A$, and thus $C^+ \Phi_L \not\vdash A$.

$(\supseteq)$ The claim (1) can be written as $\theta^+_L(A) = |A|$. So we can argue as in the proof of 1.7.5. Viz., consider a formula $A \in L_\pi[k]$, and a valuation $\psi$ in $C^+ \Phi_L$. Choose $B_i$ such that $\psi(p_i) = |B_i| = \theta^+_L(B_i)$, and consider a substitution $S := [B_i, \ldots, B_k/p_1, \ldots, p_k]$. Then $SA \in L$, so $\psi(A) = \theta^+_L(SA) = C^+ P_L$ by 1.2.9 and (2). ■

We also have an analogue of Theorem 1.7.13:

**Theorem 7.5.4** Let $L$ be an m.p.l. with constant domains. Consider the modal algebra $\text{Lind}(\overline{L(S_0)})$ of all $S_0$-sentences up to the equivalence $\vdash_L A \equiv B$. Then

$$\text{Lind}(\overline{L(S_0)})_+ \cong C^+ \Phi_L.$$

**Lemma 7.5.5** $C^+ \Phi_L$ is descriptive.
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Proof It is again similar to the propositional case.

\( C^+ \Phi_L \) is differentiated, since \( \Gamma \neq \Delta \) implies \( \Gamma \not\subseteq \Delta \), and thus \( \Gamma \in |A|, \Delta \not\in |A| \) for some \( A \).

Tightness

\[ \forall A \ (\Gamma \in \Box_i |A| \Rightarrow \Delta \in |A|) \Rightarrow \Gamma R_i \Delta \]

follows from the definition of \( R_i \) and the equality \( |\Box_i |A| | \leq |\Box_i |A| | \) we have already proved.

For compactness, note that if \( \mathcal{V} \) is a set of interior sets with non-empty finite intersections, then the theory \( \Gamma_0 := \{ A \mid |A| \in \mathcal{V} \} \) is \( L \)-consistent. In fact, if \( \Gamma_0 \vdash_L \bot \), then \( \Gamma \vdash \bigwedge_{i=1}^k A_i \) for some \( |A_1|, \ldots, |A_k| \in \mathcal{V} \), so \( \bigwedge_{i=1}^k |A_i| = |\bigwedge_{i=1}^k A_i| = \emptyset \) — a contradiction. By the Lindenbaum lemma, there exists an \( L \)-complete \( \Gamma \subseteq \Gamma_0 \), so \( \bigcap \mathcal{V} \not\supseteq \Gamma \).

Alternatively, one can apply Theorem 7.5.4 and Proposition 1.7.14.

Quite similar constructions and proofs can be made in the intuitionistic case.

Definition 7.5.6 For an s.p.l. \( L \) with constant domains, let \( C^+ P_L \) the set of all \( L \)-complete intuitionistic theories with the set of constants \( S_0 \). Then we define the pre-canonical model, the pre-canonical frame and the pre-canonical general frame of \( L \):

\[
C^+ F_L := (C^+ P_L, \subseteq),
\]

\[
C^+ M_L := (C^+ F_L, \theta^+_L),
\]

\[
C^+ \Phi_L := (C^+ F_L, \{ |A| \mid A \in IF(S_0) \}),
\]

where

\[
\theta^+_L(p_i) := \{ \Gamma \in C^+ P_L \mid p_i \in \Gamma \},
\]

\[
|A| := \{ \Gamma \in C^+ P_L \mid A \in \Gamma \}.
\]

Lemma 7.5.7 In the intuitionistic case \( C^+ \Phi_L \) is well-defined.

Proof Let us show that \( \{ |A| \mid A \in IF(S_0) \} \) is a subalgebra of \( HA(C^+ F_L) \). It is sufficient to check the equalities:

\[
\bot = \emptyset, \ |A \wedge B| = |A| \cap |B|, \ |A \vee B| = |A| \cup |B|, \ |A \supset B| = |A| \rightarrow |B|.
\]

The first one is trivial; the second and the third follow from Lemma 6.2.5.

The equality \( |A \supset B| = |A| \rightarrow |B| \) is checked as in the propositional case. In fact, \( |A \supset B| \subseteq |A| \rightarrow |B| \), since \( |A \supset B| \cap |A| \subseteq |B| \). The latter follows from the implications

\[
(A \supset B) \in \Gamma \& A \in \Gamma \Rightarrow \Gamma \vdash_L B \Rightarrow B \in \Gamma.
\]

To show the converse, suppose \( (A \supset B) \not\in \Gamma \). Then the theory \( \Gamma \cup \{ A \}, \{ B \} \) is \( L \)-consistent. For otherwise we have \( \Gamma, A \vdash_L B \), which implies \( \Gamma \vdash_L A \supset B \), by the Deduction theorem, and next \( (A \supset B) \in \Gamma \) by 6.2.5.

Thus by Lemma 7.2.3 there exists an \( L \)-complete \( \Delta \supseteq (\Gamma \cup \{ A \}, \{ B \}) \), hence \( \Gamma \subseteq \Delta \in (|A| \rightarrow |B|) \), and therefore \( \Gamma \not\in (|A| \rightarrow |B|) \).
Proposition 7.5.8 Let $L$ be an s.p.l. with constant domains. Then

1. for any intuitionistic propositional formula $A$, for any $L$-complete $\Gamma$
   
   \[ C^+M_L, \Gamma \vdash A \iff A \in \Gamma; \]

2. for any intuitionistic propositional formula $A$,
   
   \[ C^+M_L \vdash A \iff A \in L; \]

3. $IL(C^+\Phi_L) = L_\pi$.

Proof Repeats the proof of 7.5.3, with minor changes; for example, use 1.2.13 instead of 1.2.9. We leave the details to the reader.

Hence we obtain

Theorem 7.5.9 For any s.p.l. $L$ with constant domains, $C^+\Phi_L \cong \text{Lind}(L(S_0))_+$, where $\text{Lind}(L(S_0))$ is the Lindenbaum algebra of $S_0$-sentences in $L$.

Lemma 7.5.10 For any s.p.l. $L$ with constant domains, $C^+\Phi_L$ is descriptive.

Proof Similar to 7.5.5.

Lemma 7.5.11 In each of the following cases the pre-canonical frame $C^+\Phi_L$ is coatomic:

1. $L$ is an s.p.l. containing $\text{QHCK}$;

2. $L$ is an m.p.l. containing $\text{QS4CK}^+$.

Proof (1) Follows the lines of the proof of 7.3.9.

(2) Since $L \supseteq \text{S4.1}$, we can apply the same argument as in the propositional case for $\text{S4.1}$ — its canonical frame is coatomic.

Now we can prove the main completeness results of this section. We begin with a theorem from [Shimura, 1993].

Definition 7.5.12 A world $t$ in a transitive frame is called terminal if it is accessible from every nonmaximal cluster and its cluster $t^-$ is maximal and either trivial or degenerate. A finite transitive frame is called weakly directed if it contains a terminal world.

Theorem 7.5.13 Let $\Lambda$ be an intermediate propositional logic axiomatisable by cofinal subframe formulas of weakly directed finite rooted posets. Then $\text{QACK}$ is $C$-canonical.
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Proof If \( \Lambda = H + \{ \text{CSI}(F) \mid F \in \mathcal{F} \} \) for a set \( \mathcal{F} \) of weakly directed posets, \( L = QACK \), let us show that \( CF_L \models \text{CSI}(F) \) for \( F \in \mathcal{F} \). Suppose the contrary, then \( CF_L \) is cofinally subreducible to some \( F \in \mathcal{F} \).

By Theorems 1.12.22, 1.12.30, there exists \( f : G \rightarrow F \) for a cofinal \( G \subseteq CF_L \uparrow u_0 \) (for some \( u_0 \)) such that \( f^{-1}(x) \) consists only of maximal points, whenever \( x \) is maximal in \( F \).

Then let us prolong \( f \) to \( f' : G' \rightarrow F \) for a cofinal \( G' \subseteq C^+F_L \uparrow u_0 \).

Since \( KF \subseteq L \), \( C^+F_L \) is coatomic by Lemma 7.5.11, so it suffices to extend \( f \) only to maximal points of \( C^+F_L \). We can do this as follows.

Let \( y \in C^+P_L - CP_L \) be a ‘new’ maximal point (so \( y \) is a classical non-Henkin theory). We put \( f'(y) := t \), where \( t \) is a (fixed) terminal world of \( F \).

Thanks to the McKinsey property, the subframe \( G' := G \cup \) (maximal points of \( C^+F_L \)) is cofinal in \( C^+F_L \). Since \( f' \) sends maximal points to maximal points, the lift property is preserved. The monotonicity of \( f' \) follows from the definition: if \( uR^Gy \), and \( u \in G \), \( y \in G' - G \), then \( u \) is nonmaximal, so \( f(u) \) is nonmaximal by the choice of \( f \), while \( f'(y) \) is terminal; hence \( f'(u) = f(u)R^Ff'(y) \). (Here \( F^G \), \( R^F \) denote the accessibility relations in \( G \), \( F \) respectively.)

Therefore, \( C^+F_L \) is subreducible to \( F \), so \( C^+F_L \not\models \text{CSI}(F) \), and thus \( C^+F_L \not\models \Lambda \).

On the other hand, \( C^+\Phi_L \vdash \Lambda \) by 7.5.8 and \( C^+\Phi_L \) is descriptive (7.5.10), hence \( C^+F_L \vdash \Lambda \) by d-persistence (Theorem 1.12.28).

This is a contradiction.

The next lemma and theorem are also taken from [Shimura, 1993].

**Lemma 7.5.14** Let \( L \) be an s.p.l. containing \( QHK \), and assume that a cone \( CM_L \uparrow u \) has finitely many maximal worlds. Then \( C^+M_L \uparrow u \) has the same maximal worlds.

**Proof** Let \( x_1, \ldots, x_n \) be all maximal worlds in \( CM_L \uparrow u \), and suppose that \( y \) is another maximal world in \( C^+M_L \uparrow u \).

By distinguishability, there exist \( A_i \in x_i - y \); then \( \neg A_i \in y \), since \( y \) is classical. Hence \( \neg A_1 \land \ldots \land \neg A_n \in y \), and thus \( \neg(\neg A_1 \land \ldots \land \neg A_n) \not\in u \), which is equivalent to \( \neg(\forall_{i=1}^n A_i) \not\in u \), and to \( CM_L, u \not\vdash \neg(\forall_{i=1}^n A_i) \). But then \( CM_L, v \vdash \neg\forall_{i=1}^n A_i \) for some \( v \in R_L(u) \), while by the McKinsey property, \( vR_Lx_i \) for some \( i \). Then \( x_i \vdash A_i \) contradicts \( x_i \vdash \neg\forall_{i=1}^n A_i \).

**Theorem 7.5.15** Let \( \Lambda \) be a cofinal subframe intermediate propositional logic containing a formula \( \text{CSI}(V_n) \) for some finite \( n \), where \( V_n = IT_n^2 \). Then \( QACK \) is \( C \)-canonical.

**Proof** By Theorem 7.5.13, \( QHCK + \text{CSI}(V_n) \) is \( C \)-canonical, so \( CF_L \models \text{CSI}(V_n) \) (where \( L := QACK \)). This means that \( CF_L \) is not cofinally subreducible to \( V_n \), and thus every cone \( CF_L \uparrow u \) contains at most \( (n - 1) \) maximal worlds.
points, otherwise, due to the McKinsey property 7.3.9, there exists a cofinal subreduction to $V_n$.

Now we can show that $CF_L \vDash CSI(F)$ whenever $CSI(F) \in \Lambda$. In fact, otherwise $CF_L$ is cofinally subreducible to $F$ and by 1.12.27, there exists $f : G \rightarrow F$ for a cofinal $G \subseteq CF_L \uparrow u$ such that for every maximal $x \in F$, $f^{-1}(x)$ contains only maximal points of $CF_L$.

Then $G$ is cofinal in $CF_L \uparrow u$. In fact, by Lemma 7.5.14, $CF_L \uparrow u$ has the same maximal points as $CF_L \uparrow u$, and they are all in $G$, as $G$ is cofinal in $CF_L \uparrow u$. By 7.5.11, the set of maximal points is cofinal in $CF_L$.

Therefore $f$ is a cofinal subreduction from $CF_L$ to $F$. Now we can use the same argument as in the proof of Theorem 7.5.13 to show that $CF_L \vDash \Lambda$, which leads to a contradiction. 

Hence we obtain another proof of 7.3.14.

**Corollary 7.5.16** $Q\Lambda CK$ is $C$-canonical for $\Lambda = H + AJ$.

**Proof** In fact, $\Lambda = H + CSI(V_2)$ — a cofinal subreduction to $V_2$ does not exist iff a frame is confluent.

Let us now prove the main result from [Shimura, 2001] with slight additions.

**Theorem 7.5.17** Let $\Lambda$ be a modal propositional logic axiomatisable above $S4$ by cofinal subframe formulas of weakly directed posets. Then $Q\Lambda CK^+$ is $C$-canonical.

**Proof** Similar to the proof of 7.5.13.

It is sufficient to show that $CF_L \vDash CSM(F)$, provided $CSM(F) \in \Lambda$ and $F$ is a weakly directed poset. Suppose the contrary, then $CF_L$ is cofinally subreducible to $F$.

By Theorems 1.12.22, 1.12.30, there exists $f : G \rightarrow F$ for a cofinal $G \subseteq CF_L \uparrow u_0$ such that for any maximal $x \in F$, $f^{-1}(x)$ consists only of maximal points.

Then we prolong $f$ to $f' : G' \rightarrow F$ for a cofinal $G' \subseteq C^+F_L \uparrow u_0$.

Since by 7.5.11 $C^+F_L$ is coatomic, we extend $f$ to maximal points of $C^+F_L$ by sending every $y \in C^+P_L - CF_L$ to a certain terminal world of $F$. Hence $C^+F_L \not\vDash CSM(F)$.

On the other hand, $C^+\Phi_L \vDash CSM(F)$ by 7.5.3 and $C^+\Phi_L$ is descriptive (7.5.5), hence $C^+F_L \vDash CSM(F)$ by $d$-persistence (1.12.30(4), 1.12.22), which leads to a contradiction.

**Corollary 7.5.18**

(1) Let $\Lambda$ be a cofinal subframe intermediate logic containing the formula $SI(FS_3)$. Then $L = Q\Lambda CK$ is $C$-canonical.

(2) The same holds for a $\Delta$-elementary cofinal subframe $\Lambda \supseteq S4.1$ containing $SM(FS_3)$ and $L = Q\Lambda CK^+$. 

Proof

(1) As we know from 7.5.13, if CSI(F) ∈ L and F is a weakly directed finite rooted poset, then CFL ⊩ CSI(F). So assuming that CFL ∯ CSI(F), and F is not weakly directed we can show that CSI(F) ∉ Λ. By assumption, CFL is cofinally subreducible to F.

Let F = (W, R) and for any a ∈ W put ā := R(a) ∩ max(F). We claim that \{ā | a ∈ W - max(F)\} is a chain in (2\text{max}(F), ⊆). In fact, if ā, ċ are ⊆-incomparable and m₁ ∈ ā - ċ, m₂ ∈ ċ - ā, then F contains a subframe (where 0 is the root):

So F, and thus CFL, is subreducible to FS₃. But then CFL ∯ SI(FS₃), which contradicts the canonicity of QHCK + SI(FS₃) ⊆ L.

Therefore the sets ā constitute a chain, so there must be the least of them, say ā. But then every non-maximal a sees every m ∈ ā₀ which means that F is weakly directed. This is a contradiction.

(2) If CSM(F) ∈ Λ, then we may assume that F is a blossom frame (by 1.12.30), and hence a poset (since S₄.1 ⊆ Λ). Now we can repeat the argument from the proof of (1) replacing CSI with CSM.

Lemma 7.5.19

(1) A rooted S₄.1-frame is not cofinally subreducible to FS₁ iff the set of its inner points is a quasi-chain.

(2) A rooted S₄.1-frame is not cofinally subreducible to FS₂ iff every maximal point is accessible from every inner point.

Proof

(1) (If.) If f is a subreduction of G to FS₁, f(a₁) = 1, f(a₂) = 2, then both a₁, a₂ see points in f⁻¹(3), so they are inner in G. a₁, a₂ are incomparable, since 1, 2 are. Thus inner points are not in a quasi-chain.

(Only if.) If G is coatomic, has a root a₀ and two incomparable inner points a₁, a₂, then the partial map f : G → FS₁ sending aᵢ to i and every maximal point to 3 is a cofinal subreduction.
(2) (If.) If \( f \) is a cofinal subreduction of \( G \) to \( FS_2 \), \( f(a_1) = 1 \) and \( a_1 \) sees a maximal point \( a_2 \), then \( f(a_2) = 2 \). Similarly there exists a maximal \( a_3 \in f^{-1}(3) \). So \( a_1 \) is an inner point and \( a_1 \) does not see \( a_3 \).

(Only if.) Suppose \( G \) is coatomic with root \( a_0 \) and has an inner point \( a_1 \), which does not see a certain maximal point \( a_3 \). Then the map defined on \( a_0, a_1 \) and all maximal points of \( G \) is a cofinal subreduction if it sends \( a_i \) to \( i \) and all other maximal points to 2.

Proposition 7.5.20

(1) Let \( \Lambda \) be an intermediate propositional cofinal subframe logic containing \( CSI(FS_1) \) or \( CSI(FS_2) \). Then \( L = QACK \) is C-canonical.

(2) The same holds for a \( \Delta \)-elementary cofinal subframe \( \Lambda \supseteq S4.1 \) containing \( CSM(FS_1) \) or \( CSM(FS_2) \) and \( L = QACK^+ \).

Proof (1) By Theorem 7.5.15, the logics \( QHCK + CSI(FS_1) \), \( QHCK + CSI(FS_2) \) are C-canonical, so \( CF_L \vdash CSI(FS_1) \) or \( CF_L \vdash CSI(FS_2) \).

Now assuming \( CSI(F) \in \Lambda \) let us show \( CF_L \vdash CSI(F) \). In fact, if \( CF_L \not\vdash CSI(F) \), i.e. \( CF_L \) is cofinally subreducible to \( F \), then \( F \) is weakly directed. For, the validity of \( CSI(FS_i) \) is preserved by cofinal subreductions, thus \( F \not\vdash CSI(FS_1) \) or \( F \vdash CSI(F_2) \). Therefore \( F \) is weakly directed by Lemma 7.5.19. Hence \( CF_L \vdash CSI(F) \) by Theorem 7.5.15.

To prove (2), use the same argument applying 7.5.17.

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Figure 7.1. \( FS_1 \). Figure 7.2. \( FS_2 \). Figure 7.3. \( FS_3 \).

Since incompatible sets are always strongly incompatible in the canonical models of the intermediate propositional logics, we can prove that all cofinal subframe propositional logics are canonical.

Question 7.5.21 Let \( L \) be an s.p.l. with constant domains. Is every incompatible subset of \( CPL \) strongly incompatible?
Question 7.5.22 Is QA + CD + KF strongly complete for every cofinal sub-frame logic \( \Lambda \)? What happens in the case \( \Lambda = H + CSI(FS_3) \) where FS_3 is the frame in Fig. 7.3?

7.6 Natural models with constant domains

In this section we transfer the method described in section 6.4 to constant domains.

Lemma 7.6.1 Let \( M \) be a Kripke model for a modal logic \( L \) with the constant domain \( S_0 \). Then for any world \( u \in M \), for any \( S_0 \)-sentence \( A \)

\[
M, u \models A \iff CM_L, \nu_M(u) \models A.
\]

The same holds in the intuitionistic case, with obvious changes.

Lemma 7.6.2 Let \( M \) be the same as in the previous lemma, \( R_i \) the accessibility relations in \( M \). Then for any \( u, v \in M \)

\[
u_M(u) R_i \nu_M(v).
\]

Definition 7.6.3 Let \( L \) be an m.p.l.\( (=) \) containing Barcan axioms or an s.p.l.\( (=) \) containing CD, \( F = (W, R_1, \ldots, R_N) \) a propositional frame of the corresponding kind. A CDL-map based on \( F \) is a monotonic map from \( F \) to (the propositional base of) \( CF_L^{(=)} \).

Definition 7.6.4 Let \( h \) be a CDL-map based on a propositional frame \( F \). The predicate Kripke frame associated with \( h \) is \( F(h) := F \odot S_0 \).

If \( L \) is a logic with equality, the Kripke frame with equality associated with \( h \) is \( F^e(h) := (F, D, \sim) \), where

\[
c \sim_u d \iff (c = d) \in h(u).
\]

The CDL-model associated with \( h \) is \( M^{(=)}(h) := (F^{(=)}(h), \xi(h)) \), where

\[
\xi(h)_u(P^n_k) := \{c \in S^n_0 | P^n_k(c) \in h(u)\}
\]

for \( u \in F \).

Definition 7.6.5 A CDL-map \( h : F \to CF_L \) and the associated CDL-model \( M(h) \) are called natural if \( h = \nu_{M(h), L} \), i.e. for any \( u \in F \), \( A \in \mathcal{L}(u) \)

\[
M(h) \models (\|\_\_)A \iff CM_L, h(u) \models (\|\_\_)A \iff A \in h(u).
\]

Lemma 7.6.6 For a modal logic \( L \), a CDL-map \( h : F \to CF_L \) is natural iff it is selective (in the sense of Definition 6.4.7).

\(^2\text{[Shimura, 1993]}\)
Proof The same as for 6.4.8.

Definition 7.6.7 An intuitionistic CDL-map $h : F \rightarrow CF_L$ based on $F = (W, R)$ is called selective if it satisfies the condition

(\(\forall\)) if $(A_1 \supset A_2) \in (-h(u))$, then $A_1 \in h(v), \ A_2 \not\in h(v)$ for some $v \in R(u)$.

Note that now we do not postulate the analogue of condition (\(\forall\)) from Definition 6.4.9; it obviously holds for $v = u$, since $h(u)$ satisfies (Ac).

Lemma 7.6.8 An intuitionistic CDL-map $h : F \rightarrow CF_L$ is natural iff it is selective.

Proof Similar to 6.4.10. The only difference is the case $A = \forall x B$ in the 'if' part:

If $A \not\in h(u)$, then by (Ac), there exists $c \in D_u$ such that $B(c) \not\in h(u)$. By induction hypothesis, $u \not\models B(c)$. Hence $u \not\models A$.

The other way round, if $u \not\models \forall x B$, then since the domain is constant, there exists $c \in D_u$ such that $u \not\models B(c)$. Thus $h(u) \not\models B(c)$, which implies $h(u) \not\models A$.

Lemma 7.6.9 Let $L$ be a modal (resp., superintuitionistic) logic containing Barcan formulas (resp. CD), $M_1 \subseteq CM_L$ a selective submodel, $F_1$ the propositional frame of $M_1$, $h : F \rightarrow F_1$. Then $h$ is a natural CDL-map.

Proof Cf. Lemma 6.4.11.

Lemma 7.6.10

(1) Let $L$ be an $N$-modal logic containing Barcan formulas for all $\Box_i$, $\Gamma \in CPL$. Then there exists a CD-Kripke model $M$ based on a standard greedy tree $F$ such that $\nu_M(\lambda) = \Gamma$ and $M \models L$.

(2) If $L$ contains $\Diamond_i \top$ for $1 \leq i \leq N$, then one can take $F = F_N T_\omega$.

(3) If $L$ is 1-modal and $L \supseteq QS4$, then the claim holds for $F = IT_\omega$.

Proof Cf. Lemma 6.4.12.

Proposition 7.6.11

(1) $QK_N C = ML(CKF_N T_\omega)$, $QD_N C = ML(CKF_N T_\omega)$,

(2) $QK_N C^w = ML^w(CKE(F_N T_\omega))$, $QD_N C^w = ML^w(CKE(F_N T_\omega))$.

Proof Follows from the previous lemma, cf. 6.4.13.

Proposition 7.6.12 Let $\Lambda$ be an $N$-modal propositional PTC-logic. Then

(1) $Q\Lambda C = ML(CKGT(\Lambda))$. 
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(2) $\text{QAC}^- = \text{ML}^-(\text{CK}(\text{GT}(\Lambda)))$.

**Proof** Similar to Proposition 6.4.14.

(1) By 7.1.8, QAC is $\mathcal{C}K$-complete. Then by 1.11.5 and 3.12.8 we obtain

$L = \text{ML}(\text{CKV}_0(\Lambda))$, and hence $L = \text{ML}(\text{CKV}_1(\Lambda))$, by 3.3.21. (We use the same notation $V_0$, $V_1$ as in the proof of 6.4.14.) Now by 1.11.11 and 3.3.14 it follows that $\text{ML}(\text{CKGT}(\Lambda)) \subseteq L$.

(2) The proof is similar.

In the same way we obtain an analogue of Proposition 6.4.15

**Proposition 7.6.13**

(1) $\text{QKNC}^- + CE = \text{ML}^-(\text{CKF}_N T_\omega)$, $\text{QDNC}^- + CE = \text{ML}^-(\text{CKF}_N T_\omega)$.

(2) Let $\Lambda$ be a propositional PTC-logic. Then $\text{QAC}^- + CE = \text{ML}^-(\text{CKGT}(\Lambda))$.

Natural models with constant domains are a particular case of natural models described in section 6.4; only the notation is slightly simplified in this case.

**Definition 7.6.14** Let $L$ be a predicate logic, $F$ a propositional Kripke frame, and let $R^i_i (1 \leq i \leq N)$ or $R'$ in the intuitionistic case be relations on $CPL$.

An $(L, R^i_i, \ldots, R^N_N)$-model on $F$ is a mapping from $F$ to $CPL$, i.e. $N = (\Gamma_u \mid u \in F)$ where $\Gamma_u$ are $(L, S_0)$-places such that

$uR_i v \Rightarrow \Gamma_u R^i_i \Gamma_v$.

An $L$-model $N$ is called natural if the conditions from Definitions 6.4.7, 6.4.9 (for the modal or the intuitionistic case respectively) holds (with $S_0$ replacing $S_u$).

Recall that in the intuitionistic case the condition $(\forall)$ from Definition 6.4.9 holds obviously already with $v = u$, since theories in $CPL$ are $L \forall$-complete, i.e they satisfy $\text{Ac}$ from Definition 7.2.1.

As in Section 6.4, a natural $(L, R_L)$-model is called a natural $L$-model. (Recall that $\vec{R}_L^C = R_L$ in the intuitionistic case with constant domains.)

All main properties of natural models and crucial lemmas from Section 6.4 (together with their proofs) can be directly transferred to the case of constant domains: namely 6.4.8, 6.4.10, 6.4.11, etc. (now one has to use $\Gamma$ or $\Gamma_u$ instead of $L$-places $(S, \Gamma)$ or $(S_u, \Gamma_u)$ respectively). Therefore the logics with $\text{Ba}_i$ or with $\text{CD}$ (in the $N$-modal or in the intuitionistic case respectively) satisfy the main completeness results about the universal tree $T_\omega$ (or its subtrees). Let us formulate these completeness theorems for the intuitionistic case.

**Proposition 7.6.15**

(1) $\text{QH} + \text{CD} = \text{IL}(\text{CK}(\text{IT}_\omega))$.

(2) $\text{QH}^e + \text{CD} = \text{IL}^e(\text{CKE}(\text{IT}_\omega))$. 
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(3) $\text{QH}^d + CD = \text{IL}^w(\text{CK}(IT_\omega))$.

**Proposition 7.6.16** $\text{QH} + CD \land AP_h = \text{IL}(\text{CK}(IT^h_\omega))$ for $h > 0$ and similarly for the corresponding logics with equality.

Note that in the case of constant domains a natural analogue of 7.4.12 holds for any logic $L$ containing $AP_h$.

Now let us prove completeness results (w.r.t. coatomic trees) for the logic $\text{QH} + CD \land KF$. Here the situation is subtler than for varying domains. Namely, for an arbitrary path $w = (u_0, u_1, \ldots)$ in $IT_\omega$ we cannot always extend an $L$-consistent theory $\Gamma = \bigcup \Gamma_i$ to an $L\forall$-place $\Gamma_\omega$ in the same language (i.e. with constants from $S_0$). For example, if $S_0 = \{ c_i | i \in \omega \}$, and for each $i$, $P(c_i) \in \Gamma_{u_i}$, then $\Gamma_\omega$ satisfying (Ac) does not exist.

That is why we cannot prove completeness of $\text{QH} + CD \land KF$ w.r.t. the coatomic tree $IT^G_\omega$ with a constant domain. Moreover $\text{QH} + CD \land KF$ is not determined by $\mathcal{G}$, cf. Lemma 3.12.17. We do not know if this logic is complete w.r.t. $\overline{IT}_\omega \odot \omega$ with uncountable $\omega$, but anyway natural models with denumerable domains are insufficient.

Instead we can show completeness w.r.t. the class of (denumerable) coatomic trees of the form $\overline{IT}_\omega^G$ (for $G \subseteq IT_\omega^\delta$) (cf. [Gabbay, 1972]).

**Lemma 7.6.17** Let $L$ be a superintuitionistic predicate logic containing $CD \land KF$. Then for any $(L, S_0)$-place $\Gamma$ there exists a subset $G \subseteq IT_\omega^\delta$ such that $\overline{IT}_\omega^G$ is coatomic and for any $G' \subseteq G$ there exists an $L$-natural model $N = (\Gamma_u | u \in \overline{IT}_\omega^G)$ (with the constant domain $S_0$) such that $\Gamma_\omega = \Gamma$.

**Proof** First we construct an $L$-natural model $(\Gamma_u | u \in IT_\omega)$ on $IT_\omega$. Recall that a subset $X$ of $IT_\omega$ is called *dense* if $\forall u R(u) \cap X \neq \emptyset$. We put $X_B := \{ u \in IT_\omega | B \in \Gamma_u \}$ for QCL-theorems $B \in IF(\equiv)$ (without constants from $S_0$); actually it is sufficient to consider only theorems of the form $B = \forall(A \lor \neg A)$, cf. Remark 7.3.1). These sets are dense since $L \vdash \neg B$ (by the Glivenko theorem), thus $\neg B \notin \Gamma_u$ for any $u$.

Call a path *generic* if it intersects every $X_B$; let $G_\Gamma$ be the set of generic paths. Obviously, $\forall u \in IT_\omega \exists w \in G(u \in w)$, i.e. the tree $\overline{IT}_\omega^G$ is coatomic. Now for any generic path $w \in G_s$ we take the set $\Gamma_w := \bigcup_{v \in w} \Gamma_v$. Then $\Gamma_w \in CP_L$. In fact, if $L \vdash \bigwedge_{\Gamma_1 \supseteq \bigvee_{\Delta 1} \Gamma_1 \subseteq \bigvee_{\Delta 2} \Gamma_2$, then $\Gamma_1 \subseteq \Gamma_v, \Delta_1 \subseteq -\Gamma_v$ for some $v \in w$; also, if $\exists x A(x) \in \Gamma_\omega$, then $\exists x A(x) \in \Gamma_v$ and $A(c) \in \Gamma_v \subseteq \Gamma_w$ for some $v \in w, c \in S_0$. Also by genericity, $Q\text{CL} \subseteq \Gamma_w$, thus $\Gamma_w$ is $(Q\text{CL}, S_0)$-place and condition (Ac) for $\Gamma_w$ (in a natural model on $\overline{IT}_\omega^{G'}$ for $G' \subseteq G_s$) holds.

Recall that although $G_\Gamma$ may be uncountable, there always exists a denumerable subset $G \subseteq G_\Gamma$ such that $\overline{IT}_\omega^G$ is coatomic. Therefore we obtain the following completeness result.
**Proposition 7.6.18** The logic \( \text{QHCK} := \text{QH} + CD \wedge KF \) is complete (in semantics \( CK \)) w.r.t. the class of all coatomic trees \( \overline{IT}_\omega^G \); moreover one can take only denumerable coatomic trees.

However we do not know if \( \text{QHCK} \) is determined by a single coatomic tree of the form \( \overline{IT}_\omega^G \). But still it is determined by a single denumerable coatomic tree, e.g. by Smorynski's sum \( \omega \overline{IT}_\omega^G \) of all coatomic trees refuting nontheorems of the logic. Note that the sum over all denumerable atomic trees of the form \( \overline{IT}_\omega^G \) is not itself denumerable.

Another alternative is to add maximal elements not above paths, but above points of \( IT_\omega \) as explained below.

**Definition 7.6.19** A subset \( X \subseteq IT_\omega \) gives rise to a denumerable tree \( IT_\omega^X := IT_\omega \cup X^* \) where \( X^* = \{ u^* | u \in X \} \) is the set of maximal elements \( u^* := (u, -1) \) such that \( u^* \) is an immediate successor of \( u \).

Obviously the tree \( IT_\omega^X \) is coatomic iff \( X \) is a dense subset of \( IT_\omega \).

**Lemma 7.6.20** Let \( L \) be a superintuitionistic predicate logic containing \( CD \wedge K \). Then for any \( \Gamma \in CPL \) and for any \( X \subseteq IT_\omega \) there exists an \( L \)-natural model \( N \) (with constant domain) on \( IT_\omega^X \) such that \( \Gamma_X = \Gamma \).

**Proof** We simply extend an \( L \)-place \( \Gamma_u \) (for \( u \in X \)) to an \( L \)-place \( \Gamma_{u^*} \), by Lemma 8.3.6.

**Proposition 7.6.21** \( \text{QHCK} = \text{IL}(CK(IT_\omega^X)) \) for any dense \( X \subseteq IT_\omega \) (in particular, for \( X = IT_\omega \)).

A similar completeness result for logics with equality (w.r.t. \( CKE \) or w.r.t. \( K \)) is quite obvious.

### 7.7 Remarks on Kripke bundles with constant domains

Now let us make some observations on completeness in more general semantics than \( K \) or \( K\varepsilon \).

Proposition 7.6.16 clearly shows that \( \text{QH} + CD \wedge AP_h \) is determined by the tree \( IT_\omega^h \) in the semantics of Kripke sheaves or even quasi-sheaves (recall that \( AP_h \) is strongly valid in any Kripke quasi-sheaf based on a frame of depth \( \leq h \)).

In particular, \( \text{QCL} = \text{IL}(CK(IT^1)) = \text{IL}(CKE(IT^1)) = \text{IL}(KQ(IT^1)) \), where \( IT^1 \cong Z_1 \) is a one-element poset. On the other hand, for Kripke bundles the situation changes drastically:

**Proposition 7.7.1** \( \text{IL}(CKB(IT^1)) = \text{QH} + CD \).
Proof Let $M$ be a Kripke model based on $IT_\omega$ with constant domain $\omega$ refuting a nontheorem $A$, i.e. $M, \lambda \not\models A$. Take a Kripke bundle $F$ based on $\{u_0\}$ in which $D_{u_0} = \{a_u \mid u \in IT_\omega\} \cup \{b_i, c_i \mid i \in \omega\}$ is quasi-ordered by the following relation $R_1$:

$$a_u R_1 a_v \text{ iff } u \leq v \in IT_\omega, \text{ for } u, v \in IT_\omega,$$

$$b_i R_1 a_u \text{ for all } i \in \omega, v \in IT_\omega, \text{ and } b_i R_1 b_j \text{ for all } i, j \in \omega.$$ 

Obviously, $F$ has a constant domain — every $c_i$ is incomparable with all elements of $D_{u_0}$; every $a_u$ has infinitely many predecessors $b_i$ ($i \in \omega$). Consider the following map $\chi$ from $D_{u_0}$ onto $\omega$:

$$\chi(c_i) = i \text{ and } \chi(b_i) = \chi(a_u) = 0 \text{ for any } i \in \omega, u \in IT_\omega,$$

and consider the model $M'$ over $F$ for the formula $A^1$ (with an additional parameter $z$) such that:

$$M', u_0 \models P'(a_2, d_1, \ldots, d_n) \iff M, u \models P(\chi(d_1), \ldots, \chi(d_n))$$

where the predicate letter $P'$ is substituted for $P$ in $A^1$.

One can easily check that

$$M', u_0 \models B^1(a_u, d_1, \ldots, d_n) \iff M, u \models B(\chi(d_1), \ldots, \chi(d_n))$$

for any formula $B(x_1, \ldots, x_n)$ and $u \in IT_\omega$ (by induction on $B$). Thus, $M', v_0 \not\models A^1(\lambda)$, and so $A \not\in \mathbf{IL}(F)$. ■

In the same way we can simulate every Kripke model with a constant domain by a Kripke bundle model based on one-element poset, so that individuals replacing additional parameters of formula $A^1$ correspond to worlds of the original Kripke frame.

Corollary 7.7.2 $\mathbf{IL}(\mathbf{CKB}(F)) = \mathbf{QH} + \mathbf{CD}$ for any propositional base $F$.

Therefore in the semantics of Kripke bundles the propositional base is not so important as the structure of individual domains.

Remark 7.7.3 Note that the above corollary does not transfer to logics with equality — e.g. the formula $\forall x y (x = y) \supset p \lor \neg p$ is strongly valid in every Kripke bundle over a one-element base.

Remark 7.7.4 One can also try to describe the logic $\mathbf{IL}(\mathbf{KB}(IT^1))$ determined by a one-element base with varying domains. We may conjecture that this logic is $\mathbf{QH}$, but it is still unclear how to simulate arbitrary Kripke frames with varying domains within a single domain in a single world.

Remark 7.7.5 Let us also mention the semantics of Kripke quasi-sheaves. One can easily show that $\mathbf{IL}(\mathbf{CKQ}(IT_1)) = \mathbf{QH} + \mathbf{CD}$ and $\mathbf{IL}(\mathbf{KQ}(IT_1)) = \mathbf{QH}$ for the $\omega$-chain $IT_1$; in fact, every Kripke model based on $IT_\omega$ can be simulated in a Kripke quasi-sheaf over $IT_1$ similarly to Proposition 7.7.1. Branching of
individuals in a quasi-sheaf replaces the branching of worlds. For the case of varying domains, note that for constructing a natural model over $IT_\omega$ we may assume that individual domains of all worlds at the same level coincide.

Similarly, $\text{IL}(\mathcal{KQ}(F)) = \text{QHCK}$ and $\text{IL}(\mathcal{KQ}(F)) = \text{QH} + KF$ for an $(\omega + 1)$-chain $F$; atomic trees $IT_\omega^G$ or $IT_\omega^X$ can be used here. On the other hand, note that $\text{IL}(\mathcal{KQ}(F)) = \text{QH} + CD \land AP_h$ and $\text{IL}(\mathcal{KQ}(F)) = \text{LP}^+_h$ for any poset $F$ of finite height $h > 0$ (in particular, for an $h$-element chain).

### 7.8 Kripke frames over the reals and the rationals

In this section we consider specific extensions of the intermediate logic of linear ordered sets with constant domains $\text{QLCC}$, viz. the logic of frames over the rational and the real line. Both logics happen to be finitely axiomatisable. The first result is not a big surprise, but the second is a ‘happy chance’ making a contrast with classical logic. These results were first proved in [Takano, 1987] who also observed that strong completeness theorem holds in both cases.

We use the following two axioms:

Ta. $(\forall x P(x) \supset \exists x Q(x)) \supset \exists x (P(x) \supset r) \lor \exists x (r \supset Q(x)).$

Ta'. $(\forall x P(x) \supset \exists x Q(x)) \supset \exists x (P(x) \supset \exists y Q(y)) \lor \exists x (\exists y Q(y) \supset Q(x)).$

**Lemma 7.8.1** $\text{QHC} + \text{Ta} = \text{QLCC} + \text{Ta'}.$

**Proof** $(\subseteq).$ It suffices to show that $\text{QLCC} + \text{Ta'} \vdash \text{Ta}.$ So we argue in $\text{QLCC} + \text{Ta'}.$

We have

$$(Q(x) \supset r) \lor (r \supset Q(x))$$

by $AZ$, hence $(Q(x) \supset r) \lor \exists x (r \supset Q(x)), \forall x((Q(x) \supset r) \lor \exists x (r \supset Q(x))),$ and thus

(1) $$(\forall x (Q(x) \supset r) \lor \exists x (r \supset Q(x)))$$

by $CD$. Similarly we have

(2) $$\forall x (r \supset P(x)) \lor \exists x (P(x) \supset r).$$

From (1) and (2) we obtain either the conclusion of $\text{Ta}$:

$$\exists x (P(x) \supset r) \lor \exists x (r \supset Q(x))$$

or

(3) $$\forall x (Q(x) \supset r) \land \forall x (r \supset P(x)).$$
Now assume the premise of $Ta$ (congruent to the premise of $Ta'$):

(4) \hspace{1cm} \forall x P(x) \supset \exists x Q(x).

Hence by $Ta'$ we have

\[ \exists x(P(x) \supset \exists y Q(y)) \vee \exists x(\exists y Q(y) \supset Q(x)). \]

Consider the first option and assume

(5) \hspace{1cm} \exists x(P(x) \supset \exists y Q(y)).

By 2.6.15 we also have

\[ \forall x(Q(x) \supset r) \vdash \exists y Q(y) \supset r. \]

So

(3), $P(x) \supset \exists y Q(y) \vdash P(x) \supset r,$

and thus

(3), (5) $\vdash \exists x(P(x) \supset r)$.

Now assume

(6) \hspace{1cm} \exists x(\exists y Q(y) \supset Q(x)).

Since by 2.6.15

\[ \forall x(r \supset P(x)) \vdash r \supset \forall x P(x), \]

we have

(3) $\vdash r \supset \forall x P(x),$ 

hence

(3), (4) $\vdash r \supset \exists x Q(x),$ 

and thus

(3), (4) $\vdash r \supset \exists y Q(y).$

Now obviously

\[ r \supset \exists y Q(y), \exists y Q(y) \supset Q(x) \vdash r \supset Q(x), \]

hence

(3), (4), \exists y Q(y) \supset Q(x) \vdash r \supset Q(x),

and thus

(3), (4) $\vdash \exists y Q(y) \supset Q(x) \supset r \supset Q(x).$

Therefore by monotonicity

(3), (4) $\vdash \exists x(\exists y Q(y) \supset Q(x)) \supset \exists x(r \supset Q(x)),$

and eventually

(3), (4), (6) $\vdash \exists x(r \supset Q(x)).$
So we have proved the conclusion of $Ta$.

$(\supseteq)$. We have $\mathbf{QH} + Ta \vdash Ta'$, since obviously $[\exists x Q(x)/r] Ta \equiv Ta'$. To show $\mathbf{QH} + Ta \vdash AZ$, note that

$$[p, p/P(x), Q(x)] Ta = (\forall x p \supset \exists x p) \supset \exists x (p \supset r) \lor \exists x (r \supset p)$$

is clearly equivalent to $(p \supset r) \lor (r \supset p)$.

In this section we consider subsets of $\mathbb{R}$, in particular, $\mathbb{Q}$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{Q}_+ = \mathbb{R}_+ \cap \mathbb{Q}$ as propositional Kripke frames with the accessibility relation $\leq$.

**Theorem 7.8.2**

1. $\mathbf{QLCC} = \mathbf{IL(CKQ)} = \mathbf{IL}(\mathbb{Q} \odot \omega)$.

2. $\mathbf{QLCC}$ is strongly complete w.r.t. $\mathbf{CKQ}$ and $\mathbb{Q} \odot \omega$.

**Proof**

(1) The inclusion $\subseteq$ follows readily from Proposition 7.3.6. The proof of $\supseteq$ is quite similar to 6.7.3. By Lemma 6.7.2, $\mathbb{Q}_+ \rightarrow F$ for any rooted countable chain $F$. Then we can apply Propositions 7.3.6, 3.3.14.

(2) By the Shimura theorem, $\mathbf{QLCC}$ is strongly Kripke complete, hence it is also strongly complete w.r.t. the class of countable chains with domain $\omega$. By 6.7.2 we know that $\mathbb{Q}_+ \rightarrow F$ for any countable chain $F$. Hence $\mathbb{Q}_+ \odot \omega \rightarrow F \odot \omega$ by 5.1 and eventually strong completeness w.r.t. $\mathbb{Q}_+ \odot \omega$ follows by 5.1.

**Definition 7.8.3** Let $F_0 \subseteq F$ be linear Kripke frames with a constant domain $V$, $R$ the accessibility relation in $F$. Let $M = (F, \theta)$, $M_0 = (F_0, \theta_0)$ be intuitionistic Kripke models. $M$ is called a right extension of $M_0$ if for any $V$-sentence $A$

$$\theta^+(A) = R(\theta_0^+(A)). \quad (\ast r)$$

So we have

$$M, u \vdash A \iff \exists v \in F_0 \ (v R u \& M_0, v \vdash A)$$

Note that $(\ast r)$ readily implies $\theta_0^+(A) = \theta^+(A) \cap F_0$, i.e. $M_0 \subseteq M$.

A right extension is obviously unique (if it exists).

**Definition 7.8.4** Let $F_0 \subseteq F$ be linear Kripke frames with a constant domain $V$, $R$ the accessibility relation in $F$. Let $M = (F, \theta)$, $M_0 = (F_0, \theta_0)$ be intuitionistic Kripke models. $M$ is called a left extension of $M_0$ if for any $V$-sentence $A$, for any $u \in M$

$$u \in \theta^+(A) \iff R(u) \cap F_0 \subseteq \theta_0^+(A).$$
The latter condition can also be written as follows

\[ \theta^+(A) = \Box_R \theta_0^+(A), \]

where

\[ \Box_R U := \{ u \mid R(u) \cap F_0 \subseteq U \}. \]

\((*l)\) is also equivalent to

\[ M, u \models A \iff \exists v \in F_0 (uRv \& M_0, v \not\models A). \]

**Lemma 7.8.5** Let \( F \cup V \) be a linear Kripke frame over \( F = (W, R) \), \( W_0 \subseteq W \) and let \( M_0 = (F|W_0, \theta_0) \) be a Kripke model such that

1. \( R(W_0) = W \),
2. for any \( V \)-sentence \( \forall x B(x) \)

\[ \bigcap_{a \in V} R(\theta_0^+(B(a))) \subseteq R(\theta_0^+(\forall x B(x))). \]

Then there exists a right extension of \( M_0 \) over \( F \).

Note that the converse to \((2r)\) obviously holds, so \((2r)\) is equivalent to

\[ R(\theta_0^+(\forall x B(x))) = \bigcap_{a \in V} R(\theta_0^+(B(a))). \]

**Proof** We define the valuation \( \theta \) by the equality from 7.8.3

\[ \theta^+(A) = R(\theta_0^+(A)) \]

for all atomic \( V \)-sentences \( A \). \( \theta \) is obviously intuitionistic, since \( R \) is transitive. Then we check \((*r)\) for any \( V \)-sentence \( A \) by induction.

1. Consider the case \( A = B \supseteq C \). By the induction hypothesis, it suffices to show that for any \( u \)

\[ u \in R(\theta_0^+(A)) \iff \forall v \in R(u) (v \in R(\theta_0^+(B)) \Rightarrow v \in R(\theta_0^+(C))). \]

('Only if'). Assuming \( u \in R(\theta_0^+(A)) \), \( uRv \) and \( v \in R(\theta_0^+(B)) \), let us check \( v \in R(\theta_0^+(C)) \).

By assumption there exists \( u_1 Ru \) such that \( u_1 \models B \supseteq C \) and \( v_1 Ru \) such that \( v_1 \models B \) (where \( \models \) refers to \( M_0 \)). Let \( w_1 = \max(u_1, v_1) \); then \( w_1 Ru \). Now \( u_1 \models B \supseteq C \) implies \( w_1 \models B \supseteq C \); \( v_1 \models B \) implies \( w_1 \models B \). Thus \( w_1 \models C \), and so \( v \in R(\theta_0^+(C)) \).

('If'). We suppose \( u \notin R(\theta_0^+(A)) \) and show

\[ \exists v \in R(u) \forall v \in R(\theta_0^+(B)) - R(\theta_0^+(C)). \]

By assumption \((1r)\), there exists \( v_1 \in R^{-1}(u) \cap W_0 \). Since \( u \notin R(\theta_0^+(A)) \), we have \( v_1 \not\models A \). Then for some \( v_2 \in R(v_1) \), \( v_2 \models B \), but \( v_2 \not\models C \).
Now put \( v = \max(u, v_2) \). Then \( v_2 \models B \) implies \( v \models B \), and thus \( v \in R(\theta^+_0(B)) \). It remains to check that \( v \notin R(\theta^+_0(C)) \).

If \( v = v_2 \), then \( v \notin R(\theta^+_0(C)) \) follows from \( v_2 \not\models C \).

If \( v = u \), then \( u \notin R(\theta^+_0(B \supset C)) \) implies \( u \notin R(\theta^+_C(B)) \). This is because \( C \supset (B \supset C) \) is an intuitionistic tautology, so \( M_0 \models C \supset (B \supset C) \), i.e. \( \theta^+_0(C) \subseteq \theta^+_0(B \supset C) \).

(II) If \( A = B \land C \), by the induction hypothesis it suffices to show

\[
R(\theta^+_0(B \land C)) = R(\theta^+_0(B)) \cap R(\theta^+_0(C)).
\]

The inclusion \( \subseteq \) follows from \( \theta^+_0(B \land C) \subseteq \theta^+_0(B) \) and \( \theta^+_0(B \land C) \subseteq \theta^+_0(C) \).

The other way round, suppose \( u \in R(\theta^+_0(B)) \cap R(\theta^+_0(C)) \). Then \( v_1 \in \theta^+_0(B), \ v_2 \in \theta^+_0(C) \) for some \( v_1, v_2 \in R^{-1}(u) \). If \( v = \max(v_1, v_2) \), then obviously \( v \in \theta^+_0(B \land C) \), and thus \( u \in R(\theta^+_0(B \land C)) \). This proves \( \supseteq \) in (1#).

The remaining cases are left to the reader; note that for the case \( A = \forall x B(x) \) one can use the assumption (2r).

**Remark** 7.8.6 (r1), (r2) are not only sufficient, but necessary for the existence of a right extension. (r2) follows from \((*)r\) for \( A = \forall x B(x) \), (r1) from \((*)r\) for \( A = \top \).

**Lemma 7.8.7** Let \( F \odot V \) be a linear Kripke frame over \( F = (W, R) \), \( W_2 \subseteq W \) and let \( M_0 = (F \upharpoonright W_0, \theta_0) \) be a Kripke model such that

1. \( R^{-1}(W_0) = W \),

2. for any \( V \)-sentence \( \exists x A(x) \)

\[
\bigcup_{a \in V} \square R \theta^+_0(A(a)) \supseteq \square R \theta^+_0(\exists x A(x)).
\]

Then there exists a left extension of \( M_0 \) over \( F \).

**Proof** Similar to Lemma 7.8.5. Now we define

\[
\theta^+(A) := \square R \theta^+_0(A),
\]

for atomic \( V \)-sentences \( A \) and prove \((*l)\) by induction for arbitrary \( A \).

Suppose \( A = B \lor C \). By the induction hypothesis, \((*l)\) for \( A \) is equivalent to

\[
\square R \theta^+_0(B) \cup \square R \theta^+_0(C) = \square R \theta^+_0(B \lor C).
\]

The inclusion \( \subseteq \) is obvious, so let us check \( \supseteq \). Suppose \( u \notin \square R \theta^+_0(B) \cup \square R \theta^+_0(C) \). Then

\[
\exists v, w \in R(u) \ (v \notin \theta^+_0(B) \land w \notin \theta^+_0(C)).
\]

Since \( R \) is linear, \( v R w \) or \( w R v \). Then respectively, \( v \notin \theta^+_0(B \lor C) \) or \( w \notin \theta^+_0(B \lor C) \), so anyway \( u \notin \square R \theta^+_0(B \lor C) \).

\[
\square R \theta^+_0(B \lor C)
\]
Suppose $A = B \supset C$. By the induction hypothesis we can rewrite (*I) as

$$u \not\in \square_R \theta_0^+(A) \iff \exists v \in R(u) \forall \theta_0^+(B) - \square_R \theta_0^+(C)).$$

($\Rightarrow$) Suppose $u \not\in \square_R \theta_0^+(A)$. Then $v \not\in \theta_0^+(A)$ for some $w \in R(u)$, and thus $v \in \theta_0^+(B) \supset \square_R \theta_0^+(C)$ for some $v \in R(w) \subseteq R(u)$.

Obviously, $v \not\in \square_R \theta_0^+(C)$. Since $M_0$ is intuitionistic, $\theta_0^+(B) \subseteq \square_R \theta_0^+(B)$.

This proves ($\Rightarrow$).

($\Leftarrow$) Suppose $u R v$, $v \in (\square_R \theta_0^+(B) \supset \square_R \theta_0^+(C))$. Then $w \not\in \theta_0^+(A)$ for some $w \in R(v) \subseteq R(u)$ and $v \in \square_R \theta_0^+(B)$ implies $w \in \theta_0^+(B)$. Hence

$$w \not\in \theta_0^+(A), \text{ so } u \not\in \square_R \theta_0^+(A).$$

The case $A = B \land C$ is obvious. The case $A = \exists x B(x)$ follows from (2l), since its converse holds trivially.

If $A = \forall x B(x)$, we have

$$u \not\in \theta^+(A) \iff \exists a \in V u \not\in \theta^+(B(a)) = \square_R \theta_0^+(B(a)), \quad (**$$

since the domain is constant and by the induction hypothesis. But $\theta_0^+(A) \subseteq \theta_0^+(B(a))$, so $u \not\in \theta^+(A)$ implies $u \not\in \square_R \theta_0^+(A)$.

The other way round, if $u \not\in \square_R \theta_0^+(A)$, then $v \not\in \theta_0^+(\forall x B(x))$ for some $v \in R(u) \cap F_0$. So $v \not\in \theta_0^+(B(a))$ for some $a \in V$, and thus $u \not\in \theta^+(A)$ by (**).

Lemma 7.8.8 Let $M = (Q_+ \circ V, \theta)$ be an intuitionistic Kripke model, in which every instance of $T a'$ is true. Then there exists an extension $\theta^*$ of $\theta$ such that $(R_+ \circ V, \theta^*)$ is an intuitionistic model.

Proof The proof consists of two parts.

(I) First we define a model over the set

$$S := Q_+ \cup \{\sigma_B(y) \mid \exists y B(y) \text{ is a V-sentence, } M, 0 \not\models \exists y (\exists z B(z) \supset B(y))\},$$

where

$$\sigma_B(y) := \inf_\mathbb{R} \theta(\exists y B(y)).$$

We would like to define a model $M_0$ with a valuation $\theta_0$ over $S$ as a right extension of $\theta$:

$$\theta_0(A) := \leq (\theta^+(A)).$$

By Lemma 7.8.5, it suffices to prove that for the relation $\leq$ on $S$, for every $V$-sentence $\forall x A(x)$,

$$\bigcap_{a \in V} \leq (\theta^+(A(a)) \subseteq \leq (\theta^+(\forall x A(x))) \quad (7.1)$$

To prove this, we assume that for a certain $\sigma \in S$

$$\forall a \in V \exists q \leq \sigma M, q \models A(a). \quad (7.2)$$
and show that for some \( q \in \mathbb{Q}_+ \),
\[
q \leq \sigma \& M, q \models \forall x A(x). \tag{7.3}
\]

**Case 1.** \( \sigma \in \mathbb{Q}_+ \).

Then \( M, \sigma \models \forall x A(x) \), so \( \sigma \in \theta^+(\forall x A(x)) \subseteq \leq (\theta^+(\forall x A(x))) \), so \( q = \sigma \) satisfies (7.3).

**Case 2.** \( \sigma = \sigma_B(y) \notin \mathbb{Q}_+ \).

By definition, \( M, 0 \not\models \exists y(\exists z B(z) \supset B(y)) \). Now let us show
\[
M, 0 \not\models \exists x(A(x) \supset \exists z B(z)). \tag{b}
\]

Suppose the contrary. Then \( M, 0 \models A(a) \supset \exists z B(z) \) for some \( a \in V \). By the assumption (|), \( M, q_a \models A(a) \) for some \( q_a \leq \sigma \). So \( M, q_a \models \exists z B(z) \), and hence \( \sigma \leq q_a \), by the definition of \( \sigma_B(y) \). Therefore \( \sigma = q_a \in \mathbb{Q} \), and we have a contradiction. Hence (b) holds.

On the other hand, by the assumption of the lemma,
\[
M, 0 \not\models (\forall x A(x) \supset \exists y B(y)) \supset (\exists x(A(x) \supset \exists z B(z)) \supset \exists y(\exists z B(z) \supset B(y))).
\]

Hence \( M, 0 \not\models \forall x A(x) \supset \exists y B(y) \), so there is \( q \in \mathbb{Q}_+ \) such that \( M, q \models \forall x A(x) \) but \( M, q \not\models \exists y B(y) \). Hence \( q \leq \sigma \), by the definition of \( \sigma_B(y) \). Thus \( q \) satisfies (7.3). This proves (7.1).

(II) Next, we construct \( M^* \) with a valuation \( \theta^* \) over \( \mathbb{R}_+^* \) as a left extension of \( M_0 \):
\[
M^*, \alpha \models A \iff \forall \sigma \in S (\alpha \leq \sigma \Rightarrow M_0, \sigma \models A).
\]

We use Lemma 7.8.7 to show the existence of \( M^* \). So we have to check that for every \( V \)-sentence \( \exists x B(x) \)
\[
\bigcup_{a \in V} \Box_{\leq \theta^+_0(B(a))} (\exists x B(x)). \tag{7.4}
\]

To prove this, suppose \( \alpha \in \Box_{\leq \theta^+_0(\exists x B(x))} \), that is
\[
\forall \sigma \in S (\alpha \leq \sigma \Rightarrow M_0, \sigma \models \exists y B(y)). \tag{7.5}
\]

Let us show that \( \alpha \in \bigcup_{a \in V} \Box_{\leq \theta^+_0(B(a))} \), i.e. for some \( a \in V \)
\[
\forall \sigma \in S (\alpha \leq \sigma \Rightarrow M_0, \sigma \models B(a)). \tag{7.6}
\]

Now we argue in \( M_0 \).

**Case 1.** \( 0 \models \exists y(\exists z B(z) \supset B(y)) \).

Then there is \( a \in V \) such that \( 0 \models \exists z B(z) \supset B(a) \), and this \( a \) is what we need.

**Case 2.** \( 0 \not\models \exists y(\exists z B(z) \supset B(y)) \).

So \( 0 \not\models \exists y(\exists z) \supset B(y) \), and hence \( \sigma_B(y) \in S \).

Let us show that \( \sigma_B(y) \not\models \exists y B(y) \). Suppose the contrary. Then \( \sigma_B(y) \models \exists y B(y) \), so \( \sigma_B(y) \models B(a_1) \) for some \( a_1 \in V \). Then by the assumption of Case 2,
\[
0 \not\models \exists z B(z) \supset B(a_1),
\]
so there exists $\sigma \in S$ such that $\sigma \models \exists z B(z)$, and $\sigma \not\models B(a)$. Thus $\sigma < \sigma_B(y)$, since $\sigma_B(y) \models B(a)$.

Hence

$$\exists q \in Q_+: \sigma < q < \sigma_B(y),$$

and then

$$M_0, q \models \exists z B(z),$$

which implies

$$M, q \models \exists z B(z).$$

But this contradicts $q < \sigma_B(y)$.

Now, by (7.5), $M_0, \sigma \not\models \exists y B(y)$ implies $\sigma_B(y) < \alpha$.

But then there is $q \in Q$ such that $\sigma_B(y) < q < \alpha$. Since $\sigma_B(y) < q$, it follows that $M_0, q \models \exists y B(y)$. Hence $M_0, q \models B(a)$ for some $a \in V$, and this $a$ satisfies (7.6). Thus (7.4) is proved.

**Theorem 7.8.9** $\text{QHC} + Ta = \text{IL}(\mathcal{C}\mathcal{K}\mathcal{R})$.

**Proof** (\(\subseteq\)). Let us show that $\mathbb{R} \oplus V \models Ta$. So consider an arbitrary model over $\mathbb{R} \oplus V$, $\alpha \in \mathbb{R}$ and show

$$\alpha \models (\forall x P(x) \supset \exists x Q(x)) \supset \exists x (P(x) \supset r) \lor \exists x (r \supset Q(x)).$$

(7.7)

Suppose the contrary. Then there exists $\beta \leq \alpha$ such that

$$\beta \models \forall x P(x) \supset \exists x Q(x), \text{ but } \beta \not\models \exists x P(x) \supset \exists x Q(x).$$

Thus for any $a \in V$, there is $\gamma_a \geq \beta$ such that $\gamma_a \models P(a)$, $\gamma_a \not\models r$. Similarly, for any $b \in V$ there is $\delta_b \geq \beta$ such that $\delta_b \models r$, $\delta_b \not\models Q(a)$. From $\gamma_a \not\models r$ and $\delta_b \models r$ it follows that $\gamma_a < \delta_b$, for every $a, b \in V$.

Now put $\epsilon := \sup_{a \in V} \gamma_a$. Then $\beta \leq \epsilon$ and $\epsilon \models \forall x P(x)$. Also $\beta \models \forall x P(x) \supset \exists x Q(x)$, hence $\epsilon \models \exists x Q(x)$.

On the other hand, $\epsilon \leq \delta_b$ for any $b \in V$, so $\epsilon \not\models \exists x Q(x)$. This contradiction proves (7.7)

(\(\supseteq\)). We assume $\text{QLCC} + Ta \not\models A$ for a sentence $A$ and construct a countermodel for $A$ over $\mathbb{R}_+$. Since $\mathbb{R}_+$ is a generated subframe, this implies $A \not\models \text{IL}(\mathcal{C}\mathcal{K}\mathcal{R})$.

By Lemma 7.8.1, $\text{QLCC} + Ta \models A$. By Theorem 7.8.2(2), we obtain a model $M = (Q_+ \oplus V, \theta)$ and $q_0 \in Q$ such that every instance of $Ta'$ is true at $M, q_0$ and $M, q_0 \not\models A$. We may assume that $q_0 = 0$. By Lemma 3.3.18, $M' := M \upharpoonright Q_+$ is a model, in which every instance of $Ta'$ is true. Hence by Lemma 7.8.8 there is an extension $M^*$ of $M'$ over $\mathbb{R}_+$.

**Proposition 7.8.10**

(1) $\text{QLCC} + KF = \text{IL}(\mathcal{C}\mathcal{K}([0, 1] \cap Q))$.

(2) $\text{QHC} + Ta + KF = \text{IL}(\mathcal{C}\mathcal{K}[0, 1])$.

We skip the proof, since it is very similar to Theorems 7.8.2 and 7.8.9.
Proposition 7.8.11 For any linearly ordered set F, the following conditions are equivalent:

(1) F is Dedekind complete (i.e. every non-empty subset with an upper bound has a supremum);

(2) $\mathbf{IL}(CKF) \vdash Ta'$.

Proof

The proof of $(1) \Rightarrow (2)$ is the same as for $(\subseteq)$ in 7.8.9.

Let us show $(2) \Rightarrow (1)$. Suppose $(2)$, but not $(1)$. Then there exists a non-empty $X \subseteq W$ with an upper bound, but without a supremum. Let $Y$ be the (non-empty) set of upper bounds of $X$. We index both $X$ and $Y$ by elements of $V$ of cardinality $|W|$ so that $X = \{ \xi_a \mid a \in V \}$ and $Y = \{ \eta_a \mid a \in V \}$. Consider a model $M$ over $W \odot V$, in which

(i) $w \vdash P(a) \leftrightarrow \xi_a \leq w$,

(ii) $w \vdash Q(a) \leftrightarrow \eta_a < w$.

Let us show that for any $w \in W$

(3) $w \in Y$ if $w \vdash \forall xP(x) \leftrightarrow w \vdash \exists xQ(x)$.

The first equivalence is obvious:

$w \vdash \forall xP(x) \leftrightarrow \forall a \xi_a \leq w \leftrightarrow w \in Y$

by (i) and the definition of $Y$.

Next, by (ii)

$W \vdash Q(a) \leftrightarrow w > \eta_0$,

which implies $w \in Y$.

Conversely, suppose $w \in Y$. Since $w$ is not a supremum of $X$, there is $a \in V$ such that $\eta_a < w$. Hence $w \vdash Q(a)$, and thus $w \vdash \exists xQ(x)$.

Therefore

(4) $M \vdash \forall xP(x) \supset \exists yQ(y)$.

Now consider any $\xi \in X$. Let us show that

(5) $\xi \not\vdash \exists x(P(x) \supset \exists zQ(z))$.

In fact, for any $a$

$\xi \leq \eta_a$, $\eta_a \not\vdash Q(a)$, $\eta_a \vdash \exists zQ(z)$

by (3), hence

$\xi \not\vdash \exists zQ(z) \supset Q(a)$.

(6) $\xi \vdash \exists y(\exists zQ(z) \supset Q(y))$. 
In fact, let us show

$$(7) \; \; \xi \not\models P(a) \supset \exists y Q(y)$$

for any $a \in V$. Consider $\xi' = \max(\xi, \xi_a)$. Then $\xi' \in X$, $\xi \leq \xi'$, $\xi \not\models P(a)$, and $\xi' \not\models \exists y Q(y)$ by (3). This implies (7) and (6).

But (2) implies

$$\xi \not\models (\forall x P(x) \supset \exists y Q(y)) \supset \exists x (P(x) \supset \exists z Q(z)) \lor \exists y (\exists z Q(z) \supset Q(y)),$$

which contradicts (4), (5), (6). \hfill \blacksquare

**Proposition 7.8.12**

1. $\text{IL}(CKQ) \subseteq \text{IL}(CKR)$.

2. If a formula $A$ does not contain $\forall$ or $\exists$, then $\text{IL}(CKQ) \vdash A$ iff $\text{IL}(CKQ) \vdash A$.

**Proof**

1. $\text{IL}(CKQ) \subseteq \text{IL}(CKR)$ readily follows from Theorems 7.8.2 and 7.8.9, while $\text{IL}(CKQ) \neq \text{IL}(CKR)$ by Proposition 7.8.11.

2. Let us consider only formulas without occurrences of $\forall$.

Let $A(x_1, \ldots, x_n)$ be a $\forall$-free formula, and suppose $\text{IL}(CKQ) \not\models A$. Consider a Kripke model $M_0 = (Q \cap V, \theta)$, $q \in Q$, $a_1, \ldots, a_n \in V$, such that $M_0, q \not\models A(a_1, \ldots, a_n)$. We define $M$ over $R$ similarly to the proof of 7.8.4:

$$(\#) \; \; M, \alpha \models B \iff \exists q \in Q (q \leq \alpha \& M_0, q \models B)$$

for every $\alpha \in R$ and atomic $V$-sentence $B$. Now we prove $(\#)$ for any $\forall$-free $B$, but unlike Lemma 7.8.5, we do not need the assumption (7.1). Hence $M, q \not\models A(a_1, \ldots, a_n)$.

$\exists$-free formulas are considered similarly using Lemma 7.8.7 instead of Lemma 7.8.5. \hfill \blacksquare

Takano [Takano, 1987] also proved that $\text{IL}(\mathcal{K}(R - Q)) = QLC + CD$. Actually his method proves the following statement.

**Theorem 7.8.13** If $W \subseteq R$ and both $W$ and $F - W$ are dense in $R$, $F = (W, <)$, then $\text{IL}(CKF) = \text{IL}(F \cap \omega) = QLC + CD$.

**Proof** It is sufficient to show that $\text{IL}(F \cap \omega) \subseteq \text{IL}(Q \cap \omega)$. So given a sentence $B$ and a countermodel $M = (Q \cap \omega, \xi)$ for $B$, we construct a model $M_1$ over $F \cap \omega$ refuting $B$. The construction consists of two stages.
(I) At the first stage we take a denumerable subset $W_0$ of $W$ that is dense in $\mathbb{R}$, $F_0 = (W_0, <)$, and construct a model $M_0 = (F_0 \odot \omega, \xi_0)$ refuting $B$ and satisfying the following condition:

$$\inf_{\mathbb{R}} \xi_0^+(A) \not\in (W - W_0)$$

for every $\omega$-sentence $A$. \hfill (+)

Put

$$S:=\{\inf_{\mathbb{R}} \xi_0^+(A) \mid A \text{ is an } \omega\text{-sentence}\} - \mathbb{Q}$$

and enumerate the sets $\mathbb{Q} \cup S$ and $W_0$, i.e., put $\mathbb{Q} \cup S = \{u_k \mid k \in \omega\}$ and $W_0 = \{v_n \mid n \in \omega\}$. Now since $W_0$ and $\mathbb{R} - W$ are dense in $\mathbb{R}$, there exists an order embedding $h : \mathbb{Q} \cup S \rightarrow \mathbb{R}$ such that $h[\mathbb{Q}] = W_0$ and $h[S] \subseteq \mathbb{R} - W$.

To construct $h$, we can apply the back-and-forth method and define in turn $h(u_k)$ and $h^{-1}(v_n)$ for new elements of $\mathbb{Q} \cup S$ and $W_0$ respectively, preserving the order $<$. Or we can define $h(u_k)$ in the following direct way, by induction on $k$. Suppose $h(u_i)$ for $i < k$ are already defined, let

$$u_l := \max\{u_i \mid i < k, u_i < u_k\}, \quad u_r := \min\{u_i \mid i < k, u_k < u_i\}.$$

If $u_k \in \mathbb{Q}$, we put $h(u_k) = v_n$ for the least $n$ such that $v_n \in (h(a), h(u_r))$. And if $u \in S$, we choose $h(u_k)$ in $(h(u_l), h(u_r)) - W$. Then definitely every $v_n \in W_0$ is $h(u_k)$ for a suitable $k$, so to say, $n$ becomes ‘the least possible’ at some stage of the construction.

We leave the routine technical details to the reader.

Let us show that

$$\forall X \subseteq \mathbb{Q} \left( \inf_{\mathbb{R}} X \in \mathbb{Q} \cup S \Rightarrow \inf_{\mathbb{R}} h[X] = h(\inf_{\mathbb{R}} X) \right). \hfill (*)$$

In fact, let $\inf_{\mathbb{R}} X = u_k$. Then $h(u_k) \leq h(u)$ for all $u \in X$. Suppose $\inf_{\mathbb{R}} h(X) > h(u_k)$. By density, there exists $v \in W_0$ such that $h(u_k) < v < \inf_{\mathbb{R}} h(X)$; let $v = h(u), u \in \mathbb{Q}$. Then $u$ is a lower bound of $X$, so $u \leq u_k$, and hence $v = h(u) \leq h(u_k)$. This is a contradiction.

Obviously, the restriction $h_0 = h|\mathbb{Q}$ is an order isomorphism between $\mathbb{Q}$ and $W_0$, so $h_0^{-1} : W_0 \rightarrow \mathbb{Q}$ is a $p$-morphism. Now by Lemma 5.1 we obtain a model $M_0 = (W_0 \odot \omega, \xi_0)$ such that

$$\xi_0^+(A) = h[\xi_0^+(A)]$$

for every $\omega$-sentence $A$. This model is a required one, because $(*)$ readily implies $(+)$. In fact,

$$\inf_{\mathbb{R}} \xi_0^+(A) = h(\inf_{\mathbb{R}} \xi_0^+(A)) \in h[\mathbb{Q} \cup S] \subseteq W_0 \cup (\mathbb{R} - W).$$

(II) Now we apply Lemma 7.8.4 and obtain a right extension $M_1$ of $M_0$. This model is a required one over $W$. It is sufficient to check the condition $(r2)$. 


So we take an arbitrary $w_0 \in W$, assume that
\[ \forall a \in V \exists u \in W_0 \ (u \leq w_0 \& M_0, u \models A(a)), \tag{7a} \]
and find $u \in W_0$ such that
\[ u \leq w_0 \text{ and } M_0, u \models \forall xA(x). \tag{7b} \]

Put $u_0 := \inf \{ u \in W_0 \mid M_0, u \models \forall xA(x) \}$.

**Case 1.** $u_0 \in W_0$ and $M_0, u_0 \models \forall xA(x)$.

Let us show that $u_0 \leq w_0$. Suppose the contrary: $w_0 < u_0$. Then by density there exists $u_1 \in W_0$ such that $w_0 < u_1 < u_0$. On the other hand, for every $a \in V$ there exists $u \in W_0$ such that $u \leq w_0 < u_2$ and $M_0, u \models A(a)$, so $M_0, u_1 \models A(a)$ as well. Hence $M_0, u_1 \models \forall xA(x)$ and $u_1 < u_0$, which contradicts the choice of $u_0$.

Thus we conclude that $u = u_0$ satisfies (7b).

**Case 2.** $u_0 \in W_0$ and $M_0, u_0 \not\models \forall xA(x)$. Then $M_0, w_0 \not\models A(a)$ for some $a \in V$. By (7a), there exists $u \in W_0$ such that $u \leq w_0$ and $M_0, u \models A(a)$. Then $u_0 < u$, and so $M_0, u \models \forall xA(x)$ by the definition of $u_0$. Thus $u$ satisfies (7b).

**Case 3.** $u_0 \not\in W_0$. Then $u_0 \not\in W$ by (+). Let us show that $u_0 < w_0$. Suppose the contrary. Then $w_0 < u_0$, since $w_0 \in W$ and $u_0 \not\in W$. So by density, there exists $w \in W_0$ such that $w_0 < w < u_0$. Then $M_0, w \not\models \forall xA(x)$, by the definition of $u_0$. So $M_0, w \not\models A(a)$ for some $a \in V$. On the other hand, $M_0, u \models A(a)$ for some $u \leq u_0$, and hence $M_0, w \models A(a)$ as well. Since $u \leq w_0 < w$, this is a contradiction.

Now since $u_0 < w_0$, there exists $u \in W_0$ such that $u \leq w_0$ and $M_0, u \models \forall xA(x)$. This $u$ satisfies (7b).

---

**Corollary 7.8.14** $\text{IL}(\mathcal{C}(\mathbb{R} - \mathbb{Q})) = \text{IL}((\mathbb{R} - \mathbb{Q}) \cup \omega) = \text{QLC} + \text{CD}$.

**Remark 7.8.15** The above argument can be readily transferred to the logics with equality. In fact, Lemmas 7.8.5 and 7.8.7 are extended easily.

The conditions $(\tau \ast)$ and $(l \ast)$ for equality:
\[ M, u \models a = b \iff \exists v \in W_0 (vRw \& M_0, u \models a = b) \]
and
\[ M, u \not\models a = b \iff \exists v \in W_0 (wRv \& M_0, v \not\models a = b) \]
respectively follow from $(r1)$ and $(l1)$.

Thus we obtain the following analogues of Theorems 7.8.2, 7.8.9 and 7.8.12:
7.8. KRIPKE FRAMES OVER THE REALS AND THE RATIONALS

(1) If $W$ and $(\mathbb{R} - W)$ are dense in $\mathbb{R}$ then

$$IL^= (\mathcal{CE}(W)) = QLC^= + CD, \quad IL^= (\mathcal{C}(W)) = QLC^=d + CD;$$

in particular

$$IL^= (\mathcal{C}EQ) = IL^= (\mathcal{C}E(R - Q)) = QLC^= + CD,$$

$$IL^= (\mathcal{C}Q = IL^= (\mathcal{C}(R - Q)) = QLC^=d + CD.$$  

(2)

$$IL^= (\mathcal{KE}) = QLC^d + CD + Ta,$$

$$IL^= (\mathcal{K}(R)) = QLC^=d + CD + Ta.$$  

These results imply corollaries for the Kripke semantics with nested domains.

Recall that $IL(\mathcal{K}(Q)) = QLC$, $IL^= (\mathcal{KE}Q) = QLC^=, IL^= (\mathcal{K}Q) = QLC^=d$. Hence by the method from section 3.9 we obtain

**Proposition 7.8.16** If subsets $W$ and $R - W$ are dense in $\mathbb{R}$, then

$$IL(\mathcal{K}W) = QLC, \quad IL^= (\mathcal{KE}W) = QLC^=, \quad IL^= (KW) = QLC^=d.$$  

Let us finally mention some simple consequences of the above results.

**Theorem 7.8.17** The logics $IL^= (\mathcal{KR})$, $IL^= (\mathcal{KE}R)$ are recursively axiomatisable.

**Proof** Follows from 7.8.9 and 3.9.4.  

**Corollary 7.8.18** $IL(\mathcal{KR})$ is recursively axiomatisable.

**Proof** In fact, we already know that $IL^= (\mathcal{KR})$, $IL^= (\mathcal{KE}R)$ are its conservative extensions (by 3.8.6 and 2.16.13).

As we know, the completeness proof for $QLC$ can be extended to logics with equality, so

$$IL^= (\mathcal{CE}Q) = QLC^=, \quad IL^= (\mathcal{K}Q) = QLC^=d.$$  

As we know from 7.8.2 and 7.8.14

$$IL(\mathcal{K}(R - Q)) = IL(\mathcal{K}Q) = QLC + CD.$$  

Thus 7.8.18 implies

**Proposition 7.8.19**

$$IL(\mathcal{K}(R - Q)) = IL(\mathcal{K}Q) = QLC;$$

$$IL^= (\mathcal{KE}(R - Q)) = IL^= (\mathcal{KE}Q) = QLC^=;$$

$$IL^= (\mathcal{K}(R - Q)) = IL^= (\mathcal{K}Q) = QLC^=d.$$
Perhaps Takano’s proof for \( \mathbb{R} - \mathbb{Q} \) can also be transferred to the case with nested domains, but the embedding method provides the result rather easily.

As for an explicit axiomatisation for \( \text{IL}(\mathcal{KR}) \), \( \text{IL}^\neq(\mathcal{KR}) \), \( \text{IL}^\neq(\mathcal{KE} \mathcal{R}) \), one can easily see that \( Ta \) is refuted in a frame over \( \mathbb{R} \). In fact, informally speaking, \( Ta \) means the following.

Suppose \( \forall x P(x) \sqsupset \exists y Q(y) \) is true, say at world \( 0 \in \mathbb{R} \). Let \( v \) be the g.l.b. of the set \( \{ u \mid u \models r \} \) (if this set is empty, then \( Ta \) holds trivially). Now if \( v \not\models \forall x P(x) \), i.e. \( u \not\models P(a) \) for some \( u \geq v \), then \( 0 \models P(a) \sqsupset r \). Otherwise \( v \models \forall x P(x) \), and then \( v \models \exists y Q(y) \), i.e. \( r \models Q(b) \) for some \( b \), so \( 0 \models r \sqsupset Q(b) \).

If the domain is constant, both individuals \( a, b \) are in \( D_0 \), but in the case of nested domains \( a, b \) can exist only in larger domains \( D_u \) or \( D_v \), so the formula \( Ta \) is refutable. Constructing a counterexample is left as an exercise for the reader.

**Conjecture.** \( \text{IL}(\mathcal{KR}) = \text{QLC} \)

By Theorem 3.8.6, this conjecture implies the equality

\[
\text{IL}^\neq(\mathcal{KR}) = \text{QLC}^\neq,
\]

and (as one can show),

\[
\text{IL}^\neq(\mathcal{KE} \mathcal{R}) = \text{QLC}^\neq.
\]
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